EIKONAL APPROACH TO THE SCATTERING OF CHARGED PARTICLES AND TO COULOMB EXCITATION

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The eikonal scheme is used to investigate the elastic and inelastic scattering of two charged particles. The t-matrix is found to be reproduced exactly at *all* scattering angles and at *all* energies both in the on-shell situation and for $\eta \gg 1$ in the half-off-shell case. The result is a consequence of choosing the direction of the eikonal approximation and the pole of the eikonal free Green function in a particular way so as to depend on the scattering angle. In situations where the Sommerfeld parameter η $(= zZe^2(\hbar v)^{-1}$ is large, as is a feature of heavy ions, analytic formulae are obtained for the t-matrix corresponding to scattering by displaced charges. The agreement with the exact result suggests that Coulomb excitation can be treated in this manner. An analytic expression is obtained for the derivative with respect to the nuclear charge of the Coulomb-excitation matrix element and the DWBA formalism is then used. The final result can be put in terms of integrals over the charge of expressions involving the half-off-shell t-matrix. The calculations refer only to single-step processes.

1. Introduction

The phenomenon of the scattering of two charged particles was first considered from a quantal viewpoint by Wentzel [1] and Oppenheimer [2] whose estimates were based on use of the first Born approximation. Exact treatments by Mott [3] and by Gordon [4] lead to an additional phase factor in the t-matrix

$$\exp\left[i\pi+2i \arg \Gamma(1+i\eta)\right], \tag{1.1}$$

where $\eta = zZe^2 (\hbar v)^{-1}$ is the Sommerfeld parameter associated with charges ze and Ze

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which move with relative velocity v. Gordon [4] also allowed for a screening radius $\boldsymbol{R}$ and showed that this gives rise to an extra factor

$$\exp\left[-i\eta \ln\left(2mvRh^{-1}\right)\right]. \tag{1.2}$$

This factor was however ignored; but it does lead to apparent divergencies on expansion. Dalitz [5] in fact, studied the divergent parts of the first three Born terms and surmised that the exponential factor (1.2) can be built up through the Born expansion and this was confirmed by Kacser [6] who calculated these terms exactly. The factor is known [7] however to be of little physical significance in the analysis of any specific measurement and we shall therefore ignore it in our calculations.

Adopting the eikonal scheme, we show in Sect. 2 that the exact quantal result is reproduced provided that the linearization, characteristic of the eikonal approximation, is taken to be normal to the direction of the momentum transfer and in the scattering plane. Also, the pole in the eikonal Green function must be given a particular dependence on the scattering angle. We note the agreement holds for all scattering angles and for any energy (neglecting relativistic features), as was shown earlier by Glauber [8], but in a somewhat restrictive fashion. It was also re-emphasised by the work of Moore [9]: the factor (1.1) is however lost in this work. The work of Wallace [10] runs parallel to our investigation. This investigation is accomplished by comparison, term by term, of a Born--series expansion while our method is to compare the final expressions for the t-matrices. Our conclusion is the same.

Sect. 3 allows for a displacement $\boldsymbol{r}$ of one or two charges in the target. The expression for the t-matrix splits naturally into an outer and inner portion, associated, respectively, with impact parameters larger and smaller than $|\boldsymbol{r}_\perp| = s$. The expressions involve an infinite series of generalized hypergeometric functions of the ${}_1F_2$ type. Great simplification obtains if $\eta \gg 1$. The inner portion of the scattering amplitude is seen to be negligible and the outer portion deviates from the quantal result simply by the presence of an extra multiplicative factor $\exp(i\boldsymbol{s}\cdot\boldsymbol{\Delta})$, where $\boldsymbol{\Delta} = \boldsymbol{k}_f - \boldsymbol{k}_i$ lies along the direction of momentum transfer.

Off-shell effects are dealt with in Sect. 4, first by evaluating the half-off-shell t-matrix for Coulomb scattering between two charges in situations where $\eta \gg 1$. Comparison is made with the exact expression set out by Ford [11]. The method is to assert equality of the modulus of the derivative of the t-matrix element with respect to Z_T, the charge number of the target in the two cases. This leads to evaluation of a parameter specifying the eikonal direction. In this we require the solution of a transcendental equation applicable to differing momenta and angles.

The procedure logically extends to Coulomb-excitation phenomena where η values are characteristically large. Guided by the equality of the eikonal expressions with exact results in the case of elastic scattering, we expect the validity of the eikonal treatment to apply when charges are slightly displaced. We outline in Sect. 5 a technique based on the distorted-wave Born approximation which avoids the need to introduce partial waves. Using the values of a quantity, ϱ, which relates to the deflection of the eikonal direction from that in the elastic scattering case and which is determined by the half-off-shell situation, or by any other means, we arrive at an expression for the derivative of the t-matrix with respect to the nuclear charge. Numerical methods must be applied towards evaluation of this expression; but the smallness of the ϱ-value leads to the possibility of expressing the Coulomb-excitation t-matrix simply as an integral over the charge parameter of a quantity involving the half-off-shell exact t-matrix. The numerical is in the process of being done.

This procedure enables us to handle Coulomb excitation to first order. We indicate a small modification which extends the analysis to consideration of the reorientation effect and this is discussed in Sect. 5.

Sect. 6 summarizes our findings.

2. *Evaluation of the exact t-matrix for two point-like charges*

We designate the Coulomb potential as the limit of a Yukawa interaction:

$$V(r) = \lim_{\mu\to 0} \exp(-\mu r) zZe^2 r^{-1}, \tag{2.1}$$

where μ^{-1} plays the role of a screening radius. The t-matrix is evaluated by summing the infinite eikonal Born series

$$t = -(2\pi)^2 m \langle \boldsymbol{k}_f | V - V\tilde{G}_0 V + V\tilde{G}_0 V \tilde{G}_0 V - \ldots | \boldsymbol{k}_i \rangle, \tag{2.2}$$

where m is the reduced mass of the interacting particles, $\hbar$ is set equal to unity and $\tilde{G}_0$ is the eikonal form of the free Green function G_0, which is approximated as

$$\tilde{G}_0 = [m^{-1}\boldsymbol{k}\cdot(\boldsymbol{p}-\boldsymbol{k}_i)+\delta(E)-i\varepsilon]^{-1} \cong [m^{-1}k(p_{\parallel}-\kappa)-i\varepsilon]^{-1}. \tag{2.3}$$

Here, $\boldsymbol{k}$ is the vector characterizing the eikonal approximation and $p_{\parallel}$ is the projection on $\boldsymbol{k}$ of the operator $\boldsymbol{p}$ which depicts the relative motion of the projectile and target. The quantity $\delta(E)$ denotes the shift in energy which, besides the constant shift $k^2(2m)^{-1}-E$, includes an additional term representing in some way the omitted operator $(\boldsymbol{p}-\boldsymbol{k})^2(2m)^{-1}$. The constant shift vanishes in the case of elastic scattering. The eikonal approximation depicts the latter term as a c-number with some dependence on the scattering angle θ but not on $p_{\parallel}$. This dependence reflects on recoil corrections which should be more important at large angles. Some attempts [12] have already been made to represent $\delta(E)$ in the context of the Glauber formalism in applications to scattering off deuterium, where the results are expressed in terms of recoil and Fresnel corrections.

To reproduce the exact quantal result for the t-matrix, by summing the infinite series (2.2), we are led to define $\boldsymbol{k}$ as being in the scattering plane and satisfying

$$\boldsymbol{k} \perp \Delta(=\boldsymbol{k}_f-\boldsymbol{k}_i), \quad |\boldsymbol{k}| = (2mE)^{1/2},$$

and

$$\kappa = \boldsymbol{k}_i\cdot\hat{\boldsymbol{k}}, \quad \hat{\boldsymbol{k}} = \boldsymbol{k}|\boldsymbol{k}|^{-1}. \tag{2.4}$$

This then determines $\boldsymbol{k}$ uniquely. Thus,

$$\boldsymbol{k} = (2mE)^{1/2}[\boldsymbol{k}_i+\lambda(k_i, k_f; \theta)\boldsymbol{k}_f]\,|\boldsymbol{k}_i+\lambda(k_i, k_f; \theta)\boldsymbol{k}_f|^{-1},$$
$$\lambda(k_i, k_f; \theta) = (k_i^2-\boldsymbol{k}_i\cdot\boldsymbol{k}_f)(k_f^2-\boldsymbol{k}_i\cdot\boldsymbol{k}_f)^{-1}. \tag{2.5}$$

We show in Sect. 4 that the expression determining the direction of $\boldsymbol{k}$ also applies for inelastic scattering, but only if $\eta \gg 1$; the magnitudes k_i and k_f are naturally unequal in such cases. The finding is that the component of $(\boldsymbol{k}_f-\boldsymbol{k}_i)$ along the direction of $\boldsymbol{k}$ is small compared to the perpendicular component and the direction is approximately expressed for large values of η by Eq. (2.5).

In the case of elastic scattering,

$$\lambda(k_i, k_f; \theta) = 1,$$
$$\kappa(k_i, k_f; \theta) = \boldsymbol{k}_i\cdot\boldsymbol{k} = \boldsymbol{k}_f\cdot\boldsymbol{k} = (2mE)^{1/2}\cos\tfrac{1}{2}\theta. \tag{2.6}$$

We note that the deviation of κ from $k = (2mE)^{\frac{1}{2}}$ is largest at 180°, in keeping with the notion of recoil corrections. The eikonal Green function is then rewritten simply as

$$\tilde{G}_0 = m^{-1}(2mE)^{1/2}(p_{\|}-(2mE)^{1/2}\cos\sim\theta-i\varepsilon)^{-1}, \tag{2.7}$$

where E denotes the relative energy in the centre-of-mass system. The pole in $\tilde{G}_0$ is seen to be angular-dependent. We also notice that the above equations ensure the symmetry of the t-matrix with interchange of entrance and exit channels.

The differential cross section $(d\sigma/d\Omega)$ is given by $|t|^2$, which is calculated in a conventional manner by introducing its Fourier transform and evaluating the Born-series term--by-term. Allowing for the fact that $k_{i\|}$ and $k_{f\|}$ each equal κ, as is evident by Eq. (2.6), we find

$$t = -ik\lim_{\mu\to 0}\int_0^\infty bdbJ_0(b\Delta)\left\{\sum_{m=0}^\infty[-2i\eta K_0(\mu b)]^m(m!)^{-1}-1\right\}$$
$$= -ik\lim_{\mu\to 0}\int_0^\infty bdbJ_0(b\Delta)\left\{(\mu b)^{2i\eta}-1\right\}, \tag{2.8}$$

where use is made of the fact that

$$K_0(z)\xrightarrow[z\to 0]{}-\ln z. \tag{2.9}$$

This conforms with Glauber's expression for the t-matrix. With the use of a convergence factor covering the integration over large values of b, as outlined in Appendix A, we find

$$t = -\lim_{\mu\to 0}\frac{2\eta k}{\Delta^2}\left(\frac{2\mu}{\Delta}\right)^{2i\eta}e^{2i\arg\Gamma(1+i\eta)}. \tag{2.10}$$

This result is not applicable to forward scattering, as we have ignored the unit term on the right-hand side of Eq. (2.8) and the implications of this are felt only at zero degrees. Following Gordon's procedure, the screening factor

$$\lim_{\mu\to 0}\exp[-2i\eta\ln(k\mu^{-1})] \tag{2.11}$$

is ignored and we finally are led to the result

$$t(\theta) = -\frac{\eta}{2k}\left(\frac{1}{\sin\theta}\right)^{2+2i\eta} e^{2i\arg\Gamma(1+i\eta)}. \tag{2.12}$$

This is the familiar quantal result for elastic scattering, earlier derived in Ref. [4, 5]. Had we expressed $\tilde{G}_0$ more generally as

$$\tilde{G}_0 = [A(p_{\|}-B)-i\varepsilon]^{-1}, \tag{2.13}$$

we should find that the values of A and B are precisely what is set out in Eq. (2.7) if there is to be detailed correspondence with the final result. Wallace [10] did, in fact, arrive at the same result by demonstrating the equivalence of each term in the Born series with the associated quantum expressions. Our method is somewhat more direct. This starting point is based on Eq. (2.7) which further corroborates the choice of $\tilde{G}_0$.

The quantal result obtains independent of the values of k and θ, aside from forward scattering; but this hinges on $\boldsymbol{k}$ being chosen so as to be normal to Δ and by making the pole term be angular-dependent and equal to $(2mE)^{\frac{1}{2}} \cos\frac{1}{2}\theta$. Corrections to the eikonal approximation fail to appear on account of the infinite range of the Coulomb interaction. The feature was stressed early by Moore [9]. We note that our approach is not confined to the sole use of straight-line paths since the various terms appearing on the right-hand side of Eq. (2.2) each contribute to the scattering amplitude and the distortion of the state of motion. Even if we contemplate straight-line paths along a direction of linearization, this direction changes with angle and the result is non-linear motion. Nor is our method easily related to the WKB treatment for we make no use of the notion of a trajectory.

3. Scattering by one or several displaced charges

We first consider a target situated at the origin but with a charged portion of charge ze centred at a point $\boldsymbol{r}$. We view as being point-like both this portion and the projectile with relative coordinates $\boldsymbol{r}_p$. The Coulomb interaction is then

$$V(\boldsymbol{r}_p, \boldsymbol{r}) = \lim_{\mu\to 0} \frac{zZe^2}{|\boldsymbol{r}_p-\boldsymbol{r}|} e^{-\mu|\boldsymbol{r}_p-\boldsymbol{r}|}. \tag{3.1}$$

Following the procedure as outlined in Sect. 2 we find, for a fixed value $\boldsymbol{r}$, that the on-shell t-matrix is given by

$$t_r = -ik \lim_{\mu\to 0} \int_0^\infty b\,db J_0(b\Delta)\,[\{\mu(b^2+s^2-2bs\cos\varphi)^{1/2}\}^{2i\eta}-1], \tag{3.2}$$

where $\boldsymbol{s} = \boldsymbol{r}_\perp$ is the projection of $\boldsymbol{r}$ in the impact-parameter plane which is taken to be normal to $\boldsymbol{k}$ as defined in Eq. (2.5). The angle φ is the angle between Δ and $\boldsymbol{s}$ as measured in the impact-parameter plane. Choosing a proper convergence factor, as outlined in appendices A and B, we calculate for $\theta \neq 0$, using the expansion

$$(b^2+s^2-2bs\cos\varphi)^{i\eta} = \sum_{n=0} \frac{\Gamma(1+i\eta)\,(-2\cos\varphi)^n(bs)^n(b^2+s^2)^{i\eta-n}}{\Gamma(1+i\eta-n)n!}, \tag{3.3}$$

and find

$$t_r = (2ik)^{-1}\left\{(\sin\tfrac{1}{2}\theta)^{-2-2i\eta}\sum_{m=0}^{\infty}\frac{\Gamma(1+i\eta)\Gamma(1-\frac{1}{2}m+i\eta)(-s\Delta\cos\varphi)^m}{\Gamma(1+i\eta-m)m!\Gamma(\frac{1}{2}m-i\eta)}\right.$$

$$\times {}_1F_2(m-i\eta;\tfrac{1}{2}m-i\eta,\tfrac{1}{2}m-i\eta;(\tfrac{1}{2}\Delta s)^2)$$

$$+(sk)^{2+2i\eta}\sum_{m=0}^{\infty}\frac{\Gamma(1+i\eta)\Gamma(1+\frac{1}{2}m)\Gamma(\frac{1}{2}m-1-i\eta)(-2\cos\varphi)^m}{\Gamma(1+i\eta-m)m!\Gamma(m-i\eta)}$$

$$\left.\times {}_1F_2(1+\tfrac{1}{2}m;2-\tfrac{1}{2}m+i\eta,1;(\tfrac{1}{2}\Delta s)^2)\right\}. \tag{3.4}$$

The μ-dependent phase factor met in Eq. (2.11) is ignored.

Eq. (3.4) splits naturally into an "outer" and "inner" contribution. Only the former term survives as s tends to zero, in which case t_r reduces to what is given in Eq. (2.12). We therefore classify the first term as the outer term. We can check that the second term is associated with low b-values, by expanding the binomial term in the right-hand side of Eq. (3.3) and by calculating the integral in Eq. (3.2) from $b = 0$ to $b = s$. The structure of the result is found to be similar to the second term.

A similar feature obtains in situations in which the target is composed of two point-like charges separated by some displacement vector. Again, there are both inner and outer contributions.

Particular simplicities follow in situations where η is large. This is a feature of many heavy-ion reactions. As an example, we cite the process in which oxygen ions are incident on lead. The η parameter is, in this case, near 50.

Utilizing formulae such as

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}[1+O(|z|^{-1})], \qquad |\arg z| < \pi, \tag{3.5}$$

and applying similar approximations both to the other gamma functions appearing in Eq. (3.4) and the functions ${}_1F_2$, we find t_r is given by

$$t_r = -\left(\frac{\eta}{2k}\right)(\sin\tfrac{1}{2}\theta)^{-2-2i\eta}e^{2i\arg\Gamma(1+i\eta)}e^{is\cdot\Delta}\left[1+O\left(\frac{1}{\eta}\right)\right]$$

$$+\frac{\pi^{1/2}}{2\eta k}(sk)^{2+2i\eta}\sum_{m=0}^{\infty}\frac{[\exp(-\frac{1}{4}\pi i)\eta^{1/2}\cos\varphi]^m}{\Gamma(\frac{1}{2}+\frac{1}{2}m)}\left[1+O\left(\frac{1}{\eta}\right)\right]. \tag{3.6}$$

The outer contribution is essentially the Rutherford result (Eq. (2.12)), except for the presence of an additional factor

$$\exp is\cdot\Delta. \tag{3.7}$$

The inner contribution is expected to be negligible for large η-values, owing to the highly oscillatory parts involving the product sk. It is not clear however how to sum analytically the m-dependent term in Eq. (3.6). In dealing with a target composed of two charges, we find the inner contribution is, indeed, negligible.

The appearance of the factor (3.7) might be anticipated by the following argument:

$$-ik \int_0^\infty bdbJ_0(b\Delta)\,[\mu(b^2+s^2-2bs\cos\varphi)^{1/2}]^{2i\eta} = -ik(2\pi)^{-1}\int d^2b\,\exp(i\boldsymbol{b}\cdot\boldsymbol{\Delta})$$

$$\times(\mu|\boldsymbol{b}-\boldsymbol{s}|)^{2i\eta} = -ik\exp(i\boldsymbol{s}\cdot\boldsymbol{\Delta})\int_0^\infty \beta d\beta J_0(\beta\Delta)(\mu\beta)^{2i\eta}, \tag{3.8}$$

so that the effect of a displacement equal to $\boldsymbol{s}$ is to multiply the t-matrix by the factor (3.7). Caution must be given here owing to the need to include a convergence factor, as discussed in appendices A and B. A simple shift of variables fails to lead to the above result owing to the modifications needed for small b-values. These clearly are associated with the inner contribution.

Alternatively, we might account for the shift by $\boldsymbol{s}$ by introducing translation operators $\exp(-i\boldsymbol{k}_i\cdot\boldsymbol{s})$ and $\exp(-i\boldsymbol{k}_f\cdot\boldsymbol{s})$ which operate on the eigenfunctions depicting relative motion in the entrance and exit channels respectively. Once again, the factor (3.7) is obtained; however care once more is needed if $b < s$.

4. *Inelastic scattering*

We apply ourselves to inelastic scattering in the Coulomb field, dealing first with the half-off-shell t-matrix. It is opportune that the exact result is available in the form given by Ford [11]

$$t_{\text{exact}} = -2(2\pi\eta_i)^{1/2}k_i\eta_i|\boldsymbol{k}_i-\boldsymbol{k}_f|^{-2}\{(k_i^2-k_f^2)\,|\boldsymbol{k}_i-\boldsymbol{k}_f|^{-2}\}^{i\eta_i}\exp\{i[\tfrac{1}{4}\pi+\eta_i\ln(\eta_i e^{-1})]\}, \tag{4.1}$$

which is valid for situations where $k_f < k_i$, as obtains when scattering off stable nuclei. We omit, for convenience, the factor $\mu^{2i\eta_i}$ since it is not considered in the case of elastic scattering.

In determining the pole for the eikonal t-matrix, we insist, as in the case of elastic scattering, that $k_{i\parallel}$ be equal to κ. We introduce new parameters δ, Δ and ϱ which bring out the inelastic-scattering character

$$|k_{f\parallel}-k_{i\parallel}| = \delta,$$

$$|\boldsymbol{k}_{f\perp}-\boldsymbol{k}_{i\perp}| = \Delta,$$

$$\varrho = |\boldsymbol{k}_{f\perp}-\boldsymbol{k}_{i\perp}|^{-1}\,|k_{f\parallel}-k_{i\parallel}| = \delta\Delta^{-1}. \tag{4.2}$$

The ϱ-parameter is dimensionless, vanishes in the elastic limit and is plainly dependent on the scattering angle. It plays a cardinal role in the determination of the eikonal t-matrix.

Following the procedure outlined in Sect. 2, we find

$$t_{\text{eik}} = -2\eta_i k_i |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2} + 4i\eta_i k_i \int_0^\infty b\,db\,J_0(b\Delta)$$

$$\times K_0[b|k_{f_\parallel} - \kappa|] \sum_{m=0}^{\infty} (2i\eta_i \ln b)^m [(m+2)!]^{-1} K_0[b|k_{i_\parallel} - \kappa|]. \tag{4.3}$$

Taking into account Eq. (2.9), this leads to

$$t_{\text{eik}} = -2\eta_i k_i |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2} - 2\eta_i k_i \int_0^\infty b\,db\,J_0(b\Delta) K_0(b\delta) \sum_{m=0}^{\infty} (2i\eta_i \ln b)^{m+1} [(m+2)!]^{-1}. \tag{4.4}$$

The latter integral does not lend itself readily to calculation. Rather, we calculate the derivative of t_{eik} with respect to Z_T, the charge number of the target, bearing in mind that both η_i and η_f are proportional to this number. Then

$$\frac{\partial}{\partial Z_T} t_{\text{eik}} = -2\eta_i k_i Z_T^{-1} |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2} - 2\eta_i k_i Z_T^{-1}$$

$$\times \int_0^\infty b\,db\,J_0(b\Delta) K_0(b\delta) \sum_{m=0}^{\infty} (2i\eta_i \ln b)^{m+1} [(m+1)!]^{-1}$$

$$= -2\eta_i k_i Z_T^{-1} \int_0^\infty b\,db\,J_0(b\Delta) K_0(b\delta) b^{2i\eta_i}. \tag{4.5}$$

The latter impact-parameter integral is calculable analytically [15]. In limiting situations where $\eta_i \gg 1$, we find

$$\frac{\partial}{\partial Z_T} t_{\text{eik}} \underset{\eta_i \gg 1}{=} 2k_i \eta_i Z_T^{-1} (\pi\eta_i)^{1/2} |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2-2i\eta_i}$$

$$\times (\varrho^{-1/2} + i\varrho^{+1/2}) \exp[-2\eta_i \tan^{-1} \varrho + 2i\eta_i \ln(2\eta_i e^{-1})]. \tag{4.6}$$

Noteworthy is the presence of the factor $|\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2-2i\eta_i}$ which also appears in the case of elastic scattering.

The demand that for $\eta_i \gg 1$

$$\left|\frac{\partial}{\partial Z_T} t_{\text{eik}}\right| = \left|\frac{\partial}{\partial Z_T} t_{\text{exact}}\right| \tag{4.7}$$

leads to equivalence of t_{eik} with t_{exact} as expressed in Eq. (4.1). This result is established in the following way: We note first that the exact expression, as given in Eq. (4.1), is expressible as

$$t_{\text{exact}} = R \exp i\varphi,$$

$$R = A\eta_i^{3/2} = 2(2\pi)^{1/2} k_i \eta_i^{3/2} |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2},$$

$$\varphi = \tfrac{5}{4}\pi + \eta_i \ln\{\eta_i(k_i^2 - k_f^2)\,|\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2} e^{-1}\}. \tag{4.8}$$

Then,

$$\frac{\partial}{\partial Z_T} t_{\text{exact}} = A\eta_i^{3/2} Z_T^{-1}[\tfrac{3}{2} + i\eta_i \ln \{\eta_i(k_i^2 - k_f^2)\, |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2}\}] \exp i\varphi$$

$$\underset{\eta_i \gg 1}{\cong} iAZ_T^{-1}\eta_i^{5/2} \ln \{\eta_i(k_i^2 - k_f^2)\, |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2}\} \exp i\varphi, \tag{4.9}$$

whence

$$\left| \frac{\partial}{\partial Z_T} t_{\text{exact}} \right| = R\eta_i Z_T^{-1} \ln \{\eta_i(k_i^2 - k_f^2)\, |\boldsymbol{k}_i - \bar{\boldsymbol{k}}_f|^{-2}\} = |t_{\text{axact}}| \frac{\partial}{\partial Z_T} \varphi. \tag{4.10}$$

It is noteworthy that the phase in Eq. (4.8) contains the logarithmic term with argument proportional to e^{-1} and that this disappears on differentiating with respect to Z_T. The modulus of t_{exact} is, in fact, independent of the logarithmic term although this is not the case with $\left| \frac{\partial}{\partial Z_T} t_{\text{exact}} \right|$.

If we reverse the procedure and attempt to derive the t-matrix element knowing its derivative with respect to Z_T, the combination $\eta_i Z_T^{-1} \ln(\eta_i e^{-1})$ must be associated with the phase, otherwise difficulties would arise from the term $\eta_i Z_T^{-1} \ln[\eta_i(k_i^2 - k_f^2)]$ if we extend the solution to the elastic limit. This feature allows for a determination of both the magnitude and phase of the t-matrix element using knowledge only of the modulus of the derivative of the matrix element.

Insisting that Eq. (4.7) be valid, we find that ϱ must satisfy the transcendental equation

$$(1+\varrho^2)\varrho^{-1} \exp(-4\eta_i \tan^{-1} \varrho) = 2\eta_i^2 \ln^2 [\eta_i(k_i^2 - k_f^2)|\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2}]. \tag{4.11}$$

It is clear that ϱ must be very small. Moreover, it vanishes in the elastic limit, as is to be expected. The η_i in Eq. (4.11) should be understood as a large, but *fixed* parameter.

Writing t_{eik}, in anology with Eq. (4.8), as

$$t_{\text{eik}} = \tilde{R} \exp(i\tilde{\varphi}), \tag{4.12}$$

and presuming, as in the exact case, that

$$\tilde{R} \left| \frac{\partial}{\partial Z_T} \exp i\tilde{\varphi} \right| \gg \frac{\partial}{\partial Z_T} \tilde{R}, \quad \eta_i \gg 1, \tag{4.13}$$

the modulus of the derivative of t_{eik} equals the product of $\tilde{R}$ and $\frac{\partial}{\partial Z_T} \tilde{\varphi}$. Associating the derivative of the phase with the logarithmic term and identifying as in Eq. (4.10), the values of $\tilde{R}$ and $\tilde{\varphi}$ are found to be

$$\tilde{R} = 2(2\pi)^{1/2} k_i |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2} \eta_i^{3/2},$$

$$\tilde{\varphi} = \tfrac{5}{4}\pi + \eta_i \ln [\eta_i(k_i^2 - k_f^2)\, |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2} e]. \tag{4.14}$$

We thus deduce the equality

$$t_{\rm eik} = t_{\rm exact}, \qquad \eta \gg 1 \tag{4.15}$$

for arbitrary values of k_i, k_f and scattering angle provided that the Sommerfeld parameter is large.

5. *Coulomb excitation*

One application is to inelastic scattering by displaced charges in the Coulomb field and, in particular, to single-step transitions associated with Coulomb excitation. The method hinges on application of the DWBA formalism which utilizes knowledge of the Coulomb distorted eigenfunctions $\chi_c(\boldsymbol{k}_i, \boldsymbol{r}_p)$ and $\chi_c(\boldsymbol{k}_f, \boldsymbol{r}_p)$ which depict the relative motion of the two channels. The interaction Hamiltonian describing the relative motion of a point-charge projectile in the field of a charged nucleus is

$$H = Z_T e^2 |\boldsymbol{r}_p - \boldsymbol{r}|^{-1} = Z_T e^2 r_p^{-1} + |Z_T e^2 |\boldsymbol{r}_p - \boldsymbol{r}|^{-1} - Z_T e^2 r_p^{-1}|, \tag{5.1}$$

where $\boldsymbol{r}_p$ is the vector connecting the projectile and the nuclear centre. The charge in the target is considered as being displaced by the vector $\boldsymbol{r}$. The splitting of the Hamiltonian into one part which generates the distorted-wave eigenfunctions and into a residual term depending explicitly on $\boldsymbol{r}$ is characteristic of the DWBA procedure.

The DWBA expression for the transition amplitude is given by

$$T_{fi}^{\rm DWBA} = \langle \varphi_f(\boldsymbol{r}) | t_{fi}^{\rm DWBA}(\boldsymbol{r}) | \varphi_i(\boldsymbol{r}) \rangle, \tag{5.2}$$

where

$$t_{fi}^{\rm DWBA}(\boldsymbol{r}) = -(2\pi)^2 m \langle \chi_c(\boldsymbol{k}_f, \boldsymbol{r}_p) | Z_T e^2 (|\boldsymbol{r}_p - \boldsymbol{r}|^{-1} - r_p^{-1}) | \chi_c(\boldsymbol{k}_i, \boldsymbol{r}_p) \rangle. \tag{5.3}$$

The fact that $t_{fi}^{\rm DWBA}$ $(\boldsymbol{r})$ depends on $\boldsymbol{r}$ leads to the possibility of inelastic transitions between target states $|\varphi_i(\boldsymbol{r})\rangle$ and $|\varphi_f(\boldsymbol{r})\rangle$. The simplest interpretation is that $\boldsymbol{r}$ be viewed as a dynamical operator, as one meets in the characterization of vibrational-like nuclei, for example.

The reduced matrix element $t_{fi}^{\rm DWBA}(\boldsymbol{r})$ splits into four terms

$$t_{fi}^{\rm DWBA}(\boldsymbol{r}) = [t_{fi}^{(1)}(\boldsymbol{r}) - t_{fi}^{(1)}] + [t_{fi}^{(2)}(\boldsymbol{r}) - t_{fi}^{(2)}], \tag{5.4}$$

where

$$\begin{aligned} t_{fi}^{(1)}(\boldsymbol{r}) &= -(2\pi)^2 m \langle \boldsymbol{k}_f | V_r (1 - \tilde{G}_0^i V_p + \tilde{G}_0^i V_p \tilde{G}_0^i V_p - \ldots) | \boldsymbol{k}_i \rangle, \\ t_{fi}^{(1)} &= -(2\pi)^2 m \langle k_f | V_p (1 - \tilde{G}_0^i V_p + \tilde{G}_0^i V_p \tilde{G}_0^i V_p - \ldots) | \boldsymbol{k}_i \rangle, \\ t_{fi}^{(2)}(r) &= -(2\pi)^2 m \langle \boldsymbol{k}_f | V_p (-\tilde{G}_0^f + \tilde{G}_0^f V_p \tilde{G}_0^f - \ldots) V_r (1 - \tilde{G}_0^i V_p + \tilde{G}_0^i V_p \tilde{G}_0^i V_p - \ldots) | \boldsymbol{k}_i \rangle, \\ t_{fi}^{(2)} &= -(2\pi)^2 m \langle \boldsymbol{k}_f | V_p (-\tilde{G}_0^f + \tilde{G}_0^f V_p \tilde{G}_0^f - \ldots) V_p (1 - \tilde{G}_0^i V_p + \tilde{G}_0^i V_p \tilde{G}_0^i V_p - \ldots) | \boldsymbol{k}_i \rangle, \end{aligned} \tag{5.5}$$

where

$$V_r = Z_T e^2 |\boldsymbol{r}_p - \boldsymbol{r}|^{-1}, \qquad V_p = Z_T e^2 r_p^{-1}, \tag{5.6}$$

and the superscripts, associated with $\tilde{G}_0$, specify the channel energies. Both Green functions are of the form given in Eq. (2.7).

Our method is to calculate $t_{fi}^{(1)}(\boldsymbol{r})$ in terms of $t_{fi}^{(1)}$ and $t_{fi}^{(1)}(\boldsymbol{r})$ in terms of $t_{fi}^{(2)}$. The quantities $t_{fi}^{(1)}$ and $t_{fi}^{(2)}$ are just the half-off shell t-matrices and their expressions are given as in Eq. (4.1). In calculating $t_{fi}^{(1)}(\boldsymbol{r})$ for example, we proceed as in Sect. 2 and find

$$\frac{\partial}{\partial Z_T} t_{fi}^{(1)}(\boldsymbol{r}) = -2\eta_i k_i Z_T^{-1} \exp(-ir_{\parallel}\delta) \int_0^{\infty} bdbJ_0(b\Delta)K_0[(b^2+s^2-2bs\cos\varphi)^{1/2}\delta]b^{2i\eta_i} \quad (5.7)$$

where $r_{\parallel} = \boldsymbol{r}\cdot\hat{\boldsymbol{k}}$, $\boldsymbol{s} = (\boldsymbol{r})_{\perp}$ and φ is the angle between $\boldsymbol{r}_{\perp}$ and $(\boldsymbol{k}_f-\boldsymbol{k}_i)_{\perp}$.

The calculation of the impact-parameter integral is sketched in appendix C. The final result is simply put. One merely alters the quantity δ as it appears in $t_{fi}^{(1)}$ as follows:

$$\delta \to \delta[1-\eta_i^{-1}\boldsymbol{r}\cdot(\boldsymbol{k}_f-\boldsymbol{k}_i)_{\perp}]^{1/2} \quad (5.8)$$

with the understanding that Δ is unaltered. Recognizing that

$$\Delta = (1+\varrho^2)^{1/2}|\boldsymbol{k}_f-\boldsymbol{k}_i|,$$
$$\delta = \varrho(1+\varrho^2)^{-1/2}|\boldsymbol{k}_f-\boldsymbol{k}_i|, \quad (5.9)$$

the transformation reads

$$|\boldsymbol{k}_f-\boldsymbol{k}_i|^2 \to |\boldsymbol{k}_f-\boldsymbol{k}_i|^2[1-\varrho^2(1+\varrho^2)^{-1}\eta_i^{-1}\boldsymbol{r}\cdot(\boldsymbol{k}_f-\boldsymbol{k}_i)_{\perp}]. \quad (5.10)$$

This can be applied to Eq. (4.1) and takes a fixed value η_i if $\eta \gg 1$.

The smallness of the ϱ-value, as is implicit by Eq. (4.11), suggests that $|\boldsymbol{k}_f-\boldsymbol{k}_i|^2$ is essentially unaltered by the above transformation. Consequently, the following approximate relationship holds as long as $\eta \gg |$:

$$\frac{\partial}{\partial Z_T} t_{fi}^{(1)}(\boldsymbol{r}) = e^{-ir_{\parallel}\delta}\frac{\partial}{\partial Z_T} t_{\text{eik}} \quad (5.11)$$

where t_{eik} may be equated to t_{exact} as given in Eq. (4.1). Integration of Eq. (5.11) leads to

$$t_{fi}^{(1)}(\boldsymbol{r}) = \int^{Z_T} \exp(-ir_{\parallel}\delta)\frac{\partial}{\partial Z_T}(t_{\text{exact}})dZ_T = e^{-ir_{\parallel}\delta}t_{\text{exact}}. \quad (5.12)$$

In performing the integral, we suppose that $t_{fi}^{(1)}(r)$ tends to zero for small values of Z_T. We further make the expansion

$$\exp(-ir_{\parallel}\delta) = \sum_L (-ir_{\parallel})^L\delta^L(L!)^{-1} \quad (5.13)$$

and we restrict ourselves to specific multipolarities. The usual case corresponds to quadrupole excitations ($L = 2$). From Eq. (5.9)

$$\delta^L \cong \varrho^L|\boldsymbol{k}_f-\boldsymbol{k}_i|^L, \quad (5.14)$$

and a rough estimate based on Eq. (4.11) leads to

$$\delta^L \cong |\boldsymbol{k}_f-\boldsymbol{k}_i|^L[2\eta_i^2\ln^2\{\eta_i(k_i^2-k_f^2)\,|\boldsymbol{k}_f-\boldsymbol{k}_i|^{-2}\}]^{-L}. \quad (5.15)$$

A proper solution of Eq. (5.11) is best obtained numerically and we hope to report the results when ready. The solution for $t_{fi}^{\mathrm{DWBA}}(\boldsymbol{r})$ does not require evaluation of $t_{fi}^{(1)}$ and $t_{fi}^{(2)}$ as these do not contribute to inelastic processes. Nonetheless, the evaluation of $t_{fi}^{(2)}$ does point to the form of $t_{fi}^{(2)}(\boldsymbol{r})$. Thus, considering $t_{fi}^{(2)}$as it appears in Eq. (5.5), we evaluate the matrix elements of $\tilde{G}_0$ by introducing the Fourier transform of V as in the elastic case. With the help of Cauchy's theorem, we find each of the V_p-factors that occurs between Green functions brings out the factor $2\eta_i \ln b$ while the V_p-term between the last $\tilde{G}_0^f$ function and the first $\tilde{G}_0^i$ function introduces the quantity $K_0(b\delta)$. The final expression which is valid in the eikonal scheme is

$$t_{fi}^{(2)} = -2\eta_f k_f \int_0^\infty bdbJ_0(b\Delta) \sum_{m=0}^\infty (2i\eta_f \ln b)^{m+1} K_0(b\delta) \sum_{n=0}^\infty (2i\eta_i \ln b)^n [(m+n+2)!]^{-1} \tag{5.16}$$

and

$$\frac{\partial}{\partial Z_T} t_{fi}^{(2)} = -2\eta_f k_f Z_T^{-1} \int_0^\infty bdbJ_0(b\Delta)K_0(b\delta) \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(2i\eta_f \ln b)^{m+1}(2i\eta_i \ln b)^n}{(m+n+2)!}$$

$$= -2\eta_f k_f Z_T^{-1} \int_0^\infty bdbJ_0(b\Delta)K_0(b\delta) \sum_{p=1}^\infty (2i\eta_f \ln b)^p (p!)^{-1} \sum_{n=0}^\infty (\eta_i \eta_f^{-1})^n. \tag{5.17}$$

Summing first the geometric series and utilizing the fact that

$$\eta_i k_i = \eta_f k_f, \tag{5.18}$$

we are led to

$$\frac{\partial}{\partial Z_T} t_{fi}^{(2)} = -2\eta_f k_i k_f [Z_T(k_i - k_f)]^{-1} [I(\eta_f) - I(\eta_i)],$$

$$I(\eta) \equiv \int_0^\infty bdbJ_0(b\Delta)K_0(b\delta) b^{2i\eta}, \tag{5.19}$$

and therefore that

$$\frac{\partial}{\partial Z_T} t_{fi}^{(2)} = k_i(k_i - k_f)^{-1} \left[\frac{\partial}{\partial Z_T} [t_{\mathrm{eik}}(\eta_f)] - \frac{\partial}{\partial Z_T} [t_{\mathrm{eik}}(\eta_i)] \right]. \tag{5.20}$$

The expressions for $t_{\mathrm{eik}}(\eta_i)$ and $t_{\mathrm{eik}}(\eta_f)$ simply differ by the need to introduce quantities ϱ and $\bar{\varrho}$, respectively, which satisfy the same equation Eq. (4.11) except for an interchange in the suffixes i and f in the latter case. Following the procedure used in Sect. 4, we find

$$t_{fi}^{(2)} = 2(2\pi)^{1/2} k_i (k_i - k_f)^{-1} |\boldsymbol{k}_i - \boldsymbol{k}_f|^{-2} \exp\left(\tfrac{5}{4}\pi i\right)$$

$$\times \left\{ k_f \eta_f^{3/2} \exp\left[i\eta_f \ln \frac{\eta_f(k_i^2 - k_f^2)}{|\boldsymbol{k}_i - \boldsymbol{k}_f|^2 e} \right] - k_i \eta_i^{3/2} \exp\left[i\eta_i \ln \frac{\eta_i(k_i^2 - k_f^2)}{|\boldsymbol{k}_i - \boldsymbol{k}_f|^2 e} \right] \right\}. \tag{5.21}$$

$$\frac{\partial}{\partial Z_T} t_{fi}^{(2)}(\boldsymbol{r}_p) = e^{-ir_{\parallel}\delta} \frac{\partial}{\partial Z_T} t_{fi}^{(2)} \tag{5.22}$$

and the procedure based on Eq. (5.12)–(5.15) leads to evaluation of $t_{fi}^{(2)}(\boldsymbol{r})$. The rough estimate leading to Eq. (5.15) suggests the dependence of the differential cross section on $|\boldsymbol{k}_f - \boldsymbol{k}_i|^{2L-4}\, \varrho^{2L}$ with consequent isotropy for $E2$ transitions. This feature is approximately the case in the backward hemisphere particularly for small Q-values.

The procedure outlined above extends to multi-step phenomena. In particular, the two-step process leads to matrix elements of the type

$$\langle \chi_c(\boldsymbol{k}_f, \boldsymbol{r}_p)|\, (V_r - V_p) G_0^n (V_r - V_p)\, |\chi_c(\boldsymbol{k}_i, \boldsymbol{r}_p)\rangle,$$

where G_0^n is the propogator associated with the intermediate state. Besides the contribution from nuclear interactions, there are the specific electromagnetic effects. We can study the re-orientation effect which is associated with transitions between magnetic sub-states in the final nucleus. Such matrix elements are evaluated by calculating the second derivative with respect to the charge of the target. The resultant expression leads to a generalization of Eq.'s (5.7), (5.17) and the simplifications applying to the calculation of $t_{fi}^{(1)}(\boldsymbol{r})$ apply here as well.

6. *Summary*

One of our main results is that the quantum scattering amplitude is reproduced in the eikonal scheme for the Coulomb interaction between charged particles. This is so provided that the eikonal vector lies in the direction normal to the momentum transfer, but in the scattering plane. The pole of the free Green function changes with the scattering angle θ and it is set equal to $k \cos \frac{1}{2}\theta$. We can allow for discrete distributions of charge; but the procedure becomes complex if there are more than three of them. The amplitude splits into an inner and outer part; however only the latter needs to be considered if the Sommerfeld parameter is large and it differs from the Rutherford result simply by an extra multiplicative factor of the type $\exp(i\boldsymbol{s} \cdot \boldsymbol{\Delta})$.

Exponential factors of this kind can be associated with the structure of target, as in the case of a nucleus subject to strong collective excitations. The vector $\boldsymbol{s}$ is viewed as a dynamical operator linking different excited states up the target. We have ignored specific nuclear distortion effects. The eikonal technique has been extended to allow for nuclear interactions in connection with charged composite objects scattering on each other by Czyż and Maximon [16], Dar and Kirzon [17] and Kujawski [18].

Our method also applies to atomic physics phenomena involving the impact of electrons or positively charged projectiles on hydrogen and on the helium atom. Here, we possess exact knowledge of the atomic wave functions and we perform explicit integration over the internal coordinate $\boldsymbol{s}$. The equality with the exact quantal treatment at all energies and angles helps explain successes already reported [19].

Our eikonal treatment extends readily to inelastic scattering and in particular to the half-off-shell case for $\eta \gg 1$. Again, for this, the eikonal direction must be chosen specifically and the pole depends on the scattering angle in a way close to the situation for the on--shell case.

This feature is used in treating Coulomb excitation. The method is analytical, but up to a point. There is need at the end that we calculate numerically certain expressions involving the half-off-shell t-matrix; but no partial waves appear. The multiplicity of partial-waves that contribute to conventional calculations of Coulomb excitation are not seen in the final expression. The counterpart to the summation over partial waves are the impact-parameters which are evaluated analytically. Our treatment is restricted to one-step processes with the help of the DWBA. The method readily extends to considerations of the re-orientation effect; but we reserve comment on this until completion of the numerical analysis.

The treatment we give depends critically on the ϱ-values that come into the calculation. The premise is that they can be evaluated by what is needed to reproduce the half-off-shell result for inelastic scattering. It is not clear though that the t-matrix for Coulomb excitation, which involves a dynamical change from $\boldsymbol{k}_i$ to $\boldsymbol{k}_f$ in the presence of the nuclear interaction should relate to the half-off-shell t-matrix which does not involve any nuclear interaction. If, indeed, the nuclear interaction must be brought into the calculation for this should be apparent from our numerical calculations. We might then treat ϱ as an adjustable parameter; but we leave this matter open until calculations are made.

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APPENDIX

A. *The solution of the impact integral for the interaction between point charges*

We consider the impact-parameter integral

$$\int_0^\infty bdbJ_0(b\Delta)b^{2i\eta} \tag{A1}$$

which is a special cáse of an integral described by Watson [14]. To deal with (A1), we introduce a convergence factor and calculate

$$\lim_{\substack{\mathrm{Re}\,\alpha>0\\ \alpha\to 0}} \int_0^\infty bdbJ_0(b\Delta)b^{2i\eta}e^{-\alpha b}. \tag{A2}$$

From p. 711 of Ref. [15]

$$\int_0^\infty J_\nu(\beta x)x^{\mu-1}e^{-\alpha x}dx = (\tfrac{1}{2}\beta)^\nu \frac{\Gamma(\mu+\nu)}{(\alpha^2+\beta^2)^{\frac{1}{2}(\mu+\nu)}\Gamma(1+\nu)}\, {}_2F_1[\tfrac{1}{2}(\mu+\nu), \tfrac{1}{2}(1-\mu+\nu);$$

$$1+\nu; \beta^2(\alpha^2+\beta^2)^{-1}], \qquad \mathrm{Re}\,(\mu+\nu) > 0, \mathrm{Re}\,(\alpha\pm i\beta) > 0. \tag{A3}$$

Following appropriate substitutions we find

$$\lim_{\substack{\mathrm{Re}\,\alpha>0\\ \alpha\to 0}} \int_0^\infty bdbe^{-\alpha b}J_0(b\Delta)b^{2i\eta} = \Gamma(2+2i\eta)\Delta^{-2-2i\eta}{}_2F_1(1+i\eta-\tfrac{1}{2}-i\eta; 1; 1)$$

$$= \Gamma(2+2i\eta)\Delta^{-2-2i\eta}\pi^{1/2}\Gamma^{-1}(-i\eta)\Gamma^{-1}(\tfrac{3}{2}+i\eta).$$

Using standard identities related to gamma functions, we find, on taking the limits on α, that

$$\int_0^\infty bdbJ_0(b\Delta)b^{2i\eta} = -\tfrac{1}{2}\, i\eta \left(\frac{2}{\Delta}\right)^{2+2i\eta} e^{2i\arg\Gamma(1+i\eta)}.$$

B. *The solution for the scattering by a displaced charge*

We write the impact-parameter integral in the form

$$\int_0^\infty bdbJ_0(b\Delta)\,(b^2+s^2-2bs\cos\varphi)^{i\eta} = \sum_{m=0}^{\infty} \frac{\Gamma(1+i\eta)}{\Gamma(1+i\eta-m)} \frac{(-2s\cos\varphi)^m}{m!}\, b^m(b^2+s^2)^{i\eta-m}. \tag{B1}$$

The case $m = 0$ requires special care as regards convergence. Limiting ourselves to m values not equal to zero, we note the result from p. 687 of Ref. [15]

$$\begin{aligned}\int_0^\infty \frac{x^{\varrho-1}J_\nu(ax)dx}{(x^2+k^2)^{\mu+1}} &= a^\nu k^{\varrho+\nu-2\mu-2}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\varrho)\Gamma(1-\mu-\tfrac{1}{2}(\varrho+\nu))\,\{2^{1+\nu}\Gamma(1+\mu)\Gamma(1+\nu)\}^{-1}\\ &\times {}_1F_2[\tfrac{1}{2}\varrho+\tfrac{1}{2}\nu;\tfrac{1}{2}\varrho+\tfrac{1}{2}\nu-\mu,\nu+1;(\tfrac{1}{2}ak)^2]+\tfrac{1}{2}(\tfrac{1}{2}a)^{2\mu+2-\varrho}\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\varrho-\mu-1)\\ &\times\{\Gamma(\mu+2+\tfrac{1}{2}\nu-\tfrac{1}{2}\varrho)\}^{-1}{}_1F_2[\mu+1;\mu+2+\tfrac{1}{2}(\nu-\varrho),\mu+2-\tfrac{1}{2}(\nu+\varrho);(\tfrac{1}{2}ak)^2],\\ &a>0,\qquad -\mathrm{Re}\,\nu<\mathrm{Re}\,\varrho<2\,\mathrm{Re}\,\mu+\tfrac{7}{2}.\end{aligned} \tag{B2}$$

Making appropriate substitutions we find, for $m \geqslant 1$,

$$\begin{aligned}\int_0^\infty bdbJ_0(b\Delta)b^m(b^2+s^2)^{i\eta-m} &= \tfrac{1}{2}\left(\frac{2}{\Delta}\right)^{2+2i\eta-m}\frac{\Gamma(1-\tfrac{1}{2}m+i\eta)}{\Gamma(\tfrac{1}{2}m-i\eta)}\\ &\times {}_1F_2(m-i\eta;\tfrac{1}{2}m-i\eta,\tfrac{1}{2}m-i\eta;(\tfrac{1}{2}s\Delta)^2)+\tfrac{1}{2}s^{2+2i\eta-m}\frac{\Gamma(1+\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}m-1-i\eta)}{\Gamma(m-i\eta)}\\ &\times {}_1F_2(1+\tfrac{1}{2}m;2-\tfrac{1}{2}m+i\eta,1;(\tfrac{1}{2}s\Delta)^2).\end{aligned} \tag{B3}$$

In dealing with $m = 0$, we evaluate

$$\lim_{\substack{\mathrm{Re}\,\alpha>0\\ \alpha\to 0}} \int_0^\infty bdbe^{-\alpha b}J_0(b\Delta)\,(b^2+s^2)^{i\eta}. \tag{B4}$$

We use the following representation for J_0:

$$J_0(b\Delta) = \pi^{-1}\int_0^\pi d\psi\, \exp(ib\Delta\cos\psi) \tag{B5}$$

and evaluate

$$\lim_{\substack{\mathrm{Re}\,\alpha>0\\ \alpha\to 0}} \int_0^\infty bdb \exp\left[-b(\alpha - i\Delta \cos\psi)\right] (b^2+s^2)^{i\eta} \tag{B6}$$

by performing the limiting procedure on α. We then integrate over ψ. The integral in (B6) is found by differentiating with respect to μ the integral

$$I = \int_0^\infty dx(x^2+\mu^2)^{\nu-1} \exp(-\mu x), \tag{B7}$$

which is found on p. 322 of Ref. [15]. Thus,

$$I = -(\pi)^{1/2} 2^{\nu-3/2}\Gamma(\nu)\mu^{2\nu}\left[\frac{2^{1/2-\nu}}{\pi^{1/2}\Gamma(1+\nu)} - (u\mu)^{1/2-\nu}H_{\nu+1/2}(u\mu) + (u\mu)^{1/2-\nu}N_{\nu+1/2}(u\mu)\right], \tag{B8}$$

where

$$|\arg u| < \pi, \ \ \mathrm{Re}\,\mu > 0,$$

and $H_{\nu+\frac{1}{2}}$, $N_{\nu+\frac{1}{2}}$ are, respectively, the Struve and Neumann functions. Setting $\nu = 1+i\eta$, $u = \alpha - i\Delta \cos\psi$ and $u = s$, we calculate I. The power series $H_{\nu+\frac{1}{2}}$ and $N_{\nu+\frac{1}{2}}$ are then expanded as power series and the ψ-integration is then done. This involves a simple integration of a sum of complex powers of ($\cos\psi$). The final result is just as given on the right-hand side of Eq. (B3), when m is set equal to zero. The final expression for the impact-parameter integral is as given in Eq. (3.4) aside from a trivial multiplicative factor.

C. *The impact-parameter integral* $I(s) \equiv \int_0^\infty bdb\, J_0(b\Delta)\, K_0[(b^2+s^2-2bs\cos\varphi)^{\frac{1}{2}}\delta]\, b^{2i\eta}$

The integral $I(0)$ is already tabulated [15] where one finds

$$I(0) = 2^{2i\eta}\Gamma^2(1+i\eta)\delta^{-2-2i\eta}{}_2F_1(1+i\eta, 1+i\eta; 1; -\Delta^2\delta^{-2}) \tag{C1}$$

$$= 2^{2i\eta}(\delta^2+\Delta^2)^{-1-i\eta}\Gamma^2(1+i\eta){}_2F_1(1+i\eta, -i\eta; 1; \Delta^2(\Delta^2+\varrho^2)^{-1}). \tag{C2}$$

Further evaluation is possible using the power-series expression for $K_0(z)$

$$K_0(z) = \lim_{\nu\to 0} \{\pi(2\sin\nu\pi)^{-1}[\exp(\tfrac{1}{2} i\pi\nu)(\tfrac{1}{2} iz)^{-\nu} \sum_{j=0}^\infty (\tfrac{1}{4} z^2)^j [j!\Gamma(\nu+j+1)]^{-1}$$

$$-\exp(-\tfrac{1}{2} i\pi\nu)(\tfrac{1}{2} iz)^{\nu} \sum_{j=0}^\infty (\tfrac{1}{4} z^2)^j [j!\Gamma(\nu+j+1)]^{-1}\}. \tag{C3}$$

We need the result

$$\lim_{\substack{\mathrm{Re}\,\alpha\to 0\\ \alpha\to 0}} \int_0^\infty \exp(-\alpha b) b^\gamma J_0(b\Delta) db = \tfrac{1}{2}(2\Delta^{-1})^{1+\gamma}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\gamma)\{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\gamma)\}^{-1} \tag{C4}$$

to perform the integration over b. In taking the limit as $\nu \to 0$, analytic continuation is required of the function ${}_2F_1$ and, with the help of relationships connecting various ${}_2F_1$ quantities with different arguments, we arrive at Eq. (C2).

We proceed in a similar fashion if $s \neq 0$, by using the binomial expression for the combination $(b^2+s^2-2bs\cos\varphi)$. In particular, we need

$$(1+x)^\alpha = \sum_m \Gamma(1+\alpha)\Gamma(1-\alpha-m)^{-1}x^m(m!)^{-1}, \qquad |x| < 1.$$

Inner and outer parts contribute as a result; but we ignore the former in dealing with situations where $\eta \gg 1$. The Γ functions simplify in such situations as in Eq. (3.5) for example, after some manipulation, we find

$$I(s, \Delta, \delta) = I(0, \Delta, \delta[1-\eta^{-1}\, s \cdot \Delta]^{1/2}).$$

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