MODIFIED KERMAN McMANUS AND THALER OPTICAL POTENTIAL

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It is claimed that the Kerman McManus and Thaler (KMT) optical potential in its commonly used form has to be modified in order to reproduce the correct high energy limit, i. e. the Glauber formula. It is shown that when assuming (apart from the fixed scatterer approximation) the commonly used off-shell prescription for individual projectile-nucleon t-matrices $\langle \vec{p}'|t|\vec{p}\rangle = t(\vec{p}-\vec{p}')$, one has to neglect the terms responsible for a multiple scattering of the projectile from the same nucleon in the KMT optical potential. This modification is in fact equivalent to the prescription proposed by us recently for multiple scattering calculations at medium energies. The importance of such a modification is illustrated on the example of proton scattering from He^4.

1. *Introduction*

It is customary to analyse the high energy hadron-nucleus scattering data in terms of the Glauber multiple scattering theory [1]. As is well known, for heavy nuclei the optical limit of the Glauber formula or equivalently the optical potential approach can be used, in which the optical potential $V_{\rm opt}$ describing the projectile-nucleus interaction is simply equal to the projectile-nucleon elastic scattering amplitude times the nuclear density. The advantage of the optical potential approach is that it retains its validity also at lower energies, of order, say, 1 GeV, where the Fresnel-type corrections to the eikonal propagation of the projectile in the nucleus i.e. to the Glauber model are known to be important (see, e.g., Refs [2, 3]). Since the projectile-nucleus interaction is described by a simple single particle (and usually central) potential, it is an easy task to solve the corresponding Schrödinger equation without using the eikonal approximation.

Recently a great deal of experimental data on hadron-nucleus scattering at energies of order 1 GeV has become available [4] and there has been renewed interest in the optical potential approach in the version due to Kerman, McManus and Thaler (KMT) [5]. This approach has been used even for nuclei as light as, e.g., He^4. It seems interesting,

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therefore, to test the validity and the convergence of the optical potential by comparing the KMT amplitude with the exact result, especially in a case where quantitative discussion is possible.

In the present paper we discuss the validity of the KMT approach in the high energy limit and present quantitative comparison between the exact (in potential scattering theory) amplitude, being simply the Glauber model result, and the KMT amplitude. We consider the scattering from the target composed of given N fixed scatterers "nucleons" assuming its wave function to be the product of the single Gaussians plus the centre-of-mass constraint, and the Gaussian form of the projectile-scatterer amplitude. In this framework we calculate the KMT amplitude with the optical potential taken to practically infinite order and compare it with the Glauber model amplitude.

We find that when using such an off-shell continuation of the amplitudes that they depend on the momentum transfer only one gets a significant discrepancy between the KMT and the Glauber models. This is due to the presence of the rescattering terms (i.e. terms in which the projectile is scattered more than once on a given nucleon) in the optical potential. On the other hand, as was shown by Harrington [7], these terms do not contribute to the multiple scattering series in the eikonal limit. This discrepancy can be simply eliminated by putting the rescattering terms in the optical potential equal to zero when using the above-mentioned prescription for the off-shell amplitudes. In fact, as was argued in [8], this way of calculating the multiple scattering amplitudes, i.e. assuming that the off-shell amplitudes depend on the momentum transfer only, together with neglecting the rescattering terms in the multiple scattering series seems to be a quite reasonable approximation even at medium energies.

As for the convergence of the optical potential, we find that for light nuclei, such as e.g. He^4, in order to reproduce the exact scattering amplitude the optical potential has to be taken to an order at least equal to the number of nucleons in the nucleus. For heavier nuclei it is sufficient to take only the first order term of the optical potential.

Finally we comment on the applicability of the KMT approach in the analysis of scattering from nuclei at medium energies.

2. *Basic notation*

The formal solution of the problem of scattering of a given projectile from a system of N scatterers in potential scattering theory was given by Watson [9].

The transition operator is

$$T = \sum_{j=1}^{N} \tau_j + \sum_{j \neq k} \tau_j \frac{1}{E - H_0 + i\varepsilon} \tau_k + \dots$$

where τ_j are the transition operators for the projectile scattering on bound scatterers, E is the total energy of the system and H_0 the Hamiltonian from which the interaction potential has been removed [10].

In the fixed scatterer approximation (FSA) the operator $(E-H_0+i\varepsilon)^{-1}$ can be replaced by the free projectile propagator

$$G_0 = \sum_{\vec{p},\vec{p}'} \frac{|\vec{p}'\rangle\delta(\vec{p}-\vec{p}')\langle\vec{p}|}{k^2-p^2+i\varepsilon} \tag{1}$$

where k is the projectile incident momentum in the Breit frame and the whole Watson series reduces to

$$T = \sum_{i=1}^{N} t_i + \sum_{i\neq k}^{N} t_i G_0 t_k + \sum_{i\neq k\neq j}^{N} t_i G_0 t_k G_0 t_j + \ldots \tag{2}$$

where t_i are the transition operators for the projectile scattering on free target constituents. At moderately high energy, say, above 300 MeV, the operators t_i and τ_i are very close to each other.

In what follows we shall adopt the FSA approximation and in referring to the Watson series we shall think of the formula [2].

The Glauber model is the exact solution of (2) in a limiting case of infinitely high energy. Here only one assumption is needed, i.e. the replacement of $G_0(\vec{p})$ by the eikonal propagator

$$G^{\text{EIK}}(\vec{p}) = (2\vec{k}\cdot(\vec{k}-\vec{p})+i\varepsilon)^{-1}. \tag{3}$$

Denoting the T matrix elements between the on-shell states in the eikonal limit by $T^{\text{EIK}}(\vec{\Delta}, \vec{r}_1, \ldots, \vec{r}_N)$, we have

$$T^{\text{EIK}}(\Delta, \vec{r}_1, \ldots, \vec{r}_N) = \frac{-ik}{8\pi^3}\int d^2b e^{i\vec{\Delta}\vec{b}}\Gamma(\vec{b}, \vec{s}_1, \ldots, \vec{s}_N),$$

$$\vec{r}_j = (\vec{s}_j, z_j) \tag{4}$$

where

$$\Gamma(\vec{b}, \vec{s}_1, \ldots, \vec{s}_N) = 1 - \prod_{j=1}^{N}(1-\gamma_j(\vec{b}-\vec{s}_j)) \tag{5}$$

and the operators γ_j are related to the on-shell t-matrix through the following relation:

$$t^{\text{EIK}}(\Delta) = \langle\vec{p}_2|t|\vec{p}_1\rangle = \frac{-ik}{8\pi^3}\int d^2b e^{i(\vec{p}_1-\vec{p}_2)\vec{b}}\gamma(\vec{b}). \tag{6}$$

The transition amplitude between the two nuclear states $|n\rangle$ and $|m\rangle$ will be denoted by

$$T^{\text{EIK}}_{nm} = \langle n|T^{\text{EIK}}(\Delta, \vec{r}_1, \ldots, \vec{r}_N)\,|m\rangle. \tag{7}$$

The relation between the t_j matrix elements and the elastic scattering amplitudes is

$$\langle\vec{p}_2|t_j|\vec{p}_1\rangle = -\frac{1}{2\pi^2}e^{i(\vec{p}_1-\vec{p}_2)\vec{r}_j}f(\vec{p}_2, \vec{p}_1),$$

where $\vec{r}_j$ is the position vector of i-th target constituent, and f are normalized so as to give the elastic differential cross-section

$$\frac{d\sigma}{d\Omega} = |f(\vec{p}_2, \vec{p}_1)|^2.$$

3. *Derivation of the KMT optical potential*

A detailed derivation of the KMT optical potential can be found, apart from the original paper of Kerman, McManus and Thaler [5], in Refs [6, 7]. In this approach one assumes that the projectile is distinguishable from the target nucleons, i. e. its antisymmetrization with them is not necessary. Then the optical potential is calculated, being the infinite sum of terms proportional to the given powers of individual t-matrices. One gets then

$$V_{\mathrm{KMT}} = \sum_{n=1}^{\infty} V_{\mathrm{KMT}}^{(n)}, \tag{8}$$

where each term $V_{\mathrm{KMT}}^{(n)}$ involves the t-matrix elements up to n-th power. For instance the first term is

$$V_{\mathrm{KMT}}^{(1)} = (N-1)\, \langle 0|\, \frac{1}{N} \sum_{j=1}^{N} t_j |0\rangle.$$

Here we shall present another, much simpler way of calculating V_{KMT} which gives the same result as its derivation in [5].

Following KMT let us assume that the antisymmetrization of the projectile with the target nucleons is not necessary and introduce the operator V_{KMT} through the relation

$$T_{00} = \langle 0|T|0\rangle = \frac{N}{N-1}\, V_{\mathrm{KMT}}(1-G_0 V_{\mathrm{KMT}})^{-1}. \tag{9}$$

By solving the above equation with respect to V_{KMT} we get immediately an expression for the KMT potential in terms of the matrix element T_{00}. Then writing T_{00} as

$$T_{00} = \sum_{n=1}^{\infty} T^{(n)}, \tag{10}$$

where $T^{(n)}$ is the n-th order scattering amplitude

$$T^{(n)} = \langle 0| \sum_{i_1 \neq i_2 \neq \ldots \neq i_n} t_{i_1} G_0 t_{i_2} G_0 \ldots G_0 t_{i_n} |0\rangle, \tag{11}$$

we obtain the expansion of the optical potential in the form

$$V_{\mathrm{KMT}} = \sum_{n=1}^{\infty} V_{\mathrm{KMT}}^{(n)}, \tag{12}$$

where

$$V_{\text{KMT}}^{(1)} = \frac{N-1}{N}\, T^{(1)},$$

$$V_{\text{KMT}}^{(2)} = \frac{N-1}{N}\, T^{(2)} - \left(\frac{N-1}{N}\right)^2 T^{(1)} G_0 T^{(1)},$$

$$V_{\text{KMT}}^{(3)} = \frac{N-1}{N}\, T^{(3)} - \left(\frac{N-1}{N}\right)^2 [T^{(1)} G_0 T^{(2)} + T^{(2)} G_0 T^{(1)}]$$
$$+ \left(\frac{N-1}{N}\right)^3 T^{(1)} G_0 T^{(1)} G_0 T^{(1)}, \text{ etc.}$$

Inserting Eq. (11) into these expressions, we arrive finally at the following expansion in terms of the individual t-matrices

$$V_{\text{KMT}}^{(1)} = (N-1)\, \langle 0|\, \frac{1}{N} \sum t_j |0\rangle,$$

$$V_{\text{KMT}}^{(2)} = (N-1)^2 \left[\langle 0|\, \frac{1}{N(N-1)} \sum_{j \neq k} t_j G_0 t_k |0\rangle \right.$$
$$\left. - \langle 0|\, \frac{1}{N} \sum_j t_j |0\rangle G_0 \langle 0|\, \frac{1}{N} \sum_k t_k |0\rangle \right],$$

$$V_{\text{KMT}}^{(3)} = (N-1)^3 \left\{ \langle 0|\, \frac{1}{N(N-1)(N-2)} \sum_{\substack{i \neq j \neq k \\ i \neq k}}^{N} t_i G_0 t_k G_0 t_j |0\rangle \right.$$

$$- \langle 0|\, \frac{1}{N(N-1)} \sum_{i \neq j}^{N} t_i G_0 t_j |0\rangle G_0 \langle 0|\, \frac{1}{N} \sum_{k}^{N} t_k |0\rangle$$

$$- \langle 0|\, \frac{1}{N(N-1)} \sum_{i \neq j} t_i G_0 \langle 0|\, \frac{1}{N} \sum_{k=1}^{N} t_k |0\rangle G_0 t_j |0\rangle$$

$$- \langle 0|\, \frac{1}{N} \sum_{j}^{N} t_j |0\rangle G_0 \langle 0|\, \frac{1}{N(N-1)} \sum_{i \neq k}^{N} t_i G_0 t_k |0\rangle$$

$$\left. + 2 \langle 0|\, \frac{1}{N} \sum_{i=1}^{N} t_i |0\rangle G_0 \langle 0|\, \frac{1}{N} \sum_{j=1}^{N} t_j |0\rangle G_0 \langle 0|\, \frac{1}{N} \sum_{k=1}^{N} t_k |0\rangle \right\}$$

$$+(N-1)^3\left\{\langle 0|\frac{1}{N(N-1)}\sum_{i\neq k} t_i G_0 \langle 0|\sum_{j=1}^{N} t_j|0\rangle G_0 t_k|0\rangle\right.$$

$$\left.-\langle 0|\frac{1}{N}\sum_{j=1}^{N} t_j|0\rangle G_0\langle 0|\sum_{k=1}^{N} t_k|0\rangle G_0\langle 0|\sum_{m=1}^{N} t_m|0\rangle\right\}$$

$$+(N-1)^2\langle 0|\frac{1}{N(N-1)}\sum_{i\neq j} t_i G_0 t_j G_0 t_i|0\rangle$$

$$-(N-1)^2\langle 0|\frac{1}{N}\sum_{i=1}^{N} t_i|0\rangle G_0\langle 0|\sum_{j=1}^{N} t_j|0\rangle G_0\langle 0|\sum_{m}^{N} t_m|0\rangle. \tag{13}$$

Anticipating the discussion in Sec. 5 let us stress that, in general, the successive terms $V_{\mathrm{KMT}}^{(n)}$ in the expansion (12) of the optical potential are not proportional to the two particle, three particle etc. nuclear density correlation functions. Indeed, the higher order terms $V_{\mathrm{KMT}}^{(n)}$ for $n > 2$ in general do not vanish even if there are no correlations between the nucleons. This is most easily seen by explicitly calculating a few first terms of V_{KMT}.

Let us assume for simplicity that the interactions of the projectile with the target nucleons are identical, i. e. in FSA

$$\langle \vec{k}'|t_j|\vec{k}\rangle = e^{i(\vec{k}-\vec{k}')\vec{r}_j}\langle \vec{k}'|t|\vec{k}\rangle \quad \text{for} \quad i = 1, \ldots, N. \tag{14}$$

Using Eqs (13) we have

$$\langle \vec{k}'|V_{\mathrm{KMT}}|\vec{k}\rangle = \sum_{n=1}^{\infty} \langle \vec{k}'|V_{\mathrm{KMT}}^{(n)}|\vec{k}\rangle, \tag{15}$$

where

$$\langle \vec{k}'|V_{\mathrm{KMT}}^{(1)}|\vec{k}\rangle = (N-1)F(\vec{k}-\vec{k}')\,\langle \vec{k}'|t|\vec{k}\rangle,$$

$$\langle \vec{k}'|V_{\mathrm{KMT}}^{(2)}|\vec{k}\rangle = (N-1)^2\int\frac{d^3p}{(2\pi)^3}\,\langle \vec{k}'|t|\vec{p}\rangle G_0(\vec{p})\,\langle \vec{p}|t|\vec{k}\rangle C^{(2)}(\vec{p}-\vec{k}',\vec{k}-\vec{p}),$$

$$\langle \vec{k}'|V_{\mathrm{KMT}}^{(3)}|\vec{k}\rangle = (N-1)^3\int\frac{d^3p_1}{(2\pi)^3}\,\frac{d^3p_2}{(2\pi)^3}\,\langle \vec{k}'|t|\vec{p}_2\rangle$$

$$G_0(\vec{p}_2)\,\langle \vec{p}_2|t|\vec{p}_1\rangle G_0(\vec{p}_1)\,\langle \vec{p}_1|t|\vec{k}\rangle W(\vec{p}_2-\vec{k}',\vec{p}_1-\vec{p}_2,\vec{k}-\vec{p}_1). \tag{16}$$

In these formulas

$$W(\vec{q}_1,\vec{q}_2,\vec{q}_3) = C^{(3)}(\vec{q}_1,\vec{q}_2,\vec{q}_3)+F(\vec{q}_2)C^{(2)}(\vec{q}_1,\vec{q}_3)$$

$$+\frac{1}{N-1}\left[C^{(2)}(\vec{q}_1+\vec{q}_3,\vec{q}_2)+F(\vec{q}_1+\vec{q}_3,\vec{q}_2)\right]-\frac{1}{N-1}F(\vec{q}_1)F(\vec{q}_2)F(\vec{q}_3),$$

$$F(\vec{q}) = \langle 0| \frac{1}{N} \sum_{j=1}^{N} e^{i\vec{q}\vec{r}_j}|0\rangle,$$

$$C^{(2)}(\vec{q}_1, \vec{q}_2) = \langle 0| \frac{1}{N(N-1)} \sum_{j \neq k}^{N} e^{i\vec{q}_1\vec{r}_j + i\vec{q}_2\vec{r}_k}|0\rangle - F(\vec{q}_1)F(\vec{q}_2),$$

$$C^{(3)}(\vec{q}_1, \vec{q}_2, \vec{q}_3) = \langle 0| \frac{1}{N(N-1)(N-2)} \sum_{j \neq k \neq m} e^{i\vec{q}_1\vec{r}_j + i\vec{q}_2\vec{r}_k + i\vec{q}_3\vec{r}_m}|0\rangle$$

$$-C^{(2)}(\vec{q}_1, \vec{q}_2)F(\vec{q}_3) - C^{(2)}(\vec{q}_1, \vec{q}_3)F(\vec{q}_2) - C^{(2)}(\vec{q}_2, \vec{q}_3)F(\vec{q}_1) - F(\vec{q}_1)F(\vec{q}_2)F(\vec{q}_3). \quad (17)$$

The functions $C^{(2)}$ and $C^{(3)}$ are simply the two-particle and three-particle correlation functions in the momentum space representation.

If there are no correlations, then $C^{(2)} = C^{(3)} = 0$ and

$$\langle \vec{k}'|V^{(2)}_{\mathrm{KMT}}|\vec{k}\rangle = 0$$

and the term W present in $V^{(3)}_{\mathrm{KMT}}$ reduces to

$$\frac{1}{N-1} [F(\vec{q}_1 + \vec{q}_3, \vec{q}_2) - F(\vec{q}_1)F(\vec{q}_2)F(\vec{q}_3)],$$

i. e. it contains the terms corresponding to the triple scattering on three nucleons and the triple scattering on two nucleons, i. e. the rescattering term.

4. *Modified KMT potential*

The optical potential V_{KMT} given by (12) is in general a complicated non-local operator and in the practical applications some approximations are made which make the calculations possible. Thus apart from the FSA it is commonly assumed that

(*i*) the t-matrix elements can be approximately considered as depending on the momentum transfer only, i. e.:

$$\langle \vec{p}'|t|\vec{p}\rangle = t(\vec{p} - \vec{p}'), \quad (18)$$

where $t(\vec{p} - \vec{p}')$ is the on-shell value of the t-matrix element for projectile-nucleon scatterings which are parametrized so as to be consistent with the experimental data. This approximation is in fact intimately connected with the original version of KMT theory and is always made in the applications (see, e. g., Ref. [6]).

(*ii*) it is sufficient to retain only a few terms in the expansion

$$V_{\mathrm{KMT}} = V_{\mathrm{D}} + V_{\mathrm{R}} = \sum_{n=1}^{\infty} V^{(n)}_{\mathrm{KMT}}. \quad (19)$$

For convenience we have split V_{KMT} into two parts V_{D} and V_{R} which contain the direct scattering terms and the rescattering terms, i. e. terms in which a given target nucleon is hit more than once by the projectile.

The approximation (i) is very closely connected with the problem of contributions from the rescattering terms in the Watson series or equivalently the rescattering terms in V_R. Let us note that in the infinite energy limit, where the eikonal approximation is valid, the whole Watson series calculated FSA and without the approximation (i) does not contain the rescattering terms and is finite [12]. The resulting amplitude in the KMT theory with the same approximations and with the potential taken to the infinite order differs, therefore, from the exact result by the presence of the rescattering terms in V_{opt}. Strictly speaking, the KMT amplitude is then equal to the Glauber amplitude plus the rescattering terms, which should vanish if the proper off-shell continuation (in the potential scattering theory) is used.

Therefore in order to get the correct infinite energy limit in the KMT theory with the approximation (i), one should put the rescattering terms (i. e. V_R) equal zero.

Now the problem arises whether this modification of the KMT optical potential is a reasonable one at lower energies where the eikonal approximation is no longer valid. The problem of the connection between the approximation (i) and the contribution from the rescattering terms has been studied in detail in Ref. [8]. In this paper an example was considered of scattering of given projectile off two fixed potential centres Gaussians at medium energies, in the framework of the first order eikonal expansion. It was shown that the whole infinite Watson series calculated with the proper off-shell continuation of t-matrix elements is very close to the finite multiple scattering series calculated with approximation (i) and without the rescattering terms. The main corrections to the eikonal approximation turned out to come from the free wave propagation between the successive scatterings, where the way of the off-shell continuation is inessential. An example of these non-eikonal corrections was given in Ref. [13] in the case of proton scattering from He^4.

Motivated by the above considerations, we suggest the same modification of the KMT amplitude at medium energy which consists of replacing the potential $V_{KMT} = V_D + V_R$ by the modified one equal simply

$$V_M = V_D. \tag{20}$$

It should also be noted that the modified KMT amplitude, i. e., calculated using V_M coincides with the formula recently proposed in Ref. [13], which is of the form

$$T = \sum t_i + \sum_{i \neq k} t_i G_0 t_k + \sum_{i \neq k \neq j} t_i G_0 t_k G_0 t_j + \ldots \text{(terms up to } N\text{-th order of scattering)} \tag{21}$$

where t_i are the t-matrices for the projectile-nucleon interaction calculated with the assumption (i) and G_0 is the exact free wave propagator, k being the projectile momentum.

In order to illustrate the importance of such a modification of the KMT optical potential, we shall make the quantitative comparison between the original and the modified KMT amplitudes in the case of hadron scattering off light nuclei at high energy, with the potentials taken to practically infinite order.

5. Comparison between the original and the modified optical potential

In order to simplify the discussion and avoid unnecessary calculational complications, we shall consider the elastic scattering of a given projectile from the nucleus with the independent particle model (IPM) density having the factorization property, so that the centre-of-mass correlation will be automatically included by multiplying the resulting amplitude by the corection factor Θ defined by (A.5). The ground state density is then

$$\varrho(\vec{r}_1, \dots, \vec{r}_N) = \prod_{i=1}^{N} \varrho_i(\vec{r}_i). \tag{22}$$

Let us also assume that the target-nucleon interactions with the projectile are identical, i. e. Eq. (14) holds.

We can write the average of the Watson operator with the density (22) as a sum of the direct and the rescattering term

$$T_{00} = T_{\mathrm{D}} + T_{\mathrm{R}}, \tag{23}$$

where

$$T_{\mathrm{D}} = N\langle 1\rangle + N(N-1)\langle 12\rangle + \dots + \langle 12 \dots N\rangle. \tag{24}$$

Here we have adopted a shorthand notation which can be most easily understood on the following examples:

$$\langle 1\rangle = \langle 0|t_i|0\rangle,$$

$$\langle 123\rangle = \langle 0|t_i G_0 t_k G_0 t_m|0\rangle = \langle 1\rangle G_0 \langle 1\rangle G_0 \langle 1\rangle \quad \text{for} \quad i \neq k \neq m,\ i \neq m,$$

$$\langle 121\rangle = \langle 0|t_i G_0 t_k G_0 t_i|0\rangle \quad \text{for} \quad i \neq k.$$

The optical potential can now be calculated with the help of the formula (9) and can also be decomposed into the sum of the direct and the rescattering term:

$$V_{\mathrm{KMT}} = V_{\mathrm{D}} + V_{\mathrm{R}} = \frac{N-1}{N} T_{00} \left(1 + G_0 \frac{N-1}{N} T_{00}\right)^{-1}, \tag{25}$$

where

$$V_{\mathrm{D}} = \frac{N-1}{N} T_{\mathrm{D}} \left(1 + G_0 \frac{N-1}{N} T_{\mathrm{D}}\right)^{-1}, \quad V_{\mathrm{R}} = V_{\mathrm{KMT}} - V_{\mathrm{D}}. \tag{26}$$

The term V_{D} can be easily computed if we substitute into (26) the expression

$$\frac{N-1}{N} T_{\mathrm{D}} = \sum_{n=1}^{N} a_n \langle 12 \dots n\rangle, \tag{27}$$

where

$$a_n = (N-1)^2 \frac{(N-2)!}{(N-n)!}, \quad a_{n+N} = 0 \quad \text{for } n = 1, 2, 3, \dots . \tag{28}$$

In this way we get

$$V_D = 1 - \frac{1}{1 + \sum_{n=1}^{N} a_n \langle 12 \ldots n \rangle} = \sum_{n=1}^{\infty} c_n \langle 12 \ldots n \rangle, \tag{29}$$

where the coefficients c_n can be obtained from the recurrence relations [14]:

$$c_n = \sum_{k=1}^{n} a_k c_{n-k} \tag{30}$$

with $c_0 = -1$.

The application of the above formulas yields then

$$V_D = V_D^{(1)} + V_D^{(2)} + \ldots \tag{31}$$

where

$$V_D^{(1)} = (N-1)\langle 1 \rangle, \quad V_D^{(2)} = 0, \quad V_D^{(3)} = -(N-1)^2 \langle 123 \rangle,$$
$$V_D^{(4)} = -(N-1)^2(N-3)\langle 1234 \rangle, \quad V_D^{(5)} = -(N-1)^3(N-2)(N-5)\langle 12345 \rangle \text{ etc.,} \tag{32}$$

and

$$V_R = V_R^{(3)} + V_R^{(4)} + \ldots \tag{33}$$

where

$$V_R^{(3)} = (N-1)^2 \langle 121 \rangle,$$
$$V_R^{(4)} = -2(N-1)^2 \langle 1213 \rangle + (N-1)^2 \langle 1212 \rangle + (N-1)^2 (N-2) \langle 1231 \rangle,$$
$$V_R^{(5)} = (N-1)^2(N-2)[\langle 12131 \rangle + 2\langle 12132 \rangle + \langle 12312 \rangle + \langle 12321 \rangle]$$
$$+ 2(N-1)^2(N-2)(N-4)[\langle 12314 \rangle + \langle 12134 \rangle]$$
$$+ (N-1)^2(N-2)(N-3)\langle 12341 \rangle - (N-1)^2(N-3)\langle 21314 \rangle. \tag{34}$$

Let us now assume the Gaussian parametrization of the projectile-nucleon amplitudes

$$t(\vec{q}) = \frac{ik\sigma}{-8\pi^3}(1 - i\alpha)e^{-\frac{a}{2}q^2}, \tag{35}$$

where k is the projectile momentum and σ is the total cross-section for projectile-nucleon interaction. This gives the following configuration space representation for $\tilde{t}(\vec{r})$

$$\tilde{t}(\vec{r}) = \gamma(\vec{b})\beta(z), \quad \int_{-\infty}^{+\infty} dz \tilde{t}(\vec{r}) = \gamma(\vec{b}),$$

where

$$\gamma(\vec{b}) = \frac{1 - i\alpha}{4\pi a}\sigma e^{-\frac{b^2}{2a}}, \quad \beta(z) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{z^2}{2a}} \quad \vec{r} = (\vec{b}, z). \tag{36}$$

Let us also assume the Gaussian form of the target density

$$\varrho(\vec{r}_1, \dots, \vec{r}_N) = \left(\frac{1}{\pi R^2}\right)^{\frac{3}{2}N} e^{-\frac{1}{R^2}\sum_{i=1}^{N} \vec{r}_i^{\,2}} \tag{37}$$

which is very convenient with respect to factorization of the centre-of-mass correction, which takes the simple form:

$$\Theta(\Delta^2) = e^{\frac{\Delta^2 R^2}{4N}}.$$

Now we shall compare the KMT amplitude in its original and modified versions, at high energy where the eikonal approximation can be used. In the second case we get simply the Glauber amplitude, i. e.,

$$\langle \vec{k}_f | T_M^{EIK} | \vec{k}_i \rangle = \langle \vec{k}_f | T_D^{EIK} | \vec{k}_i \rangle \tag{38}$$

and in the first case the Glauber amplitude + the rescattering term, i. e.,

$$\langle \vec{k}_f | T_D^{EIK} | \vec{k}_i \rangle + \langle \vec{k}_f | T_R^{EIK} | \vec{k}_i \rangle. \tag{39}$$

The Glauber amplitude or equivalently the direct terms can be written as

$$\langle \vec{k}_f | T_D^{EIK} | \vec{k}_i \rangle = N\langle 1 \rangle + N(N-1)\langle 12 \rangle^{EIK} + \dots + \langle 12 \dots N \rangle^{EIK}, \tag{40}$$

where

$$\langle 12 \dots n \rangle^{EIK} = \langle 0 | -\frac{ik}{8\pi^3} \int d^2 b e^{i\vec{\Delta}\vec{b}} \tilde{t}(\vec{b}-\vec{s}_1, \xi_1 - z_1)\theta(\xi_2 - \xi_1)$$

$$\dots \theta(\xi_n - \xi_{n-1}) t(\vec{b}-\vec{s}_n, \xi_n - z_n) |0\rangle = -\frac{ik}{8\pi^3}\frac{1}{n!}\langle 0 | \int d^2 b e^{i\vec{\Delta}\vec{b}} \gamma(\vec{b}-\vec{s}_1) \dots \gamma(\vec{b}-\vec{s}_n) |0\rangle \tag{41}$$

(compare [15]). The rescattering terms are somewhat more complicated, e. g.

$$\langle 121 \rangle^{EIK} = \int \langle 0 | -\frac{ik}{8\pi^3} \int d^2 b e^{i\vec{\Delta}\vec{b}} \tilde{t}(\vec{b}-\vec{s}_1, \xi_1 - z_1)\theta(\xi_2 - \xi_1)$$

$$\tilde{t}(\vec{b}-\vec{s}_2, \xi_2 - z_2)\theta(\xi_3 - \xi_2)\tilde{t}(\vec{b}-\vec{s}_1, \xi_3 - z_1) |0\rangle d\xi_1 d\xi_2 d\xi_3,$$

$$\langle 1212 \rangle^{EIK} = \int \langle 0 | -\frac{ik}{8\pi^3} \int d^2 b e^{i\vec{\Delta}\vec{b}} \tilde{t}(\vec{b}-\vec{s}_1, \xi_1 - z_1)\theta(\xi_2 - \xi_1)$$

$$\tilde{t}(\vec{b}-\vec{s}_2, \xi_2 - z_2)\theta(\xi_3 - \xi_2)\tilde{t}(\vec{b}-\vec{s}_1, \xi_3 - z_1)\theta(\xi_4 - \xi_3)$$

$$\tilde{t}(\vec{b}-\vec{s}_2, \xi_4 - z_2) |0\rangle d\xi_1 d\xi_2 d\xi_3 d\xi_4 \text{ etc.} \tag{42}$$

We have obtained analytical expressions for the rescattering terms of the third and fourth order.

The higher order terms have been calculated numerically, as they are given in the form of single, double, etc. integrals. We have applied the above formulas to the case of

$N = 2$ target and $N = 4$ target using the oscillator parameters $R = 1.61$ fm and $R = 1.37$ fm respectively, which correspond to the radii of the deuteron and He^4 nuclei.

In Figs 1, 2, 4 and 5 it is seen that the contribution from the rescattering terms appears to be comparable with the values of the direct multiple scattering terms, especially for larger momentum transfers, because the slope of the rescattering term of a given order n is in general smaller than that of the direct term of the same order.

Note also that in the limit

$$\frac{2a}{R^2} \to 0$$

the rescattering terms go to infinity; roughly speaking this is due to the fact that in this limit the operators $\gamma(\vec{b}-\vec{s}_i)$ act as Dirac delta functions of the argument $\vec{b}-\vec{s}_1$ and in the rescattering terms always one γ operator appears at least twice.

Another interesting point is that the contribution from the rescattering terms seems to be more impoitant as the number of nucleons in the target becomes largei (compare Figs 1 and 4). This arises from the fact that for heavier nuclei there is a larger number of possible rescatterings being of order relatively small as compared with the highest order of direct scattering, which is important in the whole amplitude.

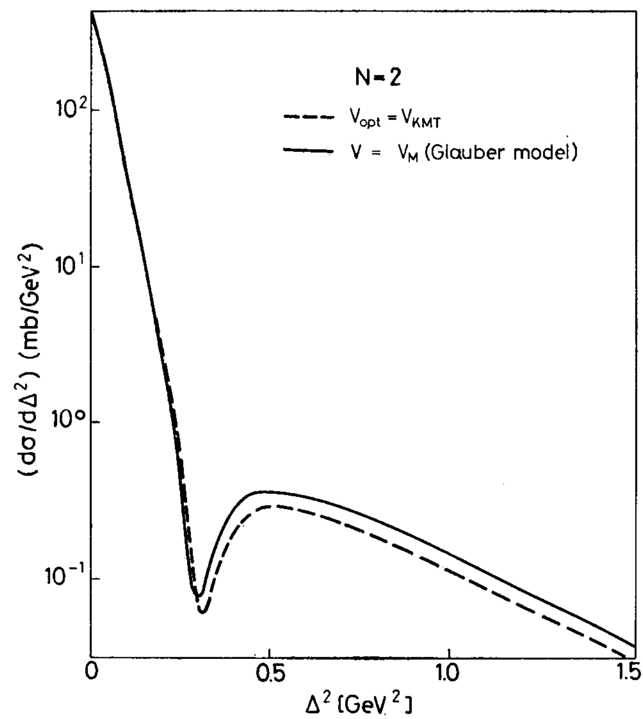

Fig. 1. The differential cross-sections for scattering from $N = 2$ target with Gaussian density $R = 1.61$ fm and with Gaussian parametrization of the amplitude calculated in the eikonal approximation with V_{KMT} and V_M taken to the infinite order. The parameters are $a = 5(\text{GeV}/c)^{-2}$, Re $t(0)$/Im $t(0) = -.33$, Im $t(0) = -k\sigma/8\pi^3$ with $\sigma = 44$ mb

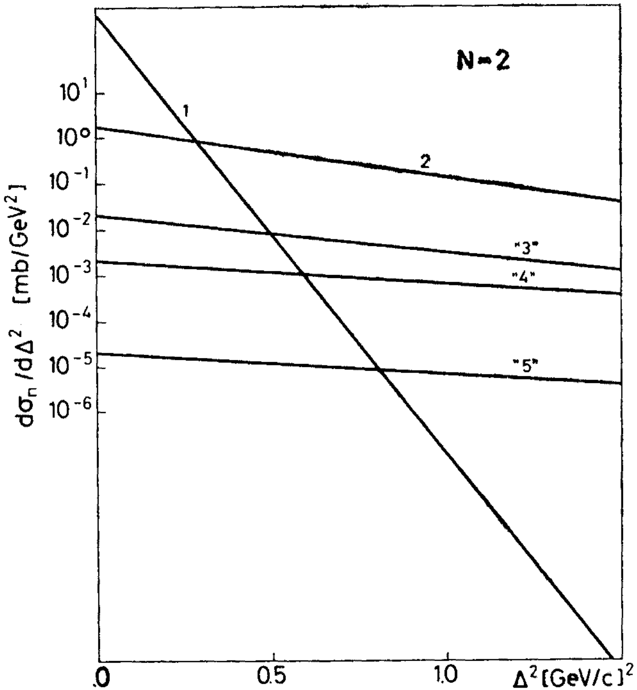

Fig. 2

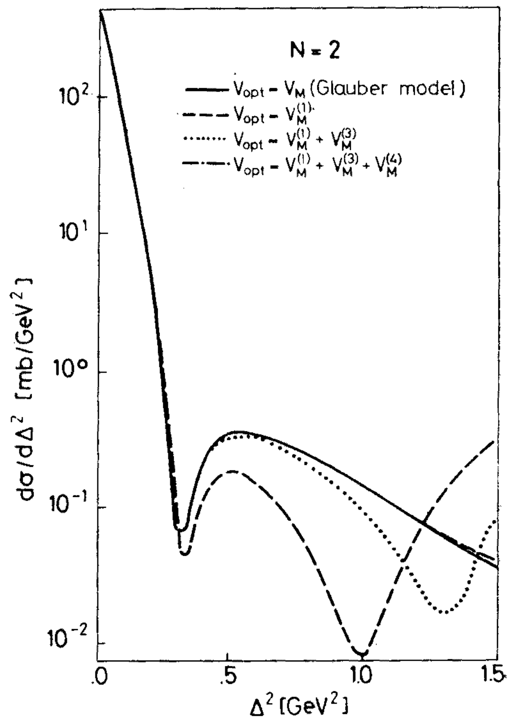

Fig. 3

Fig. 2. Moduli squared of the single, double and rescattering amplitudes for $N = 2$ target, calculated with the same parameters as in Fig. 1

Fig. 3. Convergence of the modified KMT amplitude for $N = 2$ target. In the calculation the same parameters were used as those in Fig. 1

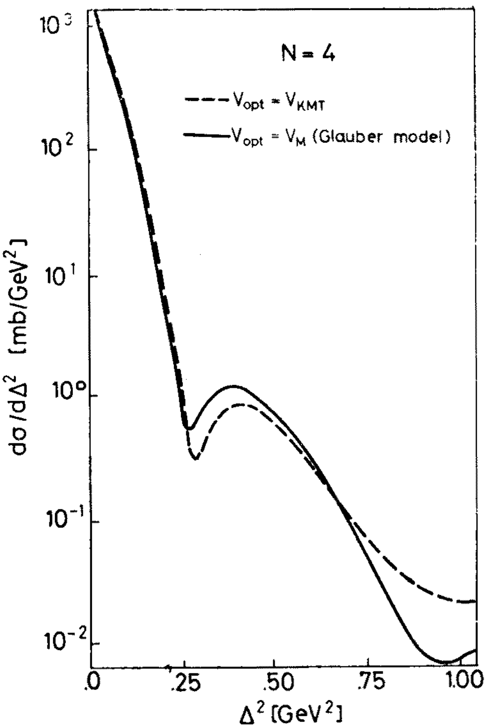

Fig. 4. The same as in Fig. 1 for $N = 4$ target, $R = 1.37$

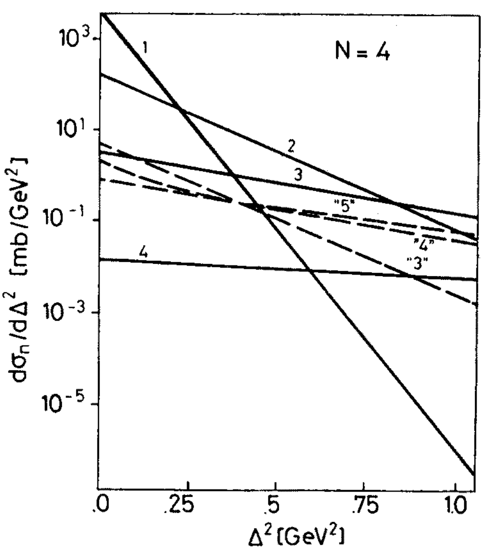

Fig. 5. The same as in Fig. 2 for $N = 4$ target

To conclude, the above results show that the suggested modification of V_{KMT} is really important. They also show the importance of the correct way of calculating the off-shell t-matrix elements if one wants to take into account the corrections coming from the rescattering terms at energies where the eikonal approximation is not valid.

6. *Convergence of the modified KMT optical potential for light nuclei*

We shall now discuss the convergence of the modified optical potential $V_{\text{M}} \equiv V_{\text{D}}$. Similarly as in the previous section we shall assume (14) and (23).

Let us denote by $V_{\text{M}}^{[j]}$ the optical potential terminated after the first j terms; according to the formula (29)

$$V_{\text{M}}^{[j]} = \sum_{n=1}^{\infty} c_n^{[j]} \langle 12 \dots n \rangle, \tag{43}$$

where

$$c_n^{[j]} = c_n \quad \text{for } n \leqslant j,$$
$$c_n^{[j]} = 0 \quad \text{for } n > j. \tag{44}$$

The amplitude $T_{\text{M}}^{[j]}$ corresponding to $V_{\text{M}}^{[j]}$ is

$$T_{\text{M}}^{[j]} = \sum_{n=1}^{\infty} d_n^{[j]} \langle 12 \dots n \rangle, \tag{45}$$

where $d_n^{[j]}$ can be calculated from the recurrence formula

$$d_n^{[j]} = \sum_{k=1}^{n} d_{n-k}^{[j]} c_k^{[j]} \quad \text{with } d_0^{[j]} = -1. \tag{46}$$

This can be done in a way analogous to that used in the derivation of V_{KMT} from the Watson series in Sec. 3. If the potential is taken to the infinite order then the corresponding coefficients $d_n^{[\infty]}$ are

$$d_n^{[\infty]} = N(N-1) \dots (N-n), \quad n = 1, \dots, N.$$

Comparing the coefficients $d_n^{[1]}$ with $d_n^{[\infty]}$, we see that their ratios

$$\eta_n^{[1]} = \frac{d_n^{[1]}}{d_n^{[\infty]}} = \frac{N(N-1)^{n-1}}{N(N-1) \dots (N-n)}, \tag{47}$$

in particular

$$\eta_1^{[1]} = \eta_2^{[1]} = 1, \quad \eta_3^{[1]} = \frac{N-1}{N-2}, \quad \eta_4^{[1]} = \frac{(N-1)^2}{(N-2)(N-3)} \text{ etc.}$$

tend to unity as $N \to \infty$. Therefore, we may expect that for heavy nuclei already the first order of V_{M} is a good approximation.

For light nuclei, such as, e. g., He^4, it will be instructive to calculate explicitly the amplitudes with $V_{\text{M}}^{[1]}$, $V_{\text{M}}^{[2]}$... etc.

Assuming now (37) and replacing G_0 by the eikonal propagator (3) we get

$$T^{[j]} = -\frac{ik}{8\pi^3}\int d^2 b e^{i\vec{\Delta}\vec{b}} \sum_{n=1}^{\infty} d_n^{[j]}\langle 0|\gamma_1 \dots \gamma_n|0\rangle. \tag{48}$$

Inserting the density (37) into the above formula we get

$$T_n^{[j]} = -\frac{ik}{4\pi^3}\sum_{n=1}^{\infty} d_n^{[j]}\frac{1}{n}(-1)^{n+1}(R^2+2a)\left[\frac{\sigma(1-i\alpha)}{R^2+2a}\right]^n e^{\frac{\Delta^2 R^2}{4N} - \frac{\Delta^2(R^2+2a)}{4n}}. \tag{49}$$

The convergence of V_M for $N = 2$ and $N = 4$ is illustıated in Figs 3, 6 where we have plotted the differential cross-sections calculated with the potential taken to the first, second, third, ... etc. orders and compared it with the exact ıesult in the eikonal approximation, i. e. the Glauber formula, or, equivalently, with the KMT amplitude with the modified potential V_M taken to the infinite order.

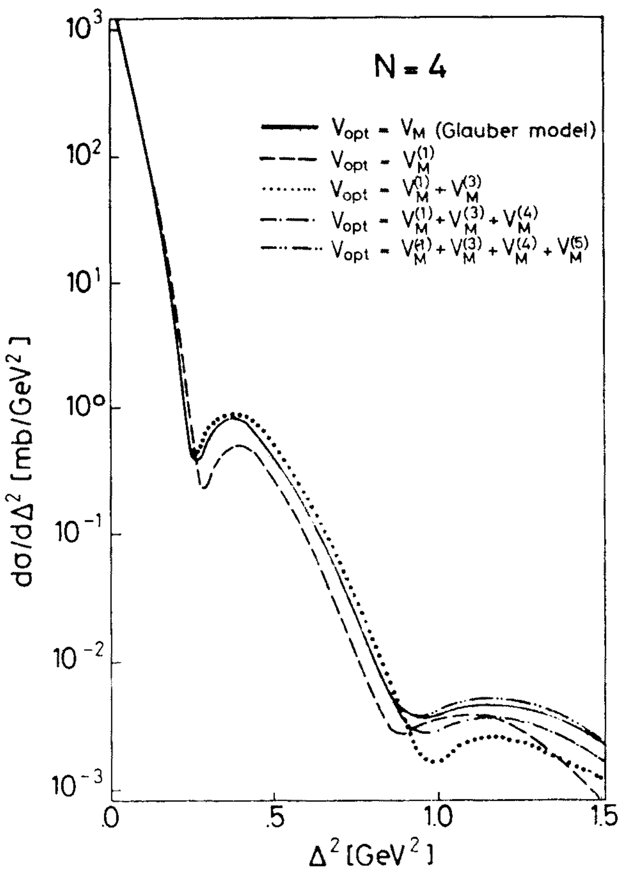

Fig. 6. The same as in Fig. 3 for $N = 4$ target

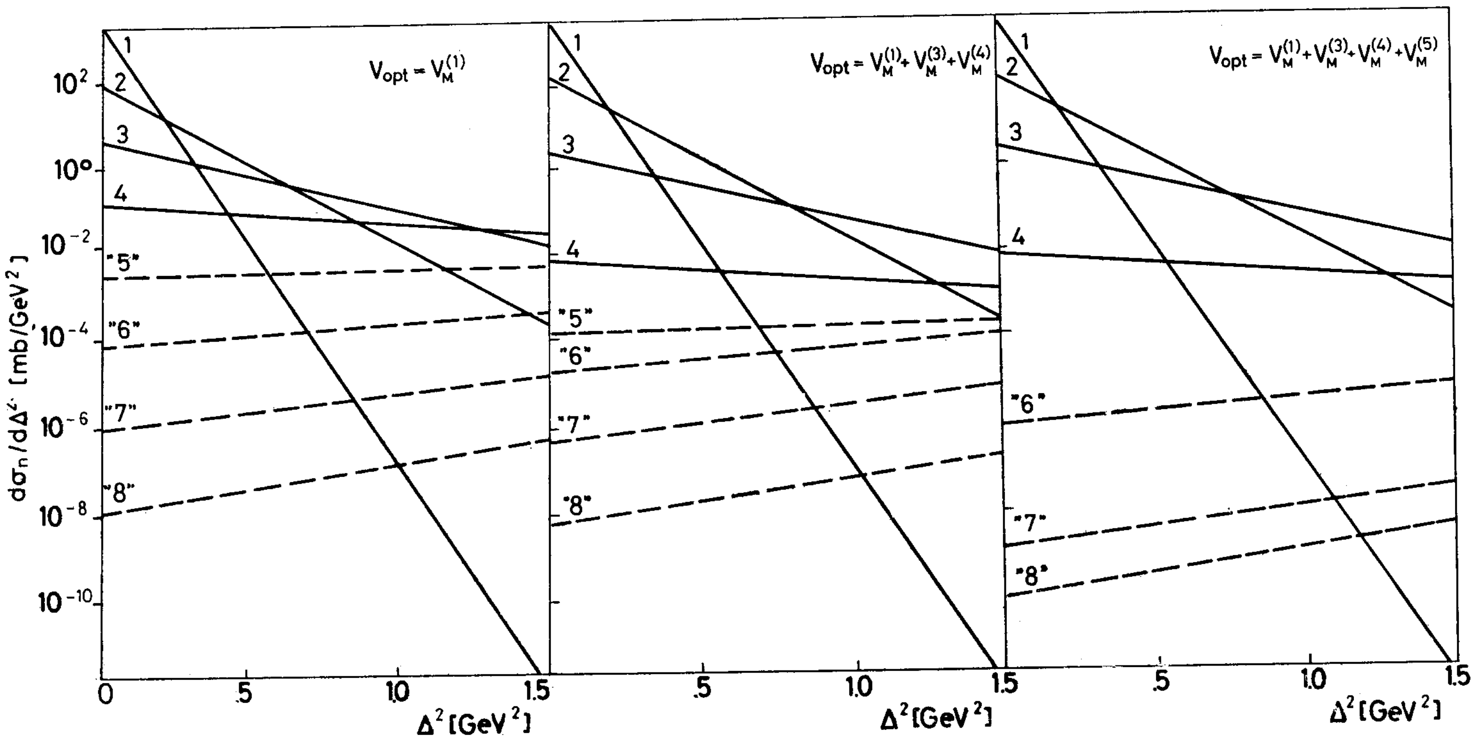

Fig. 7. Moduli squared of the successive terms in the formula for $N = 4$ target, calculated with the modified optical potential taken to the first, third, and fifth order, and with the same parameters as in Fig. 4. The indices 1, 2, 3, 4 denote the single, double, ... etc. terms and the indices "5", "6", "7", "8" denote the terms which are proportional to the fifth, sixth, ... etc. power of the t-matrix. All the terms indicated by quotation marks, vanish of course, for the potential taken to the infinite order

We see that for He^4 nucleus the first order of the optical potential is a very bad approximation already in the second maximum in the differential cross-section; it overestimates the correct result by about 100%. This is because

$$\eta_3^{[1]} = 1.5, \quad \eta_4^{[1]} = 4.5 \tag{50}$$

and in the second maximum triple scattering is very important. In order to reproduce the exact result up to the momentum transfer squared equal 1.5 GeV^2, one has to take the optical potential to the fifth order.

At medium energies the situation is not much different because of (47). For illustration we have compared in Fig. 8 the differential cross-sections for p-He^4 scatteiing calcu-

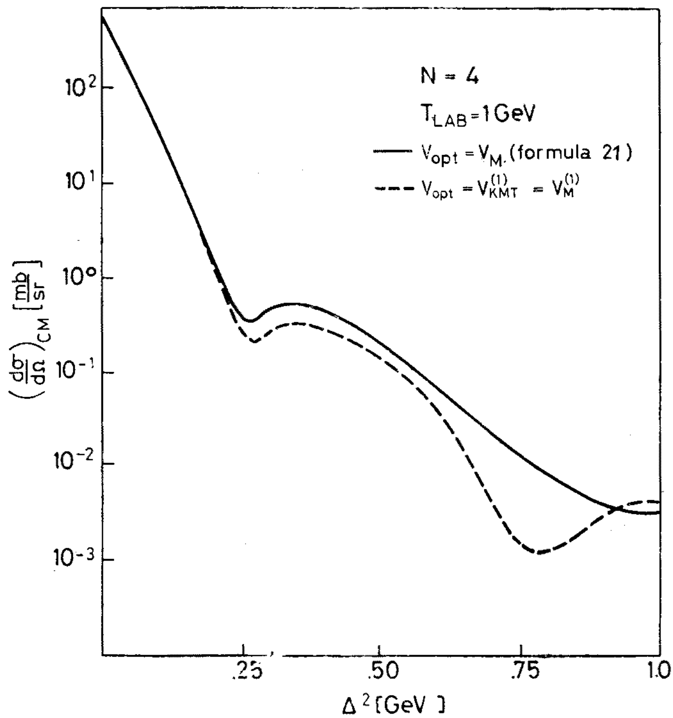

Fig. 8. Comparison between the cross-sections calculated with the modified KMT amplitude (solid line) and the KMT amplitude with the potential V_M (broken) at medium energy $k_{LAB}=1.7$ GeV with the same parameters as in Fig. 4. Here the eikonal approximation was not used

lated with the modified optical potential taken to the infinite order, or, equivalently, according to the foimula (21) [13] with the result obtained by solving the Schrödinger equation with $V_M^{[1]}(=V_{KMT}^{[1]})$. From these considerations we see that KMT approach for light nuclei is rather impractical since it should involve the optical potential of order in general greater than the number of scatterers in the target.

It is also interesting to note that if the KMT amplitude is calculated for the scattering from the target of N constituents with the potential taken to the n-th order, then the coefficients

$$d_m^{[n]} = d_m^{[\infty]} \quad \text{for } m \leqslant n,$$

$$d_m^{[n]} \neq d_m^{[\infty]} \quad \text{for } n > m \geqslant N,$$

$$d_m^{[n]} \neq 0 \quad \text{for } m = N+1, N+2, \ldots$$

i. e. there is always a non-zero contribution from the terms

$$\langle 12 \ldots m \rangle \quad \text{for } m > N$$

which become dominant at sufficiently large momentum transfer because their slope

$$\frac{R^2+2a}{4n} - \frac{R^2}{4N}, \quad \text{for } n > N$$

is negative (see Fig. 7). This explains, e. g., the behaviour at large momentum transfers of the cross-section for $N = 2$ calculated with $V_{\mathrm{KMT}}^{[1]}$, $V_{\mathrm{KMT}}^{[2]}$ etc. (Fig. 6).

7. *Conclusions*

We have studied the applicability of the KMT optical potential approach in the multiple scattering of elementary particles from nuclei at high and medium energies. It was shown how the KMT optical potential can be derived in a simple way directly from the Watson series. A modification of the KMT optical potential was then proposed, which consists of neglecting the rescattering terms when using apart from FSA such an off-shell continuation of t-matrices that they depend on the momentum transfer only. This modification was shown to be necessary in order to get the correct asymptotic high energy limit of the KMT amplitude.

For heavy nuclei the KMT amplitude with the first order optical potential is a reasonable approximation, provided that we are in such the region of momentum transfers that the dominant multiple scattering terms have combinatorial factors very close to the factors $d_m^{[\infty]}$. On the other hand, the convergence of the optical potential for light nuclei was shown to be rather poor: in order to reproduce the correct result one should use the KMT amplitude with the potential taken to the order equal at least to the number of scatterers in the target.

Therefore at medium energies, where the eikonal approximation is questionable, the KMT amplitude with the potential taken to the first order is a better approximation than the Glauber formula only for heavy nuclei. For light nuclei such as, e. g., He^4, it is much more convenient to do the multiple scattering calculations with the help of the formula recently proposed in [13] which in fact is equivalent to the KMT amplitude with the modified potential taken to the infinite order.

Thanks are due to Prof. W. Czyż for suggesting some corrections to the original version of the manuscript.

APPENDIX A

The treatment of the centre-of-mass constraint in the KMT theory

The standard way of treating the centre-of-mass correlations in the KMT optical potential is described in Ref. [6]. It consists simply of introducing the centre-of-mass correlation in the formulas [16].

Let us remark, however, that in the FSA another possibility of calculating the centre--of-mass correction arises if we have at our disposal the "auxiliary" model wave function having the factorization property

$$\langle \vec{r}_1, \ldots, \vec{r}_N | 0 \rangle = \psi(\vec{r}_1, \ldots, \vec{r}_N) = \mathscr{R}(\vec{R})\phi(\vec{r}\,'_1, \ldots, \vec{r}\,'_N) \tag{A.1}$$

where

$$\vec{R} = \frac{1}{N}\sum_{i=1}^{N} \vec{r}_i, \quad \vec{r}\,'_i = \vec{r}_i - \vec{R} \tag{A.2}$$

and $\phi(\vec{r}\,'_1, \ldots, \vec{r}\,'_N)$ is the "true" internal wave function whose arguments automatically satisfy the constraint

$$\sum \vec{r}\,'_i = 0.$$

Then using (14) we see that the Watson scattering operator given by (2) has the property that

$$\langle \vec{p}\,' | T(\vec{r}_1, \ldots, \vec{r}_N) | \vec{p} \rangle = e^{i(\vec{p}-\vec{p}\,')\vec{R}} \langle \vec{p}\,' | T'(\vec{r}\,'_1, \ldots, \vec{r}\,'_N | \vec{p} \rangle, \tag{A.3}$$

where T' is obtained from T by replacing $\vec{r}_i$ by $\vec{r}\,'_i$. Then from (A.1) and (A.3) it follows that one can calculate the transition amplitude T' with the model wave function, provided an extra correction factor is introduced as follows (see Ref. [11]):

$$\langle \phi | T' | \phi \rangle = \Theta((\vec{p}\,' - \vec{p})^2) \langle \psi | T | \psi \rangle, \tag{A.4}$$

where

$$\Theta(\Delta^2) = \int d^3R e^{i\vec{\Delta}\vec{R}} |\mathscr{R}(\vec{R})|^2. \tag{A.5}$$

The above property of the t-matrix can also be used in the calculations in the framework of the KMT theory, this is quite obvious in view of our derivation of the KMT optical potential directly from the Watson series. One can simply relate the optical potential to the t-matrix element calculated with the wave function ψ and then multiply the resulting amplitude by the correction factor Θ.

This method of treating the centre-of-mass correction is very convenient, especially in the calculation of the second order optical potential which simply vanishes if we calculate it with the wave function without correlations.

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