THE PHASE OF THE SCATTERING AMPLITUDE SATISFYING GEOMETRICAL SCALING*

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Geometrical scaling connects, at $t = 0$, the phase of the pomeron amplitude $F(s, t)$ to that of its derivative $\partial F(s, t)/\partial t$. This intrinsic phase correlation allows one to derive a dispersion sum rule relating the high- and low-energy regions.

Within axiomatic field theory it was shown few years ago [1] that, if the total cross-section of a process grows like $\ln^2 s$, the scattering amplitude $F(s, t)$ exhibits a scaling property

$$F(s_i, t)/F(s_i, 0) \to f(\tau)$$

for $|t| < \text{const}/(\ln s)^2$ and for a sequence of energies s_i tending to infinity. Here $\tau = t\ln^2 s$ and $f(\tau)$ is a nontrivial entire function of order 1/2. A phenomenological extension of this scaling property for $\sigma_{\text{tot}}(s)$ rising slower than $\ln^2 s$ was suggested [2] and subsequent consequences of the so-called geometrical scaling (GS) for the pomeron amplitude have been presented by Dias de Deus [3]. In particular, from GS he has obtained the following fixed s, t-derivative analyticity relation

$$\frac{\text{Re}\, F(s, t)}{\text{Re}\, F(s, 0)} = \frac{\partial}{\partial t}\left[t\,\frac{\text{Im}\, F(s, t)}{\text{Im}\, F(s, 0)}\right]. \tag{1}$$

As a consequence of this equation one gets in the forward direction

$$\frac{\text{Re}\, F'(s, 0)}{\text{Im}\, F'(s, 0)} = 2\,\frac{\text{Re}\, F(s, 0)}{\text{Im}\, F(s, 0)}, \quad s \text{ large}. \tag{2}$$

In what follows we make use of this specific relationship between the phase of $F(s, t)$ and that of $F'(s,t)$ at $t = 0$ by deriving a dispersion formula which could prove to be a sensitive test for the parametrizations of amplitudes satisfying GS.

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If GS holds true the pomeron amplitude $F(s,t)$, $s \leftrightarrow u$ crossing symmetric, should have the following structure [3]

$$F(s,t) \simeq isR^2\left(\xi - i\frac{\pi}{2}\right)\phi(tR^2), \quad \xi = \ln s,$$

for fixed t and large s. The functions R^2 and ϕ are "real" analytic functions; normalization yields ϕ (0) = 1. Now we assume that, according to the current trend of experimental data [4] for $\sigma_{\rm tot}(s)$,

$$R^2\left(\xi - \frac{i\pi}{2}\right) \simeq \text{const}\left(\xi - i\frac{\pi}{2}\right)^\beta, \quad 0 < \beta \leqslant 2. \tag{3}$$

Simple algebra gives then

$$\frac{\operatorname{Re} F'(s,0)}{\operatorname{Im} F'(s,0)} - 2\frac{\operatorname{Re} F(s,0)}{\operatorname{Im} F(s,0)} = 2\frac{\operatorname{Im} R^2}{\operatorname{Re} R^2} - \frac{\operatorname{Im} R^4}{\operatorname{Re} R^4} \simeq \operatorname{tg}\frac{\pi\beta}{\xi} - 2\operatorname{tg}\frac{\pi\beta}{\xi} \simeq \tfrac{1}{4}(\pi\beta/\ln s)^3. \tag{4}$$

Thus with the specific assumption (3) we get more detailed information about the asymptotic equality (2). Relation (4) is crucial for the derivation of our final result.

For definiteness let us consider the scattering of pions (mass m) on protons (mass M). It is convenient to replace the crossing symmetric scattering amplitude $F(s,t)$ by

$$G(s,t) = F(s,t) - F_{\rm B}(s,t) - F(s_0,t) - \gamma, \quad \gamma > 0,$$

where $F_{\rm B}(s,t)$ stands for the Born term and $s_0 = (m+M)^2$. Due to the positivity of $\operatorname{Im} F(s,t)$ for $s > s_0$ and $t \geqslant 0$ and our choice of γ we can easily show that $G(s,t)$, analytic in the cut s-plane, has no zeros. Therefore, introducing the symmetric variable $\nu = (s-u)^2/4 = (s+t/2-(m^2+M^2))^2$, we get

$$\frac{\sqrt{\nu-\nu_0}}{G(\nu,t)} = \frac{1}{2\pi i}\oint\frac{dz\sqrt{z-\nu_0}}{(z-\nu)G(z,t)}, \quad \nu_0 = (t/2+2Mm)^2; \tag{5}$$

the integration contour is shown in Fig. 1.

For $s = s_0$ this equation reduces to

$$4\int_{(M+m)^2}^{r}\frac{ds[s+t/2-(M^2+m^2)]Q(s,t)}{\sqrt{[s+t-(M-m)^2]\,[s-(M+m)^2]}} + \oint_{C_{r_\nu}} = 0,$$

where $Q(s,t) = D(s,t)/[D^2(s,t)+A^2(s,t)]$, $D(s,t) = \operatorname{Re}\ G(s,t)$ and $A(s,t) = \operatorname{Im}\ F(s,t)$. Differentiating both sides of this equation with respect to t and keeping r large but fixed we get in the limit $t \to 0$

$$4Mm\int_{(M+m)^2}^{r}\frac{dsQ(s,0)}{[s-(M+m)^2]^{1/2}[s-(M-m)^2]^{3/2}}$$

$$+4\int_{(M+m)^2}^{r}\frac{ds[s-(M^2+m^2)]\frac{\partial}{\partial t}Q(s,t)|_{t=0}}{[(s-(M-m)^2)\,(s-(M+m)^2)]^{1/2}} = -\frac{\partial}{\partial t}\int_0^{2\pi}\frac{d\varphi}{R^2\phi(tR^2)}\bigg|_{t=0} + O(1/r).$$

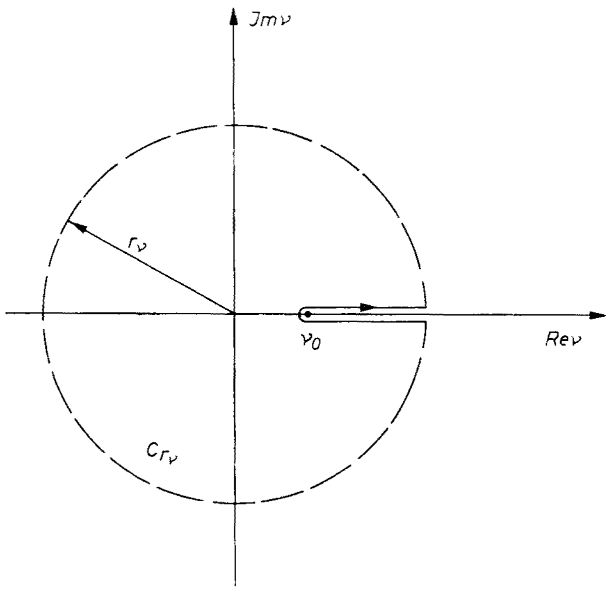

Fig. 1. Integration contour used in the derivation of Eq. (5)

Remembering that in the infinite enelgy limit the slope of the diffraction peak is given by $B(\infty)/2\sigma_{tot}(\infty) = \phi'(0)$ we finally obtain for $r \to \infty$

$$Mm \int_{(M+m)^2}^{\infty} \frac{dsQ(s,0)}{[s-(M+m)^2]^{1/2}[s-(M-m)^2]^{3/2}}$$

$$+ \int_{(M+m)^2}^{\infty} \frac{ds[s-(M+m)^2]Q'(s,0)}{[(s-(M-m)^2)(s-(M+m)^2)]^{1/2}} = \frac{\pi}{4} B(\infty)/\sigma_{tot}(\infty) \tag{6}$$

if the corresponding limits exist. There are obviously no problems with the first integral; the second one requires more attention. Its existence is related to the convergence of

$$\int^{\infty} ds \frac{A'(s,0)}{A^2(s,0)}\left[\frac{D'(s,0)}{A'(s,0)} - 2\varrho(s,0)\right] \quad \text{and} \quad \int^{\infty} ds \frac{D'(s,0)}{A^2(s,0)} \varrho^2(s,0), \tag{7}$$

where $\varrho(s, 0) = D(s, 0)/A(s, 0)$. Note that

$$\frac{A'(s,0)}{A^2(s,0)} \simeq \frac{\phi'(0)}{s\phi^2(0)} = \frac{\phi'(0)}{s}.$$

This $1/s$ damping factor is not sufficient to ensure the convergence of the first integral but the phase relation (4) helps. In a similar way we can show that the second integral converges since we expect $\varrho(s, 0) = \text{const}/\ln s$ and $\lim_{s\to\infty} D'(s, 0)/A'(s, 0) = 0$. Finally, introducing as a new integration variable the energy of the projectile in the lab system ($s = M^2+m^2+2M\omega$) we get in a more symmetric form

$$4M \int_m^\infty \frac{d\omega Q'(\omega, 0)\omega}{(\omega^2-m^2)^{1/2}} + m \int_m^\infty \frac{d\omega Q(\omega, 0)}{(\omega^2-m^2)^{1/2}(\omega+m)} = \frac{\pi}{2} \frac{B(\infty)}{\sigma_{tot}(\infty)} . \qquad (8)$$

Due to our choice of $s(=s_0)$ Eq. (8) gives a presumably strong, non-local connection between the threshold and high-energy behaviour of $F(s, 0)$. A similar relation can be obtained for $s > s_0$, emphasizing the importance of different energy regions. In general, the abundance of, e. g., πN data from the threshold up to very high energies should make a detailed numerical analysis of such a dispersion formula feasible.

So far, only the threshold value of $F(s, t)$ and the Born term were subtracted from the physical amplitude. Note, however, that all conclusions would remain unchanged if the function $G(s, t)$ were replaced by

$$G(s, t) - i\eta[(s-s_0)(s+t-(M-m)^2)]^{1/2}$$

with η small enough to not violate positivity. By varying η we can make the numerical results sensitive to the behaviour of $F(s, 0)$ and $F'(s, 0)$ around, for example, the well known high-energy dip of $\sigma_{tot}(s)$ between 50 and 100 GeV/c [5]. In particular, it would be interesting to investigate the problem of the Van Hove [6] unitary limit for the ratio B/σ_{tot} which is not yet saturated for πN scattering (but is in the pp case [7]).

To sum up, we have derived a dispersion equality whose validity could perhaps provide us with a useful information concerning the high-energy parametrization of the scattering amplitudes. The obvious advantage of our costraint against the standard dispersion relations in that, on one hand, it really depends upon the more subtle details of the pomeron amplitude's high-energy structure on the other hand it tightly connects the high- and low-energy domains.

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REFERENCES

[1] G. Auberson, T. Kinoshita, A. Martin, *Phys. Rev.* **3D**, 3185 (1971).
[2] J. Dias de Deus, *Nucl. Phys.* **59B**, 231 (1973); A. J. Buras, J. Dias de Deus, *Nucl. Phys.* **71B**, 481 (1974).
[3] J. Dias de Deus, *Nuovo Cimento* **28A** 114 (1975).
[4] A. S. Carrol et al., *Phys. Rev. Lett.* **33**, 932 (1974).
[5] R. Wit, *Nucl. Phys.* **B91**, 88 (1975).
[6] L. Van Hove, *Rev. Mod. Phys.* **36**, 655 (1964).
[7] V. Barger, Rapporteur Talk, Proc. London Conf. 1974.