RELATIVISTIC RADIAL EQUATIONS FOR TWO SPIN-1/2 PARTICLES WITH A STATIC INTERACTION*

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We derive an explicit form of the relativistic radial equations for two spin-1/2 particles with a static interaction energy. We solve these equations for Coulomb bound states with $j = 0$ in an implicit way after applying the weak-potential approximation. This solution reduces to the Sommerfeld fine-structure formula in the one-body approximation.

1. Introduction

In the mid-fifties we formulated a systematic method of evaluating the interaction-energy operator in the one-time relativistic wave equation for a many-particle system. The starting point was either [1] the Bethe-Salpeter equation for this system or [2] directly a formal field-theoretical Hamiltonian. The same problem for the S matrix was also generally solved [3]. We described at the same time a separation method of angular coordinates from the one-time relativistic wave equation for a two-spin-1/2-particle system with a general interaction energy [4].

Since recently the binding problem of two spin-1/2 particles arouses much interest [5] due to the discovery of the narrow resonances in e^+e^- channel, we publish in the present note the result of separating the angular coordinates from the one-time relativistic wave equation for such particles with a static interaction energy (cf. also [6]).

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2. *Relativistic radial equation*

Let us consider a system of two spin-1/2 particles satisfying in the center-of-mass coordinates the following wave equation

$$[E-c(\vec{\alpha}^{(1)}\cdot\vec{p}+\beta^{(1)}m^{(1)}c)-c(-\vec{\alpha}^{(2)}\cdot\vec{p}+\beta^{(2)}m^{(2)}c)-\Gamma V(r)]\psi(\vec{r})=0, \tag{1}$$

where $\vec{r}=\vec{x}^{(1)}-\vec{x}^{(2)}$, $\vec{p}=-i\hbar\partial/\partial\vec{r}$ and Γ is built of $\gamma_\mu^{(1)}$ and $\gamma_\mu^{(2)}$ in a rotationally invariant way, e. g. Γ may be $1-\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}$ or $\beta^{(1)}\gamma_5^{(1)}\beta^{(2)}\gamma_5^{(2)}$. The equation of form (1) can be derived from the quantum field theory in the static one-boson-exchange approximation (cf. e. g. [1, 2]).

Making use of the separation method of angular coordinates described in Ref. [4] we obtain from (1) the following radial equation (for its derivation see Appendix)

$$\left\{\varepsilon+i(\alpha_3^{(1)}-\alpha_3^{(2)})\left[\frac{d}{dr}+\frac{1+\frac{1}{2}(\alpha_1^{(1)}\alpha_1^{(2)}+\alpha_2^{(1)}\alpha_2^{(2)})}{r}\right]\right.$$

$$\left.-i(\alpha_1^{(1)}-\alpha_1^{(2)})\frac{\alpha_2^{(1)}\alpha_2^{(2)}\sqrt{j(j+1)}}{r}-\beta^{(1)}\kappa^{(1)}-\beta^{(2)}\kappa^{(2)}-\Gamma v(r)\right\}\psi(r)=0, \tag{2}$$

where

$$\varepsilon\equiv\frac{E}{\hbar c},\quad \kappa^{(i)}\equiv\frac{m^{(i)}c^2}{\hbar c},\quad v(r)\equiv\frac{V(r)}{\hbar c} \tag{3}$$

and j denotes the quantum number of $\vec{J}^2$, while

$$\vec{J}=\vec{r}\times\vec{p}+\frac{\hbar}{2}(\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)}) \tag{4}$$

is the total angular momentum.

Since in equation (2) the Dirac matrices $\alpha_2^{(1)}$ and $\alpha_2^{(2)}$ appear only via their product $\alpha_2^{(1)}\alpha_2^{(2)}$, we can use there the following particular 8×8 representation

$$\alpha_1^{(1)}=\sigma_3\times\sigma_1\times\mathbf{1}=\begin{pmatrix}\alpha_3 & 0\\ 0 & \alpha_3\end{pmatrix}, \tag{5}$$

$$\alpha_3^{(1)}=\sigma_2\times\sigma_1\times\mathbf{1}=\begin{pmatrix}\alpha_2 & 0\\ 0 & \alpha_2\end{pmatrix}, \tag{6}$$

$$\beta^{(1)}=\sigma_1\times\sigma_1\times\mathbf{1}=\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_1\end{pmatrix}, \tag{7}$$

$$\alpha_1^{(2)}=\mathbf{1}\times\sigma_1\times\sigma_3=\begin{pmatrix}\varrho_1 & 0\\ 0 & \varrho_1\end{pmatrix}, \tag{8}$$

$$\alpha_3^{(2)}=\mathbf{1}\times\sigma_1\times\sigma_2=\begin{pmatrix}0 & -i\varrho_1\\ i\varrho_1 & 0\end{pmatrix}, \tag{9}$$

$$\beta^{(2)} = \mathbf{1}\times\sigma_1\times\sigma_1 = \begin{pmatrix} 0 & \varrho_1 \\ \varrho_1 & 0 \end{pmatrix}, \tag{10}$$

$$\alpha_2^{(1)}\alpha_2^{(2)} = \mathbf{1}\times\sigma_3\times\mathbf{1} = \begin{pmatrix} \varrho_3 & 0 \\ 0 & \varrho_3 \end{pmatrix}, \tag{11}$$

where σ_i and $\mathbf{1}$ are 2×2 Pauli matrices, while

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & \mathbf{1} \end{pmatrix}, \quad \beta = \varrho_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \varrho_1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{12}$$

From the matrices (5)–(11) we can calculate

$$\sigma_2^{(1)} = -i\alpha_3^{(1)}\alpha_1^{(1)} = \sigma_1\times\mathbf{1}\times\mathbf{1} = \begin{pmatrix} \sigma_1^{\mathrm{D}} & 0 \\ 0 & \sigma_1^{\mathrm{D}} \end{pmatrix}, \tag{13}$$

$$\sigma_2^{(2)} = -i\alpha_3^{(2)}\alpha_1^{(2)} = \mathbf{1}\times\mathbf{1}\times\sigma_1 = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{D}} \\ \mathbf{1}^{\mathrm{D}} & 0 \end{pmatrix}, \tag{14}$$

$$\sigma_1^{(1)}\sigma_1^{(2)} = -\alpha_2^{(1)}\alpha_3^{(1)}\alpha_2^{(2)}\alpha_3^{(2)} = -\sigma_2\times\sigma_3\times\sigma_2, \tag{15}$$

$$\sigma_3^{(1)}\sigma_3^{(2)} = -\alpha_1^{(1)}\alpha_2^{(1)}\alpha_1^{(2)}\alpha_2^{(2)} = -\sigma_3\times\sigma_3\times\sigma_3, \tag{16}$$

but not $\sigma_1^{(1)}, \sigma_3^{(1)}$ and $\sigma_1^{(2)}, \sigma_3^{(2)}$ separately. In (13) and (14) we use the notation

$$\sigma_1^{\mathrm{D}} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \mathbf{1}^{\mathrm{D}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{17}$$

From (13)–(16) we obtain also

$$\vec{S}^2 = \frac{\hbar^2}{4}(\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)})^2 = \frac{\hbar^2}{2}(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}+3)$$

$$= \frac{\hbar^2}{2}(-\sigma_2\times\sigma_3\times\sigma_2+\sigma_1\times\mathbf{1}\times\sigma_1-\sigma_3\times\sigma_3\times\sigma_3+3) \tag{18}$$

and

$$S_3^2 = \frac{\hbar^2}{4}(\sigma_3^{(1)}+\sigma_3^{(2)})^2 = \frac{\hbar^2}{2}(\sigma_3^{(1)}\sigma_3^{(2)}+1) = \frac{\hbar^2}{2}(-\sigma_3\times\sigma_3\times\sigma_3+1). \tag{19}$$

By means of (19) we get for the quantum number m_s^2 of S_3^2 the following values:

ψ_α	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8
m_s^2	0	1	1	0	1	0	0	1

, (20)

where

$$\psi = (\psi_\alpha) \tag{21}$$

is given in the representation (5)–(11). The operator

$$S_3 = \frac{\hbar}{2}(\sigma_3^{(1)}+\sigma_3^{(2)}) \tag{22}$$

cannot, however, be written in the 8×8 representation (5)–(11). So, m_s is not determined for ψ_α if $m_s^2 \neq 0$ (it means that in the case of $m_s^2 = 1$ signs of m_s are mixed).

Using the matrix (18) we deduce the following conditions to be satisfied by eigenstates of $\vec{S}^2$ corresponding to $s = 0$ and $s = 1$:

$$\psi_1^{s=0} = -\psi_6^{s=0}, \quad \psi_4^{s=0} = -\psi_7^{s=0}, \quad \psi_2^{s=0} = \psi_5^{s=0} = \psi_3^{s=0} = \psi_8^{s=0} = 0 \tag{23}$$

and

$$\psi_1^{s=1} = \psi_6^{s=1}, \quad \psi_4^{s=1} = \psi_7^{s=1}. \tag{24}$$

In general we have

$$\psi = \psi^{s=0} + \psi^{s=1} \tag{25}$$

because of the spin-orbit coupling involved in equation (1) or (2).

3. A representation of the radial equation

Making use of the representation (5)–(11) and the conditions (23) and (24) we obtain from equation (2) the system of eight equations which we present in Table I, introducing the notation

$$f_1 \equiv \psi_1^{s=0}, \quad f_2 \equiv \psi_4^{s=0}, \quad f_3 \equiv \psi_1^{s=1}, \quad f_4 \equiv \psi_4^{s=1}, \quad g_1 \equiv \frac{\psi_3^{s=1} - \psi_8^{s=1}}{2},$$

$$g_2 \equiv \frac{\psi_2^{s=1} - \psi_5^{s=1}}{2}, \quad g_3 \equiv \frac{\psi_3^{s=1} + \psi_8^{s=1}}{2}, \quad g_4 \equiv \frac{\psi_2^{s=1} + \psi_5^{s=1}}{2} \tag{26}$$

and putting $\Gamma = 1$. Notice that for $j = 0$ this system splits into two independent subsystems (of four equations each) for f's and g's separately. Then, in the equations for g's (which all have $s = 1$) there is no mixing of $s = 1$ with $s = 0$. It implies that all g's correspond to $l = 0$ (to avoid spin-orbit coupling). In this case, however, all g's must be equal to zero because $j = 0$ cannot be built from $l = 0$ and $s = 1$. Thus, for $j = 0$ we are left with the system of four equations for f's. By eliminating f_1 and f_2 we then obtain from Table I the set of two equations for $j = 0$:

$$\frac{d}{dr} f_2 + \tfrac{1}{2}\left[\varepsilon - v - \frac{(\kappa^{(1)} + \kappa^{(2)})^2}{\varepsilon - v}\right] f_3 = 0,$$

$$-\left(\frac{d}{dr} + \frac{2}{r}\right) f_3 + \tfrac{1}{2}\left[\varepsilon - v - \frac{(\kappa^{(1)} - \kappa^{(2)})^2}{\varepsilon - v}\right] f_2 = 0. \tag{27}$$

Obviously, f_2 and f_3 correspond to ($s = 0$, $m_s = 0$, $l = 0$, $m_l = 0$) and ($s = 1$, $m_s = 0$, $l = 1$, $m_l = 0$), respectively (because of $j = 0$ and $m = 0$).

If we introduce the notation

$$\Delta\varepsilon \equiv \varepsilon - \kappa^{(1)} - \kappa^{(2)}, \quad \frac{1}{\kappa} \equiv \frac{1}{\kappa^{(1)}} + \frac{1}{\kappa^{(2)}}, \tag{28}$$

TABLE I

Equation (2) in the representation (5)–(11)

$$\frac{d}{dr}f_2-\frac{\varkappa^{(1)}+\varkappa^{(2)}}{2}f_4+\frac{\varepsilon-v}{2}f_3=0,$$

$$-\left(\frac{d}{dr}+\frac{2}{r}\right)f_3-\frac{\varkappa^{(1)}-\varkappa^{(2)}}{2}f_1+\frac{\varepsilon-v}{2}f_2+\frac{i\sqrt{j(j+1)}}{r}g_4=0,$$

$$-\frac{\varkappa^{(1)}-\varkappa^{(2)}}{2}f_2+\frac{\varepsilon-v}{2}f_1=0,$$

$$-\frac{\varkappa^{(1)}+\varkappa^{(2)}}{2}f_3+\frac{\varepsilon-v}{2}f_4+\frac{i\sqrt{j(j+1)}}{r}g_2=0,$$

$$\left(\frac{d}{dr}+\frac{1}{r}\right)g_2-\frac{\varkappa^{(1)}+\varkappa^{(2)}}{2}g_4+\frac{\varepsilon-v}{2}g_3=0,$$

$$-\left(\frac{d}{dr}+\frac{1}{r}\right)g_3-\frac{\varkappa^{(1)}-\varkappa^{(2)}}{2}g_1+\frac{\varepsilon-v}{2}g_2-\frac{i\sqrt{j(j+1)}}{r}f_4=0,$$

$$-\frac{\varkappa^{(1)}-\varkappa^{(2)}}{2}g_2+\frac{\varepsilon-v}{2}g_1=0,$$

$$-\frac{\varkappa^{(1)}+\varkappa^{(2)}}{2}g_3+\frac{\varepsilon-v}{2}g_4-\frac{i\sqrt{j(j+1)}}{r}f_2=0.$$

we get from equations (27) in the limit of $(\Delta\varepsilon-v)/(\kappa^{(1)}+\kappa^{(2)})\to 0$ the radial non-relativistic Schrödinger equations

$$\left[-\frac{1}{2\kappa}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+v-\Delta\varepsilon\right]f_2=0,$$

$$\left[-\frac{1}{2\kappa}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2}{r^2}\right)+v-\Delta\varepsilon\right]f_3=0 \tag{29}$$

corresponding to $l=0$ and $l=1$, respectively.

4. The case of weak potential

For practical calculations it may be of interest to consider equations (27) in the case when v^2/ε^2 can be neglected. Then $1/(\varepsilon-v)\simeq\left(1+\frac{v}{c}\right)\Big/\varepsilon$ and these equations take approximately the form

$$\frac{d}{dr}f_2-\left(\frac{1}{a_1}+k_1v\right)f_3=0,\quad \left(\frac{d}{dr}+\frac{2}{r}\right)f_3-\left(\frac{1}{a_2}-k_2v\right)f_2=0, \tag{30}$$

where

$$\frac{1}{a_1} \equiv \frac{\varepsilon}{2}\left[\left(\frac{\kappa^{(1)}+\kappa^{(2)}}{\varepsilon}\right)^2 - 1\right], \quad \frac{1}{a_2} \equiv \frac{\varepsilon}{2}\left[1 - \left(\frac{\kappa^{(1)}-\kappa^{(2)}}{\varepsilon}\right)^2\right],$$

$$k_1 \equiv \tfrac{1}{2}\left[1 + \left(\frac{\kappa^{(1)}+\kappa^{(2)}}{\varepsilon}\right)^2\right], \quad k_2 \equiv \tfrac{1}{2}\left[1 + \left(\frac{\kappa^{(1)}-\kappa^{(2)}}{\varepsilon}\right)^2\right]. \tag{31}$$

From equations (30) with the substitution

$$f_{2,3} = \frac{1}{r} e^{-\frac{r}{a}} u_{2,3} \tag{32}$$

we come to

$$\left(\frac{d}{dr} - \frac{1}{r} - \frac{1}{a}\right) u_2 - \left(\frac{1}{a_1} + k_1 v\right) u_3 = 0,$$

$$\left(\frac{d}{dr} + \frac{1}{r} - \frac{1}{a}\right) u_3 - \left(\frac{1}{a_2} - k_2 v\right) u_2 = 0. \tag{33}$$

If $v \to 0$ for $r \to \infty$, the asymptotic behaviour is $f_{2,3} \sim e^{-r/a}$, where

$$\frac{1}{a} = \frac{1}{\sqrt{a_1 a_2}} = \tfrac{1}{2}\sqrt{-\varepsilon^2 + 2(\kappa^{(1)2} + \kappa^{(2)2}) - \frac{(\kappa^{(1)2} - \kappa^{(2)2})^2}{\varepsilon^2}}. \tag{34}$$

So, the bound-state condition $a > 0$ implies that $|\kappa^{(1)} - \kappa^{(2)}| < \varepsilon < \kappa^{(1)} + \kappa^{(2)}$.

5. *Coulomb bound states*

In the case of bound states in the Coulomb potential,

$$v = -\frac{\alpha}{r}, \quad \alpha \equiv \frac{e^2}{4\pi\hbar c}, \tag{35}$$

equations (33) can be solved by the standard polynomial method (cf. [7]). Inserting

$$u_{2,3} = \sum_{\nu=0}^{n_r} c_\nu^{(2,3)} r^{\nu+\gamma} \tag{36}$$

into (33) (with (35) applied) we evaluate

$$\gamma = \sqrt{1 - k_1 k_2 \alpha^2} = \sqrt{1 - \frac{\alpha^2}{4}\left[1 + 2\frac{\kappa^{(1)2} + \kappa^{(2)2}}{\varepsilon^2} + \left(\frac{\kappa^{(1)2} - \kappa^{(2)2}}{\varepsilon^2}\right)^2\right]} \tag{37}$$

and

$$\frac{n_r + \gamma}{\alpha} = \frac{a}{2}\left(\frac{k_1}{a_2} - \frac{k_2}{a_1}\right), \quad n_r = 0, 1, 2, \ldots . \tag{38}$$

The algebraic equation (38), where $a_i = a_i(\varepsilon)$, $k_i = k_i(\varepsilon)$, $a = a(\varepsilon)$ and $\gamma = \gamma(\varepsilon)$ are given by (31), (34) and (37), is an involved condition determining the Coulomb energy levels $\varepsilon = \varepsilon_{n_r j=0}$.

Two limiting cases are of interest in order to establish the relation to the Dirac equation (and Sommerfeld formula) from one side and the Schrödinger equation (and Balmer formula) from the other.

In the first case we write

$$\varepsilon = \kappa^{(2)} + \varepsilon_D, \qquad \frac{\kappa^{(1)}}{\kappa^{(2)}} \to 0. \tag{39}$$

Then, from (38) we obtain after some calculations

$$\frac{n_r + \gamma_D}{\alpha} = \frac{1}{\sqrt{\dfrac{\kappa^{(1)2}}{\varepsilon_D^2} - 1}}, \qquad n_r = 0, 1, 2, \ldots, \tag{40}$$

where

$$\gamma_D = \sqrt{1-\alpha^2}. \tag{41}$$

Hence the Sommerfeld formula follows:

$$\varepsilon_D = \frac{\kappa^{(1)}}{\sqrt{1 + \left(\dfrac{\alpha}{n_r + \gamma_D}\right)^2}}. \tag{42}$$

Notice that in Sommerfeld formula we have in general

$$\gamma_D = \sqrt{(j_D + \tfrac{1}{2})^2 - \alpha^2}, \qquad j_D = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \tag{43}$$

and the principal quantum number is

$$n \equiv n_r + j_D + \tfrac{1}{2} = 1, 2, 3, \ldots \tag{44}$$

Thus, in our case $j_D = 1/2$.

In the second we put

$$\varepsilon = \kappa^{(1)} + \kappa^{(2)} + \Delta\varepsilon, \qquad \frac{\Delta\varepsilon}{\kappa^{(1)} + \kappa^{(2)}} \to 0. \tag{45}$$

Then, from (38) we evaluate

$$\frac{n}{\alpha} = \tfrac{1}{2}\sqrt{\frac{2\kappa}{-\Delta\varepsilon}}, \qquad n = n_r + 1 = 1, 2, 3, \ldots \tag{46}$$

Hence the Balmer formula follows:

$$\Delta\varepsilon = -\frac{\alpha^2 \kappa}{2n}. \tag{47}$$

6. *The case of strong potential*

If $\kappa^{(1)}+\kappa^{(2)}$ is very large in comparison with ε, then the binding energy $\Delta\varepsilon$ is very large too (and negative). It means that in this case we have a very strong potential (which, considered globally, is attractive and has the negative sign) and, therefore, ε^2/v^2 can be neglected in (27). Then $1/(\varepsilon-v) \simeq -\left(1+\frac{\varepsilon}{v}\right)\Big/v$ and we get approximately the equations

$$\frac{d}{dr}f_2+\tfrac{1}{2}\left[\varepsilon-v+\frac{(\kappa^{(1)}+\kappa^{(2)})^2}{v}+\frac{(\kappa^{(1)}+\kappa^{(2)})^2\varepsilon}{v^2}\right]f_3=0, \tag{48}$$

$$-\left(\frac{d}{dr}+\frac{2}{r}\right)f_3+\tfrac{1}{2}\left[\varepsilon-v+\frac{(\kappa^{(1)}-\kappa^{(2)})^2}{v}+\frac{(\kappa^{(1)}-\kappa^{(2)})^2\varepsilon}{v^2}\right]f_2=0.$$

The counterpart of the radial Schrödinger equations would have now for $\kappa^{(1)}=\kappa^{(2)}$ the form (neglecting terms with dv/dr):

$$\left[-\frac{1}{2\kappa_{\mathrm{eff}}(r)}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+v_{\mathrm{eff}}(r)-\varepsilon\right]f_2=0,$$

$$\left[-\frac{1}{2\kappa_{\mathrm{eff}}(r)}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2}{r^2}\right)+v_{\mathrm{eff}}(r)-\varepsilon\right]f_3=0, \tag{49}$$

where

$$\kappa_{\mathrm{eff}}(r)\equiv\frac{|v(r)|}{4},\qquad v_{\mathrm{eff}}(r)\equiv\frac{v(r)}{2}-\frac{\kappa^{(1)2}+\kappa^{(2)2}}{v(r)}. \tag{50}$$

Thus, in the case of strong (attractive) Coulombic potential,

$$v(r)=-\frac{\alpha_s}{r}, \tag{51}$$

the "effective potential" is

$$v_{\mathrm{eff}}(r)=-\frac{\alpha_s/2}{r}+\frac{\kappa^{(1)2}+\kappa^{(2)2}}{\alpha_s}\,r \tag{52}$$

containing the celebrated linear term [5]. In equations (49) the reduced mass κ is replaced, however, by the "effective mass"

$$\kappa_{\mathrm{eff}}(r)=\frac{\alpha_s/4}{r}. \tag{53}$$

Obviously, in the case of (51) ε^2/v^2 cannot be neglected locally for too large r, nevertheless it may happen that it can be neglected globally in bound states.

APPENDIX

Separation of angular variables in equation (1)

Introduce in equation (1)

$$p_i = -i\hbar \frac{\partial}{\partial x_i}, \tag{A.1}$$

rewrite the corresponding differential equation in spherical variables

$$x_1 = r \sin\vartheta \cos\varphi, \quad x_2 = r \sin\vartheta \sin\varphi, \quad x_3 = r\cos\vartheta \tag{A.2}$$

and multiply it on the left by the operator

$$U(\vartheta, \varphi) = \exp\left[\tfrac{1}{2}(\alpha_3^{(1)}\alpha_1^{(1)} + \alpha_3^{(2)}\alpha_1^{(2)})\vartheta\right] \exp\left[\tfrac{1}{2}(\alpha_1^{(1)}\alpha_2^{(1)} + \alpha_1^{(2)}\alpha_2^{(2)})\varphi\right]. \tag{A.3}$$

In this way one obtains the equivalent equation

$$\begin{aligned}
&\left\{\varepsilon + i(\alpha_3^{(1)} - \alpha_3^{(2)})\left[\frac{\partial}{\partial r} + \frac{1 + \frac{1}{2}(\alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)})}{r}\right]\right. \\
&+ \frac{i\alpha_2^{(1)}}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \tfrac{1}{2}\alpha_1^{(2)}\alpha_2^{(2)}\operatorname{ctg}\vartheta - \alpha_1^{(1)}\alpha_2^{(1)}\left(\frac{\partial}{\partial\vartheta} + \tfrac{1}{2}\operatorname{ctg}\vartheta\right)\right] \\
&- \frac{i\alpha_2^{(2)}}{r}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \tfrac{1}{2}\alpha_1^{(1)}\alpha_2^{(1)}\operatorname{ctg}\vartheta - \alpha_1^{(2)}\alpha_2^{(2)}\left(\frac{\partial}{\partial\vartheta} + \tfrac{1}{2}\operatorname{ctg}\vartheta\right)\right] \\
&\left. - \beta^{(1)}\kappa^{(1)} - \beta^{(2)}\kappa^{(2)} - \Gamma v(r)\right\}\bar{\psi}(r, \vartheta, \varphi) = 0,
\end{aligned} \tag{A.4}$$

where

$$\varepsilon = \frac{E}{\hbar c}, \quad \kappa^{(i)} = \frac{m^{(i)}c^2}{\hbar c}, \quad v(r) = \frac{V(r)}{\hbar c} \tag{A.5}$$

and

$$\bar{\psi} = U\psi. \tag{A.6}$$

In the derivation we have made use of the relation

$$[\Gamma, U] = 0 \tag{A.7}$$

which is valid for all rotationally invariant systems (cf. [4]).

Introduce now the spinorial spherical harmonics

$$Z_j^0(\cos\vartheta) = \tfrac{1}{2}(1 + \alpha_1^{(1)}\alpha_2^{(1)}\alpha_1^{(2)}\alpha_2^{(2)})P_j^0(\cos\vartheta) + \tfrac{1}{2}(1 - \alpha_1^{(1)}\alpha_2^{(1)}\alpha_1^{(2)}\alpha_2^{(2)})P_j^1(\cos\vartheta) \tag{A.8}$$

and verify easily that it satisfies equation

$$\begin{aligned}
&\alpha_2^{(1)}\left[-\tfrac{1}{2}\alpha_1^{(2)}\alpha_2^{(2)}\operatorname{ctg}\vartheta - \alpha_1^{(1)}\alpha_2^{(1)}\left(\frac{\partial}{\partial\vartheta} + \tfrac{1}{2}\operatorname{ctg}\vartheta\right)\right] Z_j^0(\cos\vartheta) \\
&= \sqrt{j(j+1)}\, Z_j^0(\cos\vartheta)\alpha_2^{(1)}\alpha_1^{(2)}\alpha_2^{(2)}
\end{aligned} \tag{A.9}$$

and the corresponding equation with exchanged upper indices. Due to rotational invariance the result of the separation of angular variables cannot depend on m and, therefore, we can use the function

$$\bar{\psi}(r, \vartheta, \varphi) = Z_j^0(\cos \vartheta)\psi(r) \tag{A.10}$$

as an "Ansatz" for the solution of (A.4). We obtain in this way for $\psi(r)$ the equation (2).

The general theory of separation of angular variables in the two-spinor problem can be found in Ref. [4].

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