NONLINEARITY AND TORSION*

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The present paper is concerned with the dynamics of the spinor wave fields in space-time with torsion. It is shown that torsion can both induce the minimal nonlinearity of the spinor equations and compensate it.

In the general relativistic theory of the gravitational interactions which includes the spinning properties of matter in its dynamic scheme, the gravitational field is described by means of two independent tensor fields: a metric field g_{ij} and a torsion field $Q^i{}_{jk}$ [3, 7, 9].

The torsion tensor $Q^i{}_{jk}$ is a third rank tensor and is antisymmetric in the lower indices. One can expand it into three irreducible parts [6]: the tracefree part, trace and pseudotrace. It is easily noticed that the spinor wave field, which is described by either (a) the linear covariant Dirac equation or (b) the nonlinear covariant Ivanenko–Heisenberg equation [4, 5], is the natural physical source of the pseudotrace of the torsion tensor.

(a) The general covariant Lagrangian, corresponding to the Dirac particle with mass m, has the form

$$\mathscr{L}_{\mathrm{D}} = -\frac{hc}{2}(\nabla_k\psi^+\gamma^k\psi-\psi^+\gamma^k\nabla_k\psi-2\mu\psi^+\psi). \tag{1a}$$

Here ψ is the Dirac 4-spinor; "+" denotes the Dirac conjugation; γ^k are the Dirac matrices; $\mu = mc/h$; ∇_k is the covariant derivative of the spinor function, $\nabla_i\psi = \partial_i\psi-\Gamma_i\psi$, Γ_i is any solution of the equation

$$\nabla_k\gamma_i \equiv \partial_k\gamma_i-\Gamma^m{}_{ik}\gamma_m+\gamma_i\Gamma_k-\Gamma_k\gamma_i = 0, \tag{2}$$

where $\Gamma^m{}_{ik}$ is the Cartan connection [6].

By varying the action functional with Lagrangian (1a) with respect to ψ^+ we obtain the Dirac equation in space-time with torsion

$$\gamma^k\nabla_k\psi+\mu\psi = 0. \tag{3}$$

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The spin of the Dirac particles (the spin density) is described by the pseudovector $\check{S}^i = \psi^+\gamma^i\gamma_5\psi$.

(b) We write the general covariant Lagrangian of the Ivanenko–Heisenberg spinor field as follows

$$\mathscr{L}_{\rm IH} = -\frac{\hbar c}{2}(\psi^+\gamma^k\nabla_k\psi - \nabla_k\psi^+\gamma^k\psi \pm l_0^2\check{S}^i\check{S}_i), \qquad l_0 = \text{const.} \tag{1b}$$

By varying the action functional with Lagrangian (1b) with respect to ψ^+ we obtain the Ivanenko–Heisenberg nonlinear spinor equation in the space-time with torsion

$$\gamma^k\nabla_k\psi \pm l_0^2\check{S}^i\gamma_i\psi = 0. \tag{4}$$

In our case the gravitational field equations become

$$G_{ij} = \kappa t_{ij}, \tag{5a}$$

$$Q^i{}_{jk} = \kappa\check{S}^i{}_{jk}, \tag{5b}$$

where $\kappa = 8\pi k/c^4$ is the Einstein gravitational constant; t_{ij} is the canonical energy-momentum tensor of either the Dirac matter, $t_{ij} = t_{ij}^{\rm D}$, or the nonlinear Ivanenko–Heisenberg equation, $t_{ij} = t_{ij}^{\rm IH}$; $S^i{}_{jk}$ [1] is the spin tensor of matter; G_{ij} is the Einstein tensor of the connection [1]

$$\Gamma^i{}_{jk} = \left\{ {i \atop j\ k} \right\} - \frac{\kappa}{2} S^i{}_{jk}. \tag{6}$$

One can expand the coefficients Γ_i into the Riemannian and nonriemannian parts by using definition (2) and relation (6) as well,

$$\Gamma_k = \{k\} - \frac{\kappa}{4} S^l{}_{ik}\gamma^i\gamma_l. \tag{7}$$

We can write equation (3) with the help of (7), as

$$\gamma^k\tilde{\nabla}_k\psi - l^2\check{S}^i\gamma_i\gamma_5\psi + \mu\psi = 0. \tag{8}$$

The symbol "$\sim$" denotes objects related to the Riemannian connection; $l^2 = \frac{3}{8}\hbar c\kappa$.

This equation for the particular case of space-time of absolute parallelism with torsion was first arrived at by Rodichev [8] and later on (for arbitrary spaces in nonholonomic frames) by Hehl and Datta [2].

Equation (4) can also be written as

$$\gamma^k\tilde{\nabla}_k\psi - l^2\check{S}^i\gamma_i\gamma_5\psi \pm l_0^2\check{S}^i\gamma_i\gamma_5\psi = 0. \tag{9}$$

Let us assume the plus-sign before the third term in (9) and consider $l_0^2 = l^2$. Then

$$\gamma^k\tilde{\nabla}_k\psi = 0.$$

[1] The spinor tensor is equal to the one introduced in [1–3] multiplied by the velocity of light.

This is the equation of the Dirac neutrino in Riemannian space-time.

On the other hand the Dirac neutrino in space-time with torsion is described by equation (8) (with mass equal to zero) which is consistent with (4) (with the lower sign) but in Riemannian space-time.

Due to (6) the torsion can be eliminated from the Einstein tensor of the connection Γ [1]

$$G_{ij} = \tilde{G}_{ij} - \frac{\hbar^2 c^2 \kappa^2}{16} \check{S}^k \check{S}_k g_{ij} - \frac{\hbar^2 c^2 \kappa^2}{8} \check{S}_i \check{S}_j. \tag{10}$$

One can expand the canonical energy-momentum tensor into the Riemannian and nonriemannian parts

$$t^{D}_{ij} = \tilde{t}^{D}_{ij} + \frac{\hbar^2 c^2 \kappa}{8} \check{S}^j \check{S}_j g_{ik} - \frac{\hbar^2 c^2 \kappa}{8} \check{S}_i \check{S}_k, \tag{11a}$$

$$t^{IH}_{ik} = \tilde{t}^{IH}_{ik} + \frac{\hbar^2 c^2 \kappa}{8} \check{S}^j \check{S}_j g_{ik} - \frac{\hbar^2 c^2 \kappa}{8} \check{S}_i \check{S}_k, \tag{11b}$$

where

$$\tilde{t}^{D}_{ik} = \frac{\hbar c}{2} (\tilde{\nabla}_{(i} \psi^+ \gamma_{k)} \psi - \psi^+ \gamma_{(i} \tilde{\nabla}_{k)} \psi),$$

$$\tilde{t}^{IH}_{ik} = \tilde{t}^{D}_{ik} \pm l_0^2 \frac{\hbar c}{2} \check{S}^j \check{S}_j g_{ik}.$$

With the help of (10), (11a) and (11b) one can write the field equation (5a) as

$$\tilde{G}_{ik} = \kappa(\tilde{t}^{D}_{ik} - \tfrac{1}{2} l^2 \hbar c \check{S}^j \check{S}_j g_{ik}), \quad \text{or} \quad \tilde{G}_{ik} = \kappa \tilde{t}^{IH}_{ik} \tag{12a}$$

(in the case of the Dirac matter),

$$\tilde{G}_{ik} = \kappa(\tilde{t}^{IH}_{ik} + \tfrac{1}{2} l^2 \hbar c \check{S}^j \check{S}_j g_{ik}), \quad \text{or} \quad \tilde{G}_{ik} = \kappa \tilde{t}^{D}_{ik} \tag{12b}$$

(in the case of the nonlinear spinor matter).

The results obtained can be expressed as the following theorems:

Theorem I. The dynamics of the nonlinear spinor field (the Ivanenko–Heisenberg type of nonlinearity) in its own gravitational field in space-time with torsion is equivalent to the neutrino dynamics in its own gravitational field in Riemannian space-time.

Theorem II. The dynamics of the linear spinor field (the Dirac neutrino) in its own gravitational field in space-time with torsion is equivalent to the dynamics of the nonlinear Ivanenko–Heisenberg spinor field in its own gravitational field in Riemannian space-time[2].

It is also worth noting that Eqs (12a) and (12b) point to the possibility of describing the gravitational interactions of physical systems in the Einstein–Cartan theory in terms of the Riemannian geometry but with the new energy-momentum tensor.

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[2] Similar results have been obtained for the electromagnetic field.

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