HIGH ENERGY LIMIT OF DUAL MULTILOOP DIAGRAMS FOR CPT UNITARISATION SCHEME

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Simple asymptotic expressions for the four-point N-loop dual amplitude in a specific high energy limit are given. A reasonable approximation of the obtained formula, which can be used in the Chan-Paton-Tsou unitarisation program, is proposed.

1. Introduction

Recently Chan, Paton and Tsou (CPT) have proposed a half-phenomenological method of unitarising the S-matrix based on the dual model (Ref. [1]). They calculated elastic diffractive scattering as a shadow of nondiffractive production processes. The unitarity of the S-matrix written in terms of T

$$i(T^{+}-T) = T^{+}T \tag{1}$$

is a basic formula in their approach. Theys ketched an ambitious program of solving this equation by means of the step by step iteration method. In this way, it should be possible to calculate all higher order corrections to the nondiffractive particle production. In the first step, the T on the right-hand side of Eq. (1) is regarded as a usual dual amplitude without loops. In order to compute the imaginary part of the elastic scattering amplitude in the high energy limit, one should find, according to Eq. (1), the asymptotic expressions for the diagrams depicted in Fig. 1.

In the limit $s \to \infty$ Chan, Paton and Tsou analysed the simplest 4-point 1-loop dual amplitude. Using the factorisation property they were able to write the corresponding expressions for multiloop diagrams. For inelastic diffractive scattering one wants to know the asymptotic form of different kinds of planar and nonplanar multiloop diagrams in different limits.

In this note we are interested in a specific high energy limit of a particular 4-point multiloop diagram visualized in Fig. 2. Correct treating of other diagrams requires still further study.

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The general methods of the dual perturbation theory give a relatively simple formula for n-point 1-loop amplitude (Ref. [2]). The expression for a multiloop diagram has also been found (Ref. [3]). Unfortunately, the mathematical form of the formula is highly complicated and in spite of its elegance it is not simple to make use of it. In particular,

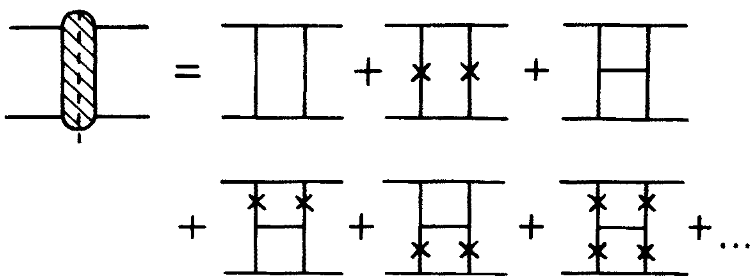

Fig. 1. The imaginary part of the elastic scattering amplitude as a sum of dual multiloop diagrams in the CPT scheme

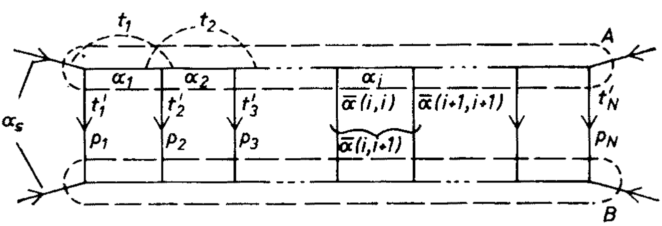

Fig. 2. Four point $N-1$ loop dual diagram

for multiloop diagrams it is not clear, whether one can define in a simple way the corresponding Chan variables x_i. This fact makes it difficult to calculate the high energy limit for such a diagram. It turns out, however, that for some diagrams analysis of the B_N-function helps considerably in deducing this limit. The results of analysis of the 1-loop diagram (Ref. [1]) we obtain in a simplified way.

For multiloop diagrams in the limit visualized in Fig. 2, we show that the parts A and B are given by the B_{N+2}-functions in which the transfers $\bar{\alpha}(i, i+1)$ are replaced by $\bar{\alpha}(i, i+1)-\bar{\alpha}(i, i)-\bar{\alpha}(i+1, i+1)$ (the notation is defined in Fig. 2). From the structure of B_N-functions it follows immediately that it is not possible to factorize such an amplitude. This fact is related to the existence of correlations between all reggeons. We propose an approximation in which all correlations between nonsubsequent reggeons are neglected.

2. *One loop case*

Let us consider the one loop four point diagram (Fig. 3). The methods of the dual perturbation theory give the following expression for the corresponding amplitude (Ref. [2]) in the high energy limit

$$L(1) = \int d^4k \int dx_1 dx_2 dx_3 dx_4 x_1^{-\alpha_1-1} x_2^{-\alpha_2-1} x_3^{-\alpha_3-1} x_4^{-\alpha_4-1}$$
$$\times (1-x_1)^{-2p_1\cdot p_2}(1-x_2)^{-2p_2\cdot p_3}(1-x_3)^{-2p_3\cdot p_4}(1-x_4)^{-2p_4\cdot p_1}$$

$$\times(1-x_1x_2)^{-2p_1\cdot p_3}(1-x_2x_3)^{-2p_2\cdot p_4}(1-x_3x_4)^{-2p_1\cdot p_3}(1-x_1x_4)^{-2p_2\cdot p_4}$$

$$\times(1-x_2x_3x_4)^{-2p_1\cdot p_2}(1-x_1x_2x_4)^{-2p_3\cdot p_4}. \tag{2}$$

In the limit $p_1 \cdot p_2$, $p_3 \cdot p_4 \to -\infty$ the only important contribution comes from the $x_1 \cdot x_3 \approx 0$ integration region. Let us denote $\alpha_{ik} = 1+(p_i+p_k)^2$. Applying the standard substitutions

$$x_1 = -\frac{z_1}{\alpha_{12}}, \quad x_3 = -\frac{z_3}{\alpha_{34}},$$

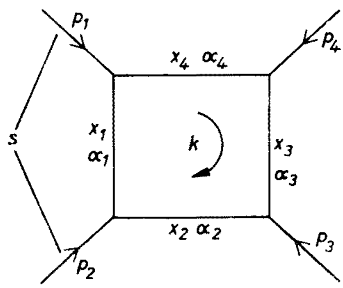

Fig. 3. Four point one loop ~~dual~~ diagram

after taking the limit $\alpha_{12}, \alpha_{34} \to -\infty$ we obtain

$$L(1) = \int d^4k \int_0^1 dx_2 dx_4 \int_0^\infty dz_1 dz_3 z_1^{-\alpha_1-1} z_3^{-\alpha_3-1} x_2^{-\alpha_2-1} x_4^{-\alpha_4-1}$$

$$\times(-\alpha_{12})^{\alpha_1}(-\alpha_{34})^{\alpha_3}(1-x_2)^{-\alpha_{23}-1}(1-x_4)^{-\alpha_{14}-1}\exp\left[-z_1(1-x_2)(1-x_4)\right.$$

$$\left.-z_3(1-x_2)(1-x_4)\right].$$

Integration over z_1, z_3 can be done yielding

$$L(1) = \int d^4k\Gamma(-\alpha_1)(-\alpha_{12})^{\alpha_1}\Gamma(-\alpha_3)(-\alpha_{34})^{\alpha_3}B_4(\alpha_2, \alpha_{23}-\alpha_1-\alpha_3)B_4(\alpha_4, \alpha_{14}-\alpha_1-\alpha_3). \tag{3}$$

The loop amplitude shows the expected factorization into two amplitudes connected by Regge pole exchanges. Moreover, one can simply compute the discontinuity of the elastic scattering amplitude coming from the cut due to the normal threshold for resonance pair production $(\alpha_4, \alpha_2) = (n_1, n_2)$.

The contribution of the (n_1, n_2)-cut is

$$\text{cut}\,(n_1, n_2) \sim \int d^4k\ \text{Res}\,(M)_{n_1,n_2}\delta(\alpha_4-n_1)\delta(\alpha_2-n_2),$$

where M in the integrand in Eq. (3).

We have

$$\text{cut}\,(n_1, n_2) \sim \int d^4k\Gamma(-\alpha_1)(-\alpha_{12})^{\alpha_1}\Gamma(-\alpha_3)(-\alpha_{34})^{\alpha_3}$$

$$\times \frac{1}{2\pi i}\oint \frac{dz_2}{z_2} z_2^{-n_2}(1-z_2)^{\alpha_1+\alpha_3-\alpha_{23}-1} \frac{1}{2\pi i}\oint \frac{dz_4}{z_4} z_4^{-n_1}(1-z_4)^{\alpha_1+\alpha_3-\alpha_{14}-1}.$$

In such a way we were able to obtain the CPT result without any summation over the internal variables.

3. *Multiloop case*

In order to explain the method we will first find the one loop result from the analysis of the B_6-function (see Fig. 4). Not losing generality we take the contribution to the one loop diagram at $\alpha_2 = 0$. According to the general rules for constructing dual amplitudes this is equal to the familiar B_6-amplitude

$$B_6 = \int dx_1 dx_2 dx_3 x_1^{-\alpha(0,1)-1} x_2^{-\alpha(0,2)-1} x_3^{-\alpha(0,3)-1}$$

$$\times (1-x_1)^{-\alpha(1,2)-1}(1-x_2)^{-\alpha(2,3)-1}(1-x_3)^{-\alpha(3,4)-1}$$

$$\times (1-x_1x_2)^{-\alpha(1,3)+\alpha(1,2)+\alpha(2,3)}(1-x_2x_3)^{-\alpha(2,4)+\alpha(2,3)+\alpha(3,4)}$$

$$\times (1-x_1x_2x_3)^{-\alpha(1,4)-\alpha(2,3)+\alpha(1,3)+\alpha(2,4)}$$

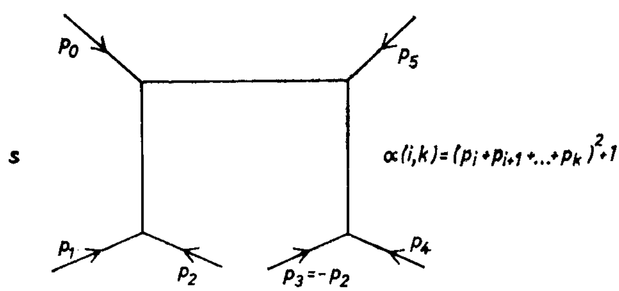

Fig. 4. High energy limit for B_6 ($s \to \infty$)

In the limit $\alpha(0, 1), \alpha(0, 3) \to -\infty$ we substitute $x_1 = 1 + \dfrac{z_1}{\alpha(0, 1)}$, $x_3 = 1 + \dfrac{z_3}{\alpha(0, 3)}$ and obtain

$$B_6 = \int_0^1 dx_2 \int_0^\infty dz_1 dz_3 \exp(-z_1 - z_3) z_1^{-\alpha(1,2)-1} z_3^{-\alpha(3,4)-1}(-\alpha(0, 1))^{\alpha(1,2)}$$

$$\times (-\alpha(0, 3))^{\alpha(3,4)} x_2^{-\alpha(0,2)-1}(1-x_2)^{\alpha(1,2)+\alpha(3,4)-\alpha(1,4)-1}$$

$$= \Gamma(-\alpha(1, 2))(-\alpha(0, 1))^{\alpha(1,2)}\Gamma(-\alpha(3, 4))(-\alpha(0, 3))^{\alpha(3,4)} B_4(\alpha(0, 2), \alpha(1, 4) - \alpha(1, 2) - \alpha(3, 4)). \tag{4}$$

Comparing this with formula (3) we see that we are able to compute the diagram shown in Fig. 2 by means of analysis of dual amplitudes at the tree level (see Fig. 5).

For the amplitude B_{2N+2} (the notation is explained in Fig. 5) we get (Ref. [4])

$$B_{2N+2} = \int_0^1 dx_1 \dots dx_{2N-1} \prod_{i=1}^{2N-1} x_i^{-\alpha(0,i)-1}(1-x_i)^{-\alpha(i,i+1)-1}$$

$$\times \prod_{i=1}^{2N-2} \prod_{j=i+2}^{2N} \Big(1-\prod_{k=i}^{j-1} x_k\Big)^{-\alpha(i,j)-\alpha(i+1,j-1)+\alpha(i,j-1)+\alpha(i+1,j)}. \tag{5}$$

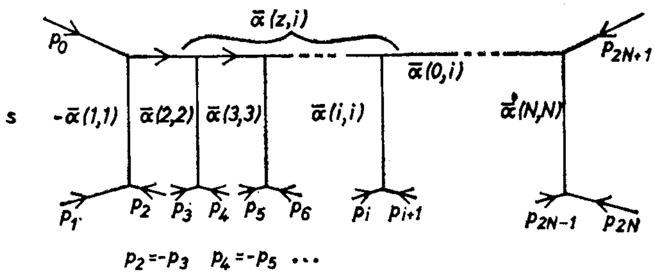

Fig. 5. High energy limit for B_{2N+2} $(s \to \infty)$

In the limit $\alpha(0, i) \to -\infty$ for odd i, keeping all remaining variables $\alpha(k, l)$ fixed and substituting $x_i = 1 + \dfrac{z_i}{\alpha(0, i)}$ for odd i, after some manipulations we obtain

$$B_{2N+2} = \prod_{i=0}^{N-1} \Gamma(-\bar{\alpha}(i+1, i+1))\,(-\alpha(0, 2i+1))^{\bar{\alpha}(i+1,i+1)}$$

$$\times \int_0^1 \prod_{i=1}^{N-1} dy_i y_i^{-\alpha(0,i)}(1-y_i)^{-\bar{\alpha}(i,i+1)+\bar{\alpha}(i,i)+\bar{\alpha}(i+1,i+1)-1}$$

$$\times \prod_{i=1}^{N-2} \prod_{j=i+2}^{N} \Big(1-\prod_{k=i}^{j-1} y_k\Big)^{-\bar{\alpha}(i,j)-\bar{\alpha}(i+1,j-1)+\bar{\alpha}(i,j-1)+\bar{\alpha}(i+1,j)}, \tag{6}$$

where $y_i = x_{2i}$, $\bar{\alpha}(k, l) = \alpha(2k-1, 2l)$, $\bar{\alpha}(0, i) = \alpha(0, 2i) = \alpha_i$. We recognize immediately the integral in Eq. (6) as a function B_{N+2} with the prescription $\bar{\alpha}(i, i+1) \to \bar{\alpha}(i, i+1) - \bar{\alpha}(i+1, i+1) - \bar{\alpha}(i, i)$. For $\bar{\alpha}(i, i) = 0$ we come back to the standard formula for B_{N+2} (Ref. [4]). From these considerations, we obtain for the $N-1$ loop amplitude, corresponding to the diagram in Fig. 2, the following simple expression

$$L(N-1) = B_{N+2}(A)B_{N+2}(B) \prod_{i=1}^{N} (-\alpha_s)^{\alpha(i,i)}\Gamma(-\bar{\alpha}(i, i)), \tag{7}$$

where $\alpha_s \to \infty$ and $B_{N+2}(A)\,(B_{N+2}(B))$ are to be constructed according to Eq. (6). In practice it is not easy to deal with B_N-functions, so we shall simplify Eq. (7) by neglecting correlations between reggeons. Let us notice that these factors in B_{N+2} which contain expressions $\bar{\alpha}(i, j-1) + \bar{\alpha}(i+1, j) - \bar{\alpha}(i, j) - \bar{\alpha}(i+1, j-1) = -2p_i \cdot p_j$ are responsible for correlations between reggeons. Thus putting the product $\prod_{i=1}^{N-2}\prod_{j=i+2}^{N}$ in Eq. (6) equal to 1 we neglect the correlations between nonsubsequent reggeons and obtain the decomposition of the B_{N+2}-function into

$$B_{N+2} \to B_4(\alpha_1, \alpha_{t_1}-\alpha_{t'_1}-\alpha_{t'_2})B_4(\alpha_2, \alpha_{t_2}-\alpha_{t'_2}-\alpha_{t'_3}) \ldots \tag{8}$$

In the CPT program we are mainly interested in calculating discontinuities coming from resonance pair production. The residue of B_{N+2} for $\alpha_1 = n_1$, $\alpha_2 = n_2, \ldots$ is

$$\sim \frac{\Gamma(\alpha_{t_1}-\alpha_{t'_1}-\alpha_{t'_2}+n_1+1)}{n_1!\Gamma(\alpha_{t_1}-\alpha_{t'_1}-\alpha_{t'_2}+1)} \frac{\Gamma(\alpha_{t_2}-\alpha_{t'_2}-\alpha_{t'_3}+n_2+1)}{n_2!\Gamma(\alpha_{t_2}-\alpha_{t'_2}-\alpha_{t'_3}+1)} \cdots \tag{9}$$

Because of the summation over resonance masses ($\leqslant 6$ GeV^2) and exploiting semi-local duality instead of Eq. (9) we choose rather its asymptotic form for $n_1, n_2 \ldots \to \infty$, which is

$$\sim n_1^{\alpha_{t1}-\alpha_{t'1}-\alpha_{t'2}} n_2^{\alpha_{t2}-\alpha_{t'2}-\alpha_{t'3}} \ldots \tag{10}$$

From this expression we see that the amplitudes

reggeon + resonance $\to$ reggeon + resonance
reggeon + particle $\to$ reggeon + particle

are not distinguishable, which seems to be a good approximation for $s/m_{\text{res}}^2 \to \infty$. To be convinced that in this limit we really may neglect the correlations between nonsubsequent reggeons let us consider the amplitude B_{2N+2} (Eq. (6)) for $N = 3$. If we restrict ourselves to the resonance masses $n_1, n_2 \leqslant 6$ GeV^2, we find that the residuum of B_5 at $\alpha_1 = n_1, \alpha_2 = n_2$ is

$$\text{res } B_5 = \sum_{k \leqslant \min(n_1, n_2)} \frac{\Gamma(\gamma+k)}{k!\Gamma(\gamma)} \frac{\Gamma(1+\beta_1+n_1-k)}{\Gamma(1+\beta_1)\Gamma(n_1-k+1)} \frac{\Gamma(1+\beta_2+n_2-k)}{\Gamma(1+\beta_2)\Gamma(n_2-k+1)}, \tag{11}$$

where

$$\beta_1 = \alpha_{t_1}-\alpha_{t'_1}-\alpha_{t'_2}, \qquad \beta_2 = \alpha_{t_2}-\alpha_{t'_2}-\alpha_{t'_3},$$

$$\gamma = \alpha_{t_{12}}+\alpha_{t'_2}-\alpha_{t_1}-\alpha_{t_2} = t_{12}+t'_2-t_1-t_2.$$

The approximate form of Eq. (9) corresponds to the $k = 0$ term in Eq. (11). We observe that the contribution from $k \neq 0$ terms introduce a multiplicative factor γ. In the limit $s \to \infty$ the amplitude is strongly damped for $t_{12}, t'_2, t_1, t_2 \neq 0$ and we may put γ equal to zero. Thus, the only important term in Eq. (11) is that with $k = 0$.

4. *Summary*

We found a simple asymptotic expression for some kinds of multiloop diagrams which, after neglecting the correlations between nonsubsequent reggeons, can be used in practical calculations.

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