MOTION OF A PHOTON IN GRAVITATIONAL FIELD

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Equations of motion for a non-quantum-theoretical "photon" in curved spacetime are obtained by use of the Noether theorem relations and normal coordinates. Simple cases of photon world lines in certain gravitational fields are considered.

1. Introduction

It is shown in papers [1–6] that the velocity and momentum of a classical particle possessing intrinsic angular momentum ("spin") are non-collinear. This peculiarity is of no importance in the Newtonian theory of gravitation (in contrast to General Relativity) since the force acting on a particle there does not depend on its spin. Studies of electron motion [7–10] with the help of the WKB method (a quasiclassical approximation of Dirac's equation) show that an electron bahaves as a classical particle possessing intrinsic angular momentum. It is also worth investigating photon motion from this standpoint. The momentum and Poynting vectors (the latter represents the velocity direction of the electromagnetic wave) were found to be non-collinear for light in paper [4].

2. Noether theorem relations

We begin the derivation of the equations describing photon motion in curved space-time with an analysis of the Noether theorem relations in the form using covariant derivatives as was done by one of the present authors in [11] ("g-covariant relations"). We racall that in that paper Lagrangian $\mathscr{L}$ was defined as a function of fields' potentials $\mathscr{A}_B$, their 1st derivatives $\mathscr{A}_{B;\alpha}$, and the metric tensor $g_{\mu\nu}$, B being a collective index

$$\mathscr{L} = \mathscr{L}(\mathscr{A}_B, \mathscr{A}_{B;\alpha}, g_{\mu\nu}). \tag{1}$$

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As in [11], we define spin density, $\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\sigma}$, canonical energy-momentum tensor, $\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma}$, and metric (symmetric) energy-momentum tensor, $\mathfrak{T}^{\mu\nu}$ (in fact, tensor densities) as

$$\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\sigma} = \frac{\partial \mathscr{L}}{\partial \mathscr{A}_{B;[\alpha}} (\mathscr{A}_B|^{\tau]\cdot}_{\cdot\sigma} - \mathscr{A}_B|^{\cdot\tau]}_{\sigma\cdot}) - \frac{\partial \mathscr{L}}{\partial \mathscr{A}^{;\sigma}_B} \mathscr{A}_B|^{[\alpha\tau]}, \tag{2}$$

$$\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma} = \frac{\partial \mathscr{L}}{\partial \mathscr{A}_{B;\alpha}} \mathscr{A}_{B;\sigma} - \mathscr{L}\delta^{\alpha}_{\sigma}, \tag{3}$$

$$\mathfrak{T}^{\mu\nu} = -2\frac{\delta \mathscr{L}}{\delta g_{\mu\nu}} \equiv -2\left(\frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} - \left(\frac{\partial \mathscr{L}}{\partial g_{\mu\nu,\alpha}}\right)_{,\alpha}\right). \tag{4}$$

Here, $\mathscr{A}_B|^{\nu\cdot}_{\cdot\mu}$ are the transformation coefficients of the potentials $\mathscr{A}_B$ under infinitesimal coordinate transformations $'x^\mu = x^\mu + \xi^\mu(x)$; $'\mathscr{A}_B('x) - \mathscr{A}_B(x) = \mathscr{A}_B|^{\nu\cdot}_{\cdot\mu}\xi^\mu_{,\nu}$. Using these definitions, the Noether theorem relations take the form

$$-\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma;\alpha} + \tfrac{1}{2}\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\varrho} R^{\varrho}_{\cdot\sigma\alpha\tau} = 0, \tag{5}$$

$$\mathfrak{T}^{\mu}_{\nu} = \mathfrak{t}^{\mu\cdot}_{\cdot\nu} - \mathfrak{M}^{\tau\mu\cdot}_{\cdot\cdot\nu;\tau}, \tag{6}$$

$$\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\sigma} = -\mathfrak{M}^{\tau\alpha\cdot}_{\cdot\cdot\sigma}. \tag{7}$$

This corresponds to Eqs (67–69) of Ref. [11] for the case where the bispin vanishes. The Riemann-Christoffel tensor is used in (5) in its standard form, $\mathscr{A}_{B;[\alpha;\beta]} = -\tfrac{1}{2}\mathscr{A}_B|^{\nu\cdot}_{\cdot\mu} R^{\mu}_{\cdot\nu\alpha\beta}$. It is worth mentioning that Eq. (5) follows from (6) by virtue of covariant conservation of the metric energy-momentum tensor. Using the definition of antisymmetrization, $\mathscr{A}_{[\mu\nu]} = \tfrac{1}{2}(\mathscr{A}_{\mu\nu} - \mathscr{A}_{\nu\mu})$, Eqs (5) and (6) yield the relations:

$$-\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma;\alpha} + \mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\varrho} R^{\varrho}_{\cdot\tau\alpha\sigma} = 0, \tag{8}$$

$$-\mathfrak{t}^{[\mu\nu]} + \mathfrak{M}^{\tau[\mu\nu]}_{\quad ;\tau} = 0, \tag{9}$$

which is the starting point of the following analysis.

First, the spin density, canonical and metric energy-momentum tensors of the electromagnetic field described by the Lagrangian

$$\mathscr{L} = -\frac{\sqrt{-g}}{4} F^{\mu\nu}F_{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} = \mathscr{A}_{\nu;\mu} - \mathscr{A}_{\mu;\nu}, \tag{10}$$

are to be represented by:

$$\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\sigma} = \sqrt{-g}\,\mathscr{A}_\sigma F^{\alpha\tau}, \tag{11}$$

$$\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma} = \sqrt{-g}\,F^{\mu\alpha}\mathscr{A}_{\mu;\sigma} - \mathscr{L}\delta^{\alpha}_{\sigma}, \tag{12}$$

$$\mathfrak{T}^{\mu\nu} = \sqrt{-g}\,(\tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu} - F^{\alpha\mu}F^{\cdot\nu}_{\alpha\cdot}). \tag{13}$$

Details of derivation of the Noether theorem relations can be found in references [12, 13], as well as in the previously mentioned paper [11].

3. *Classical characteristics of a photon*

We will now define the dynamical quantities characterizing a photon (from the classical, not quantum point of view). Admitting that the electromagnetic field is concentrated in a small region of spacetime (wave packet), we define linear and angular momenta as

$$P_\mu(y) = \int_\sigma \mathfrak{t}^{\alpha\cdot}_{\cdot\mu}\tau_\alpha d\sigma, \tag{14}$$

$$S^{\mu\nu}(y) = -2\int_\sigma (\mathfrak{t}^{\alpha[\mu}x^{\nu]}+\mathfrak{M}^{\alpha[\mu\nu]})\tau_\alpha d\sigma, \tag{15}$$

where $y^\mu = y^\mu(t)$ describes the world line (parametrized by t) of a photon if it is taken at the origin of the normal coordinates x^μ, and τ_α is a unit timelike vector orthogonal to the 3 dimensional integration element $d\sigma$. For generality, the "orbital" angular momentum is included in $S^{\mu\nu}$ (i. e., the 1st term on the right-hand side of Eq. (15)), but its presence there does not influence the final form of the equations. For these equations (which will completely determine the world line $y^\mu(t)$) it will be assumed that $S^{\mu\nu}$ has spin origin only.

We give here only the main points of the derivation of the photon equations of motion since a very similar approach has been used to get the equations of the motion of non-light-like particles endowed with spin (see Ref. [3]).

From the definitions of the normal coordinates it follows that

$$x'^\mu = x^\mu - tv^\mu - ta^\mu + O(t^2);$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - ta^\mu_{,\nu} + O(t^2), \tag{16}$$

where $v^\mu = dy^\mu/dt$ is the photon velocity, $a^\mu_{,\nu}$ vanishes on the world line, and the primed quantities correspond to $t \neq 0$ whereas non-primed ones — to $t = 0$.

The definition (14) gives

$$P'_\mu(y) = \int_{\sigma'} {}'\mathfrak{t}^{\alpha\cdot}_{\cdot\mu}{}'\tau_\alpha d\sigma' \tag{17}$$

which with the help of Eq. (16) can be written as

$$P'_\mu(y) = \int_{\sigma'} \mathfrak{t}^{\alpha\cdot}_{\cdot\mu}\tau_\alpha d\sigma + t\int_{\sigma'} a^\beta_{,\mu}\mathfrak{t}^{\alpha\cdot}_{\cdot\beta}\tau_\alpha d\sigma + O(t^2). \tag{18}$$

Then by integrating Eq. (8) and using Gauss' theorem, we get

$$\int \mathfrak{t}^{\alpha\cdot}_{\cdot\mu}\tau_\alpha d\sigma - P_\mu(y) - \int (\Gamma^\beta_{\mu\alpha}\mathfrak{t}^{\alpha\cdot}_{\cdot\beta} + \mathfrak{M}^{\alpha\beta\cdot}_{\cdot\cdot\gamma}R^{\gamma\cdot}_{\cdot\beta\alpha\mu})(dx) = 0. \tag{19}$$

Adding (18) and (19) and dividing by t we have

$$\frac{d}{dt}P_\mu - \int a^\beta_{,\mu}\mathfrak{t}^{\alpha\cdot}_{\cdot\beta}\tau_\alpha d\sigma - \int (\Gamma^\beta_{\mu\alpha}\mathfrak{t}^{\alpha\cdot}_{\cdot\beta} + \mathfrak{M}^{\alpha\beta\cdot}_{\cdot\cdot\gamma}R^{\gamma\cdot}_{\cdot\beta\alpha\mu})\frac{(dx)}{dt} = 0. \tag{20}$$

Quite similarly from (9) and (15), we come to the equation

$$\frac{d}{dt}S^{\mu\nu}+2P^{[\nu}v^{\mu]}+2\int(\mathfrak{t}^{\alpha[\nu}a^{\mu]}-a^{\sigma,[\nu}x^{\mu]}\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma}+a^{,\nu}_{\varrho}\mathfrak{M}^{\alpha[\mu\varrho]}-a^{\mu}_{,\tau}\mathfrak{M}^{\alpha[\tau\nu]})\tau_{\alpha}d\sigma$$

$$-2\int(\mathfrak{t}^{\alpha\cdot}_{\cdot\varrho}\Gamma^{\varrho[\nu}_{\alpha\cdot}x^{\mu]}+\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\varrho}R^{\varrho\cdot\cdot[\nu}_{\cdot\tau\alpha\cdot}x^{\mu]}+\Gamma^{\mu}_{\varrho\tau}\mathfrak{M}^{\tau[\varrho\nu]}-\Gamma^{\varrho\nu}_{\tau\cdot}\mathfrak{M}^{\tau[\mu\cdot}_{\cdot\cdot\varrho]})\frac{(dx)}{dt}=0, \tag{21}$$

when Eqs (16) are taken into account. We recall that normal coordinates have the following properties:

$$a^{\mu}=\frac{2}{3!}R^{\mu}_{\cdot\alpha\beta\gamma}x^{\alpha}x^{\beta}v^{\gamma}+\ldots, \tag{22}$$

$$\Gamma^{\alpha}_{(\beta\gamma,\ldots,\delta)}=0, \tag{23}$$

$$\Gamma^{\alpha}_{\beta\gamma,\delta}\xi^{\beta\gamma}=-\tfrac{2}{3}R^{\alpha}_{\cdot\beta\gamma\delta}\xi^{\beta\gamma}. \tag{24}$$

Here Eqs (23) and (24) are written at the origin of normal coordinates, and $\xi^{\beta\gamma}=\xi^{(\beta\gamma)}\equiv\frac{1}{2}(\xi^{\beta\gamma}+\xi^{\gamma\beta})$ is an arbitrary quantity symmetrical on both indices.

For future use it is essential to assume that all moments of $\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\delta}$, except the zeroth one, and of $\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma}$, except the zeroth and the first ones, vanish. Then one of the desired equations follows at once from Eq. (21):

$$\frac{D}{dt}S^{\mu\nu}=P^{\mu}v^{\nu}-P^{\nu}v^{\mu}. \tag{25}$$

In order to find the next equation one has to make use of relations

$$(\mathfrak{M}^{\tau[\alpha\sigma]}x^{\varrho})_{,\tau}-\mathfrak{M}^{\varrho[\alpha\sigma]}+x^{\varrho}\Gamma^{\alpha}_{\lambda\tau}\mathfrak{M}^{\tau[\lambda\sigma]}-x^{\varrho}\Gamma^{\sigma}_{\tau\lambda}\mathfrak{M}^{\tau[\alpha\lambda]}-x^{\varrho}\mathfrak{t}^{[\alpha\sigma]}=0, \tag{26}$$

$$(\mathfrak{t}^{\alpha\cdot}_{\cdot\sigma}x^{\beta}x^{\gamma})_{,\alpha}-\mathfrak{t}^{\beta\cdot}_{\cdot\sigma}x^{\gamma}-\mathfrak{t}^{\gamma\cdot}_{\cdot\sigma}x^{\beta}-\mathfrak{t}^{\nu\cdot}_{\cdot\mu}\Gamma^{\mu}_{\nu\sigma}x^{\beta}x^{\gamma}+x^{\beta}x^{\gamma}\mathfrak{M}^{\alpha\tau\cdot}_{\cdot\cdot\varrho}R^{\varrho}_{\cdot\tau\alpha\sigma}. \tag{27}$$

These equations are Eqs (8) and (9) written in normal coordinates. Integration of (26), (27) leads to a sequence of relations similar to (17)-(20) in which it is possible to exclude terms of the form $\int\ldots(dx)/dt$. Finally, we get the desired equation in the form

$$\frac{D}{dt}P_{\mu}=\tfrac{1}{2}S^{\alpha\beta}v^{\gamma}R_{\alpha\beta\gamma\mu} \tag{28}$$

with

$$\frac{D}{dt}S^{\mu\nu}=2P^{[\mu}v^{\nu]}. \tag{29}$$

The system (28), (29) is evidently incomplete, so we add to it an auxiliary condition

$$S^{\mu\nu}P_{\nu}=0, \tag{30}$$

thus completing our description of the world line of the classical "photon" motion. A discussion of a similar problem for non-light-like particles is given in references [3, 14], and massless particles are discussed in paper [15].

4. Equations of massless particle motion relative to a given frame of reference

Eqs (28), (29) are invariant under changes of the parameter t; this property can be traced in the process of derivation of the equations. Changes in quantities v^μ, P_μ, $S^{\mu\nu}$ resulting from the translation $t(t \to t')$ are given as:

$$v'^\mu = \frac{dx^\mu}{dt'} = \frac{dt}{dt'} v^\mu,$$

$$P'^\mu(x) = P^\mu(x); \qquad S'^{\mu\nu}(x) = S^{\mu\nu}(x) \tag{31}$$

(see (14), (15)). With the help of these relations one can gauge the parameter t by the condition $\tau_\mu v^\mu = 1$ (or $v_\mu v^\mu = 1$ if the world line is not lightlike). We give priority to the first gauge of this parameter. In this case, with the help of Eq. (29), the momentum can be expressed as

$$P^\mu = \mathscr{E} v^\mu + \dot{S}^{\mu\nu}\tau_\nu, \tag{32}$$

where the notation $\mathscr{E} = P_\mu \tau^\nu$ (photon energy) is used, and dot means the covariant derivative with respect to the parameter. Then the system (28), (29) takes the form

$$\frac{D}{dt}(\mathscr{E} v^\mu + \dot{S}^{\mu\nu}\tau_\nu) = \tfrac{1}{2} S^{\alpha\beta} v^\gamma R_{\alpha\beta\gamma\cdot}{}^{\mu}, \tag{33}$$

$$\frac{D}{dt} S^{\mu\nu}(\delta^\alpha_\mu - \tau_\mu v^\alpha)(\delta^\beta_\nu - \tau_\nu v^\beta) = 0. \tag{34}$$

Returning to the spin tensor, we represent it without loss of generality by using the monad vector τ^μ:

$$S_{\mu\nu} = 2S_{[\mu}\tau_{\nu]} - E_{\alpha\beta\mu\nu}\tau^\alpha s^\beta. \tag{35}$$

Here $E_{\alpha\beta\mu\nu}$ is the axial tensor of Levi-Cività, $s^\alpha = S^{\alpha\beta}_{*}\tau_\beta \equiv \frac{1}{2} E^{\alpha\beta\mu\nu}\tau_\beta S_{\mu\nu}$ is 3-spin, $S_\mu = S_{\mu\nu}\tau^\nu$ is a vector characterizing the position of the energy center in the monad frame τ^μ. The auxiliary condition (30) leads to relations

$$S^\mu P_\mu = 0, \tag{36}$$

$$S^{\mu\nu} S^{*}_{\mu\nu} = -4S_\mu s^\mu = 0, \tag{37}$$

$$S_\mu = E_{\alpha\beta\mu\nu}\tau^\alpha s^\beta P^\nu / \mathscr{E}, \tag{38}$$

thus the invariant $S^{\mu\nu}S_{\mu\nu}$ is conserved such that

$$S^{\mu\nu}S_{\mu\nu} = 2(S^\mu S_\mu - s^\mu s_\mu) = \text{const.} \tag{39}$$

The change of the momentum modulus along the trajectory is written as

$$\frac{d}{dt} M^2 = \frac{d}{dt}(P_\mu P^\mu) = S^{\alpha\beta} v^\gamma P^\delta R_{\alpha\beta\gamma\delta} = -\tfrac{1}{4} \frac{D}{dt}(S^{\alpha\beta} S^{\gamma\delta}) R_{\alpha\beta\gamma\delta}, \tag{40}$$

or

$$\frac{d}{dt}(M^2 + \tfrac{1}{4} S^{\alpha\beta} S^{\gamma\delta} R_{\alpha\beta\gamma\delta}) = \tfrac{1}{4} S^{\alpha\beta} S^{\gamma\delta} R_{\alpha\beta\gamma\delta;\lambda} v^\lambda. \tag{41}$$

From expression (38) it can be seen that the angle between 3-spin and 3-momentum equals arc sin ($|\vec{S}|\mathscr{E}/|\vec{P}|\cdot|\vec{s}|$). This angle vanishes if spacetime is flat, and the gauge is Coulombian (see, e.g., [13]); then $S^\mu = 0$. In order to retain this property under another gauge of the monad (frame of reference) it is necessary to pass to a new Coulombian gauge, but if spacetime is curved then this can be done in general at the initial moment of time only, and later τ_μ, 3-spin, and 3-momentum vectors change their mutual orientation along the trajectory (the monad field being given).

On can make use of arbitrariness in the monad gauge and put on the trajectory the condition

$$S^\mu = 0. \tag{42}$$

This frame of reference is analogous to a co-moving system in dynamic of particles with intrinsic angular momentum, when the corresponding condition can be written as $P^{[\mu}\tau^{\nu]} = 0$. It is worth noticing that in flat spacetime, condition (42) determines the Coulombian choice of the field gauge. In the case of (42) with $v^\mu \tau_\mu = 1$ and from Eqs (32), (35) it follows that

$$S^{\mu\nu} = \gamma E^{\alpha\beta\mu\nu} \tau_\alpha P_\beta, \tag{43}$$

$$P^\mu = \mathscr{E} v^\mu - \gamma E^{\alpha\beta\nu\mu} \tau_\alpha \dot{\tau}_\beta P_\nu \tag{44}$$

or

$$(1-\gamma^2 \dot{\tau}_\alpha \dot{\tau}^\alpha) P^\mu = \mathscr{E} v^\mu - \gamma \mathscr{E} E^{\alpha\beta\nu\mu} \tau_\alpha \dot{\tau}_\beta v_\nu - \gamma^2 \mathscr{E}(\dot{\tau}_\alpha \dot{\tau}^\alpha \tau^\mu - \dot{\tau}^\alpha v_\alpha \dot{\tau}^\mu), \tag{45}$$

$$(1-\gamma^2 \dot{\tau}_\alpha \dot{\tau}_\alpha) S^{\mu\nu} = \gamma \mathscr{E} E^{\alpha\beta\mu\nu} \tau_\alpha (v_\beta - \gamma^2 \dot{\tau}_\varrho v^\varrho \dot{\tau}_\beta) + 2\mathscr{E}\gamma^2 (\dot{\tau}^{[\mu} v^{\nu]} + \tau^{[\mu} \dot{\tau}^{\nu]}). \tag{46}$$

So the momentum and the spin in Eqs (28), (29) can be expressed through τ^μ, v^μ, γ. In this case the motion integral (39) takes the form

$$s_\mu s^\mu = \gamma^2 (M^2 - \mathscr{E}^2) = \text{const}. \tag{47}$$

For the case $S^{\alpha\beta} v^\gamma R_{\alpha\beta\gamma\mu} = 0$ the obvious solution is $\dot{P}_\mu = 0$, $\dot{S}_{\mu\nu} = 0$, $P_\mu = \mathscr{E} v_\mu$. Such a solution is adequate in spacetime systems of constant curvature, $R_{\mu\nu\lambda\varrho} = K(g_{\mu\lambda} g_{\nu\varrho} - g_{\mu\varrho} g_{\nu\lambda})$, and in particular in flat spacetime. It is also valid for radial motion in the Schwarzschild field, motion along the symmetry axis in the Kerr field, radial motion in the Friedmann spacetime. In these cases light propagates along lightlike *geodesics*. All these assertions are easily verified by direct calculation. Moreover, it is not difficult to show from (28), (29), (32) that if the photon energy is large and varies slowly along the trajectory[1], Eqs (28), (29) yield the geodesic motion equations (the geometrical optics limit):

$$\frac{D}{dt} P^{\mu} = 0, \qquad P^{[\mu}v^{\nu]} = 0. \tag{48}$$

These correspond to the bicharacteristics of the Maxwell equations, in that they contain no information about the spin. Notably, Eqs (48) do not depend on the frame of reference, though the small parameter $\gamma\dot{\tau}^{\alpha}$ depends on the monad τ_{α}.

5. *Motion of a photon in the Schwarzschild field*

Consider a more complicated example of photon motion in General Relativity, namely the scattering of light by the Schwarzschild field. For this we use homogeneous coordinates,

$$ds^2 = \left(\frac{1-\frac{C}{r}}{1+\frac{C}{r}}\right)^2 dt^2 - \left(1+\frac{C}{r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \tag{49}$$

as well as the well known solution of the geodesic equation in the "equatorial" plane. The first integrals of this solution can be written as

$$\left(\frac{d}{d\varphi}\frac{1}{r}\right)^2 + \frac{1}{r^2} = \frac{1}{\beta^2}\frac{(1+C/r)^6}{(1-C/r)^2}, \qquad \mathscr{E}\left(\frac{1-C/r}{1+C/r}\right) = \alpha = \text{const},$$

$$\mathscr{E} r^2 (1+C/r)^4 \frac{d\varphi}{dt} = \alpha\beta = \text{const}. \tag{50}$$

They correspond in the 1st approximation to the hyperbolic trajectory

$$\frac{1}{r} \cong \frac{4C}{\beta^2}(1+e\cos\varphi), \tag{51}$$

the eccentricity being $e = (1+\beta^2/16C^2)^{1/2}$. This solution can be seen, e.g., in paper [16].

We now take the equations for l^{μ}, the deviation of a particle with spin from a given world line (see Ref. [17]):

$$\frac{D}{dt} P_{1\mu} = -R^{\alpha}_{\cdot\,\mu\beta\gamma} P_{0\alpha} v^{\beta} l^{\gamma} + \tfrac{1}{2} S_1^{\alpha\beta} v^{\gamma} R_{\alpha\beta\gamma\mu}$$

$$+\tfrac{1}{2} S_0^{\alpha\beta} l^{\gamma} R_{\alpha\beta\gamma\mu} + \tfrac{1}{2} S_0^{\alpha\beta} v^{\gamma} l^{\delta} R_{\alpha\beta\gamma\mu;\delta},$$

$$\frac{D}{dt} S_1^{\mu\nu} + S_0^{\alpha\nu} R^{\mu}_{\cdot\,\alpha\beta\gamma} l^{\beta} v^{\gamma} + S_0^{\mu\alpha} v^{\gamma} l^{\beta} R^{\nu}_{\cdot\,\alpha\beta\gamma} - 2P_1{}^{[\mu}v^{\nu]} - 2P_0{}^{[\mu}\dot{l}^{\nu]} = 0,$$

$$\frac{D}{dt} S_0^{\mu\nu} = 2P_0{}^{[\mu}v^{\nu]}, \tag{52}$$

[1] For the choice of monad (42) this corresponds to the value of the dimensionless parameter $|\gamma\dot{\tau}_{\alpha}| \ll 1$.

where

$$P_1^\mu = P_0^\mu + \Delta P^\mu = \mathscr{E} v^\mu + \Delta P^\mu, \qquad S_1^{\mu\nu} = S_0^{\mu\nu} + \Delta S^{\mu\nu}.$$

If l_μ, $\Delta S_{\mu\nu}$, ΔP_μ are small (it is natural to suppose here that the deviation from a geodesic is small), and restricting ourselves in Eqs (52) to linear terms in l_μ, $\Delta S_{\mu\nu}$, and ΔP_μ, we get

$$\frac{D}{dt} P_{1\mu} = \tfrac{1}{2} \Sigma^{\alpha\beta} v^\gamma R_{\alpha\beta\gamma\mu}, \qquad \frac{D}{dt} \Sigma^{\mu\nu} = 2P_1{}^{[\mu} v^{\nu]}. \tag{53}$$

Here $\Sigma^{\mu\nu} = 2\mathscr{E} l^{[\mu} v^{\nu]} + S_0^{\mu\nu}$ is the sum of the orbital (relative to the world line under consideration) and spin angular momenta.

If the parameter t is determined by the condition $\tau_\mu v^\mu = 1$, the momentum equals

$$P_{1\mu} = \mathscr{E} v_\mu + \dot{\Sigma}_{\mu\nu} \tau^\nu. \tag{54}$$

Using now the auxiliary condition (30), one can write the resulting angular momentum as

$$\Sigma^{\mu\nu} = \mathscr{E}(\gamma \tau_\alpha v_\beta E^{\alpha\beta\mu\nu} + 2 l^{[\mu} v^{\nu]}). \tag{55}$$

From (30), (53), (55) follows the motion integral

$$\Sigma_{\mu\nu} \Sigma^{\mu\nu} = \mathscr{E}^2 \gamma^2 = \text{const} \tag{56}$$

or

$$\gamma = \alpha \gamma_0 / \mathscr{E}, \qquad \gamma_0 = \text{const}. \tag{57}$$

Multiplying the first Eq. (53) by $\mathscr{E} v^\mu$ and making use of Eqs (48) (equivalent to (50)), we have

$$P_{1\mu} \mathscr{E} v^\mu = \mathscr{E}^2 \dot{l}^\nu v_\nu = \text{const}. \tag{58}$$

We demand, then,

$$l^\mu v_\mu = 0, \qquad l^\mu \tau_\mu = 0 \tag{59}$$

(this agrees with the initial data), and define the monad field as

$$\tau^\mu = \delta_0^\mu \frac{1 + C/r}{1 - C/r}. \tag{60}$$

Returning to the 2nd equation of system (53) and rewriting it as

$$\dot{\Sigma}^{\mu\nu} (\delta_\mu^\alpha - \tau_\mu v^\alpha)(\delta_\nu^\beta - \tau_\nu v^\beta) = 0, \tag{61}$$

insertion of $\Sigma^{\mu\nu}$ from Eq. (55) into (61) gives

$$\gamma \dot{\tau}_\alpha v_\beta (E^{\alpha\beta\mu\nu} + 2\tau_\lambda E^{\alpha\beta\lambda[\mu} v^{\nu]}) = 0. \tag{62}$$

This equation is satisfied identically in the 0-th approximation in C/r when Eqs (59) are taken into account. Only the first equation of system (53) remains to be considered. Since in this equation the spin force

$$F^{\mu}_{S} = -\tfrac{1}{2} S^{\alpha\beta}v^{\nu}R^{\mu}_{\cdot\,\nu\alpha\beta} = \frac{\alpha\gamma_0}{r^2}(v_1 v^3 R^{2}_{\cdot\,323} + v_3 v^1 R^{2}_{\cdot\,112})\delta^{\mu}_{2} \tag{63}$$

acts on the photon only orthogonally to the equatorial plane, we take the form

$$l^{\mu} = l\delta^{\mu}_{2} \tag{64}$$

for the solution.

Somewhat tedious calculations (though only in the 1st order in C/r) lead to the single equation

$$\frac{d^2 l}{d\varphi^2} + l + \frac{6C\gamma_0}{\beta r} v^1 = 0. \tag{65}$$

Since in the 0th approximation Eq. (51) gives $\frac{1}{r} = \frac{1}{\beta}\cos\varphi$, $v^1 = \sin\varphi$, Eq. (65) is easily integrated to give

$$l = \frac{C\gamma_0}{\beta^2}\sin 2\varphi. \tag{66}$$

We see therefore that the deviation of a photon from the equatorial plane is negligibly small and symmetric for opposite orientations of spin, i.e., with the approximation used, the plane of polarization does not rotate. Far more interesting theoretically is the 2nd approximation in spin since

$$\mathscr{E}^2 v^2 = (1-\gamma^2\dot{\tau}_{\alpha}\dot{\tau}^{\alpha})M^2 + \gamma^2\mathscr{E}^2[\dot{\tau}_{\alpha}\dot{\tau}^{\alpha} + (\dot{\tau}_{\alpha}v^{\alpha})^2], \qquad 1-\gamma^2\dot{\tau}_{\alpha}\dot{\tau}^{\alpha} > 0. \tag{67}$$

This implies the possibility that for the next approximation $v^2 \not\equiv 0$, thus leading to the rotation of the polarization plane.

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