NONSTATIC FLUID SPHERES IN GENERAL RELATIVITY

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A class of exact interior solutions for a spherically symmetric perfect fluid distributions with nonhomogeneous density and pressure is obtained. It is shown that in some cases the intial outward motion is reversed and finally the system collapses to a singularity of infinite matter density. There are also cases when an initially collapsing sytem may bounce back so that the ultimate catastrophe can be avoided. The metric obtained is nonsingular and satisfies the conditions that the pressure and matter density within the distribution is nonnegative.

1. Introduction

Spherically symmetric relativistic fluid spheres consisting of perfect fluid were discussed in the literature by several workers with different ways of approach. One of them is to impose certain symmetry conditions and restrictions on the metric and to find some analytical solutions (McVittie 1967, Thompson and Whitrow 1967, Bonnor and Faulkes 1967, Bondi 1969). Most of them considered a fluid of uniform density. Nonuniform models were studied by Nariai (1967) and Faulkes (1969). We give here a new class of exact interior solutions for the radial motion of a perfect fluid sphere with nonuniform density and pressure distributions. Although the solutions are, to some extent, similar to those given by Nariai, they are different and have different properties. Our solutions include cases of collapse to a singularity as well as those of bounce. The interior solutions can be matched at the boundary with the outside Schwarzschild metric (Cocke 1966). From the boundary conditions one can study the behaviour of the model at different instants of time, the only restriction being that the matter density ϱ and the pressure p satisfy the following conditions

$$\begin{aligned} &\varrho > 0 \quad \text{in the region} \quad 0 \leqslant r \leqslant r_0, \\ &p > 0 \quad \text{in the region} \quad 0 \leqslant r \leqslant r_0, \\ &p = 0 \quad \text{at} \quad r = r_0, \end{aligned}$$

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so that the models are physically realistic. It is an interesting property of the solution that it closely resembles the cosmological Robertson–Walker line element and indeed goes over to the latter in a special case.

2. Field equations and their solutions

We assume that the sphere consists of a perfect fluid and its motion is shear free, so that the line element can be written in the isotropic form

$$ds^2 = e^{\nu}dt^2 - e^{\omega}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2). \tag{1}$$

Using co-moving co-ordinates we further get

$$v^{\mu} = e^{-\nu/2}\delta_4^{\mu} \tag{2}$$

and

$$T_1^1 = T_2^2 = T_3^3 = -p, \quad T_4^4 = \varrho. \tag{3}$$

The field equations give the following two relations (Faulkes 1969)

$$e^{-\nu/2}\dot{\omega} = C(t) \tag{4}$$

and

$$\frac{\partial^2 R}{\partial \kappa^2} = \Gamma(\kappa) R^2, \tag{5}$$

where $R = e^{-\omega/2}$ and $\kappa = r^2$. $\Gamma(\kappa)$ is an arbitrary function of κ. As was pointed out by Faulkes, one may obtain the case of uniform density by choosing $\Gamma(\kappa) = 0$. On the other hand, when Γ is a constant one arrives at Faulkes' solution. In this paper we give a new class of exact solutions with Γ as a chosen function of κ. The solutions are

$$e^{\omega/2} = \frac{(T + a/y)^2}{y}$$

and

$$e^{\nu} = \frac{1}{(T + a/y)^2}, \tag{6}$$

where $y = B\kappa + C' = Br^2 + C'$. T is a function of time and a, B, C are arbitrary constants. Here $\Gamma(\kappa) = \dfrac{6aB^2}{(B\kappa + C')^5}$. The solutions for $g_{00} = e^{\nu}$ are obtained from (4) by a suitable choice of constant of integration $C(t)$. One can now eliminate one of the arbitrary constants C' by a transformation of the radial co-ordinate, so that the line element may be written as

$$ds^2 = \frac{1}{\left(T + \dfrac{a}{1+kr^2}\right)^2} dt^2 - \frac{\left(T + \dfrac{a}{1+kr^2}\right)^4}{(1+kr^2)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2). \tag{1a}$$

From (1a) by a suitable adjustment of the scale factors for the radial and time co-ordinates, one can choose k as $k = \pm 1$ or 0. The line-element (1a) closely resembles the Robertson–Walker cosmological line element and actually reduces to the latter when $a = 0$. The case $k = 0$ corresponds to the open model of Einstein and de Sitter, where the space-time becomes spatially flat and spatially infinite in extent.

One can now evaluate directly from the field equations the matter density ϱ and pressure p as

$$8\pi\varrho(r, t) = 12\dot{T}^2 + \frac{12k}{(T+a/y)^4} + \frac{24ak}{(T+a/y)^5}\frac{(1-kr^2)}{(1+kr^2)} \tag{7}$$

and

$$8\pi p(r, t) = -4\ddot{T}(T+a/y) - 12\dot{T}^2 - \frac{4ak}{(T+a/y)^5}\frac{(1-kr^2)}{(1+kr^2)} - \frac{4k}{(T+a/y)^4}, \tag{8}$$

where y is now written for $(1+kr^2)$.

We obtain two more relations from the divergence equations $T^{\mu\nu}{}_{;\nu} = 0$, that is

$$p' = -\frac{(\varrho+p)2akr}{(T+a/y)y^2}, \tag{9}$$

$$\dot{\varrho} = -\tfrac{3}{2}(\varrho+p)\dot{\omega}, \tag{10}$$

with

$$\dot{\omega} = \frac{4\dot{T}}{(T+a/y)}.$$

3. *The boundary conditions and the behaviour of the model*

The metric (1a) is matched to the Schwarzschild metric across the moving boundary (Cocke 1966) provided that

$$p(r_0, t) = 0,$$

$$2m = \tfrac{1}{4}\dot{\omega}_0^2 r_0^3 e^{(3\omega_0/2-\nu_0)} - \tfrac{1}{4}\omega_0'^2 r_0^3 e^{\omega_0/2} - r_0^2\omega_0' e^{\omega_0/2}, \tag{11}$$

where the subscript "0" indicates the values at the boundary $r = r_0$ and m stands for the Schwarzschild mass. In fact the constancy of m, that is the conservation of the gravitational mass, is equivalent to the requirement of the vanishing of pressure at the boundary (Raychaudhuri 1953).

In view of metric (1a) the condition (11) reduces to

$$4\dot{T}^2 = \frac{2m}{r_0^3}\frac{y_0^3}{(T+a/y_0)^6} - \frac{4k}{(T+a/y_0)^4} - \frac{8ak}{(T+a/y_0)^5}\frac{1}{y_0} + \frac{8ak^2r_0^2}{(T+a/y_0)^5}\frac{1}{y_0} + \frac{16a^2k^2r_0^2}{(T+a/y_0)^6}\frac{1}{y_0^2} \tag{12}$$

with $y_0 = (1+kr_0^2)$.

Putting $(T+a/y_0) = S_0$, since $\dot{T}^2 \geqslant 0$ one can readily obtain the relation

$$\left[\frac{2m}{r_0^3}(1+kr_0^2)^3+\frac{16a^2k^2r_0^2}{(1+kr_0^2)^2}\right]-8ak\frac{(1-kr_0^2)}{(1+kr_0^2)}S_0-4kS_0^2 \geqslant 0. \tag{13}$$

The equality sign corresponds to the case $\dot{T} = 0$, which refers to the turning point in the motion.

Again from (8) and (12) remembering that $p = 0$ at $r = r_0$ one gets for $\dot{T} = 0$ (or $\dot{S}_0 = 0$) the relation

$$2(\ddot{S_0}) = 4S_0\ddot{S}_0 = -\frac{1}{S_0^6}\left[2kS_0^2+\tfrac{1}{2}\left\{\frac{2m}{r_0^3}(1+kr_0^2)^3+\frac{16a^2k^2r_0^2}{(1+kr_0^2)^2}\right\}\right]. \tag{14}$$

When $k = +1$ the right side is negative and $d^2(S_0^2)/dt^2 < 0$ for $d(S_0^2)/dt = 0$. This is equivalent to stating that S_0^2 will have a maximum and no minimum. The system in such a case with an initial outward motion will reach a maximum of the proper volume, will reverse its motion and finally will collapse to a singularity $S_0 \to 0$; i.e. to a singular state of infinite density. A condition necessary but not sufficient for bounce from initially collapsing state may be given by $k = -1$.

4. *Case of collapse to zero volume*

Case I:

Let $k = +1$, $a > 0$ and $(1-r_0^2) > 0$. Putting

$$P = \frac{8a(1-r_0^2)}{(1+r_0^2)} \quad \text{and} \quad Q = \left[\frac{2m}{r_0^3}(1+r_0^2)^3+\frac{16a^2r_0^2}{(1+r_0^2)^2}\right],$$

the relation (13) gives

$$X = 4S_0^2+PS_0-Q \leqslant 0. \tag{15}$$

The equality sign corresponds to the instant $\dot{T} = 0$ when the motion is reversed and this occurs for two values of S_0 given by

$$S_0 = \frac{-P\pm(P^2+16Q)^{1/2}}{8}. \tag{16}$$

The negative value of S_0 is not allowed because of the physical requirement that p' at the boundary $r = r_0$ should be negative (see Eq. (9)), which is again a consequence of the fact that $p = 0$ at $r = r_0$ and $p > 0$ in the region $0 \leqslant r < r_0$. Thus there is a maximum of S_0 given by

$$(S_0)_{\max} = \frac{-P+(P^2+16Q)^{1/2}}{8},$$

where the initial outward motion is reversed and finally collapses to singularity of zero proper volume at the boundary. One should note that at this moment, although the proper volume at the boundary tends to zero when $\dot{S}_0 \to 0$, the corresponding quantity remains finite in the region $0 \leqslant r < r_0$. The matter density increases to infinity at all points at the final stage of collapse because then $\dot{T} \to \infty$ (see Eq. (12)). In view of (7) and (9) $\varrho > 0$ throughout the region $0 \leqslant r \leqslant r_0$ and $p' < 0$ at the boundary $r = r_0$. In view of the fact that at the boundary $p = 0$ and $p' = 0$ and also that everywhere within the distribution the matter density ϱ is positive one can conclude readily from the relation (9) that p must be monotonically decreasing function of r. Thus we get a physically realistic collapsing model with $\varrho > 0$ and $p > 0$ in the interior region.

Case II:

$k = +1$, $a < 0$, and $(1-r_0^2) > 0$. In this case the relation (15) reduces to

$$4S_0^2 - PS_0 - Q \leqslant 0, \tag{15a}$$

where P and Q are again positive with P in (15a) being given by

$$P = \frac{8a(r_0^2-1)}{(r_0^2+1)},$$

Q has the same value as before.

It is also a collapsing model with the same characteristics as in case I with S_0 being replaced by $-S_0$, without any change, however, in S_0^2 which is a measure of the dimension of the system. Here the negative root of (15a) is admissible. It can be shown in the same manner as before that the matter density ϱ and pressure p remain greater than zero in the interior and vanish exactly at the boundary.

5. *A possible case for bounce*

Case III:

Let $k = -1$. Now since $y = (1-r^2)$, $1 > r_0^2$ in order that there is no singularity of the type $y = 0$ in the metric for any value of r in the interior of the system.

We further choose $a < 0$ so that $a = -|a|$. The condition (13) can now be written in the form

$$4S_0^2 - 8|a|\left(\frac{1+r_0^2}{1-r_0^2}\right)S_0 + \left[\frac{2m}{r_0^3}(1-r_0^2)^3 + \frac{16a^2r_0^2}{(1-r_0^2)^2}\right] \geqslant 0 \tag{17}$$

with

$$S_0 = \left(T - \frac{|a|}{1-r_0^2}\right).$$

The equality sign corresponds to the turning points in the motion of expansion or contraction of the fluid sphere or equivalently to the situation $\dot{T} = 0$. Putting

$$P_1 = 8|a|\left(\frac{1+r_0^2}{1-r_0^2}\right) \quad \text{and} \quad Q_1 = \left[\frac{2m}{r_0^3}(1-r_0^2)^3 + \frac{16a^2r_0^2}{(1-r_0^2)^2}\right]$$

relation (17) reduces to

$$X = 4S_0^2 - P_1 S_0 + Q_1 \geqslant 0 \tag{18}$$

which is again equivalent to

$$X = (S_0 - S_1)(S_0 - S_2) \geqslant 0. \tag{18a}$$

The quantities S_1 and S_2 correspond to two roots of the quadratic equation $X = 0$. It turns out from the relation (18a) that S_0 cannot lie between S_1 and S_2, because in this case $X < 0$. So either the system may collapse from a maximum volume corresponding to

$$S_1 = \frac{P_1 - (P_1^2 - 16Q_1)^{1/2}}{8}$$

to a singularity $S_0 \to 0$ (where the matter density becomes infinitely large at all points of the sphere) or it may expand from a minimum volume corresponding to

$$S_2 = \frac{P_1 + (P_1^2 - 16Q_1)^{1/2}}{8}.$$

The latter case corresponds to a bounce for an initially collapsing model turning back at the point $S_0 = S_2$.

We next examine the magnitude of density and pressure of the fluid in the case III. It turns out from (7) and (12) that

$$\tfrac{8}{3}\pi\varrho(r_0, t) = \left[\frac{2m}{r_0^3}\frac{(1-r_0^2)^3}{S_0^6} + \frac{16a^2r_0^2}{S_0^6(1-r_0^2)^2}\right]. \tag{19}$$

The relation (19) immediately gives for the matter density at the boundary $r = r_0$, the condition $\varrho(r_0, t) > 0$ at all instants.

Further taking derivatives of both sides of (7) with respect to the radial co-ordinate one obtains

$$T\varrho' = \frac{30a^2 r}{\left(T - \dfrac{|a|}{1-r^2}\right)^6}\frac{(1+r^2)}{(1-r^2)^3} \tag{20}$$

which shows that the matter density ϱ is a monotonically increasing function of the radial co-ordinate. It turns out, therefore, that the problem of obtaining a positive density everywhere reduces to the problem of making the central density ϱ_c (at $r = 0$) positive by a suitable choice of the constant parameters.

In fact, this can be shown in the following way. The condition that $\varrho_c > 0$ at the moment $\dot{T} = 0$ requires in view of (7)

$$T < 3|a|, \tag{21}$$

where

$$T = S_2 + \frac{|a|}{1-r_0^2} \quad \text{or} \quad T = S_1 + \frac{|a|}{1-r_0^2}.$$

Since $S_2 > S_1$, it is sufficient to show that

$$S_2 + \frac{|a|}{1-r_0^2} < 3|a|, \tag{22}$$

so that at the points of reversal of motion the central density remains positive. The inequality (22) is, in fact, equivalent to the form

$$r_0 < 1/2 \tag{23}$$

which is a restriction on the dimension of the system. Again the condition that S_1 and S_2 are real (i.e. $P_1^2 > 16Q_1$) puts a restriction on the Schwarzschild mass of the fluid in the form

$$m < \frac{2r_0^3|a|^2}{(1-r_0^2)^3}\,. \tag{24}$$

If the above inequalities (23) and (24) are satisfied one gets a positive density distribution during the whole regime of collapse in case III, as is evident from (7). Regarding the example of a bouncing model the density distribution is a positive function at least close to the moment of bounce. How long after the bounce the density will remain non negative everywhere will depend on the magnitude of a.

Case IV:

$k = -1$ and $a > 0$. We get in this case basically the same models as those discussed in case III with S_0 being replaced by $-S_0$ and they do not need any separate discussion.

6. *Conclusion*

We have given above the exact solution for a perfect fluid sphere which collapses to a point from a maximum volume or bounces back from a minimum volume depending on the parameter chosen. For an external observer in the Schwarzschild co-ordinates the relation between the radial co-ordinate $\bar{r}$ and the co-moving co-ordinate r is from (11)

$$\bar{r}(r, t) = re^{\omega/2}. \tag{25}$$

Also

$$e^{-\nu_0}(\dot{\bar{r}})^2(r, t) = \left(\frac{r\bar{r}'}{\bar{r}}\right)_0^2 - \left(1 - \frac{2m}{\bar{r}_0}\right), \tag{26}$$

where

$$\dot{\bar{r}} = \frac{\partial \bar{r}}{\partial t}, \qquad \bar{r}' = \frac{\partial \bar{r}}{\partial r}.$$

Equation (26) shows that at the turning points $\bar{r}_0 > 2m$. One of us (Banerjee 1971) has shown that sometimes we can have a peculiar situation when $(\bar{r}'/\bar{r})_0$ is negative, i.e.

although $\bar{r}_0 > 2m$, $\bar{r}$ may equal $2m$ outside the boundary. Just outside the sphere of matter the circumferences of the spherical surfaces decrease as the radial distances increase in this case. The condition that $(\bar{r}'/\bar{r})_0 < 0$ at the turning points is seen to be

$$S_0(1-k^2 r_0{}^4) < 4akr_0^2. \tag{27}$$

Substituting the values of S_0 in the different cases we find that this codition cannot be satisfied in the case of bounce, but it may be satisfied for the cases of collapse by giving suitable values to the parameters. This means that a sphere always bounces back before reaching the Schwarzschild radius as expected.

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