ON TWO KINDS OF SPINORS

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Time and space reflexions of spinors are investigated. It is assumed that the connection between spinors and null bivectors, discovered by Cartan and Whittaker, is valid in all reference frames. It follows that both time and space reflexions of spinors are antilinear operations and that there are two kinds of spinors, just as there are two kinds of Euclidean vectors. Penrose's picture of a spinor on the sphere of complex numbers can be drawn only for one of the two spinors.

1. *Introduction*

This investigation originated from our desire to understand the notion of spinor flag introduced by Penrose [1]. It turns out that there are two kinds of spinors, whose flags behave in a geometrically different way under reflexions. This result seems to be new, at least Penrose does not mention the second spinor which — unlike the first one — cannot be drawn on the sphere of complex numbers.

Our method consists in the following. We start from the connection — discovered by Cartan [2] and Whittaker [3] — between spinors and null bivectors and assume that this connection is valid in all reference frames, including those which differ from the original one by space or time reflexion. As a consequence we obtain that both space and time reflexions of spinors are antilinear operations i.e. they involve complex conjugation.

Several authors noted [4], [5] that it is possible to represent reflexions of spinors by antilinear operations. However, as far as we know, it has not been noted that within this approach there are two geometrically different spinors.

2. *Three kinds of densities*

The Lorentz transformations are linear substitutions

$$x^{\mu'} = A^{\mu'}_{\nu} x^{\nu}$$

such that the quadratic form

$$xx = (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2$$

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is left invariant:

$$(x^{0'})^2-(x^{1'})^2-(x^{2'})^2-(x^{3'})^2 = (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2.$$

The Lorentz transformations are called:

$$\begin{aligned}
&\text{proper transformations if } A_0^{0'} \geqslant +1,\ \det(A_\nu^{\mu'}) = +1,\\
&\text{time reflexions} \quad \text{if } A_0^{0'} \leqslant -1,\ \det(A_\nu^{\mu'}) = -1,\\
&\text{space reflexions} \quad \text{if } A_0^{0'} \geqslant +1,\ \det(A_\nu^{\mu'}) = -1,\\
&\text{total reflexions} \quad \text{if } A_0^{0'} \leqslant -1,\ \det(A_\nu^{\mu'}) = +1.
\end{aligned}$$

Vectors are geometric quantities whose components transform like cartesian coordinates. There are three geometrically different transformation laws which coincide with that of a vector for the proper Lorentz transformations:

$$x^{\mu'} = \operatorname{sign}(A_0^{0'})A_\nu^{\mu'}x^\nu,$$

$$x^{\mu'} = \operatorname{sign}(A_0^{0'})\det(A_\beta^{\alpha'})A_\nu^{\mu'}x^\nu,$$

$$x^{\mu'} = \det(A_\beta^{\alpha'})A_\nu^{\mu'}x^\nu.$$

Geometric quantities with these transformation laws will be called respectively densities of the first, second and third kind.

There is a conceptual difference between purely analytical operation of time reflexion and the operation which some German authors call *Bewegungsumkehr*. We postulate, however, that analytically defined time reflexion should be numerically identical with the result of *Bewegungsumkehr*. For example, it is clear that under *Bewegungsumkthr* energy of a particle remains positive, which means that the so called energy and momentum vector is not actually a vector but a density of the first kind:

$$p_{\mu'} = \operatorname{sign}(A_{0'}^{0})A_{\mu'}^{\nu}p_\nu.$$

TABLE I

Four kinds of vectors

Quantity	Transformation law	Physical examples
vector	$x^{\mu'} = A_\nu^{\mu'}x^\nu$	translation
vectorial density of the first kind	$p^{\mu'} = \operatorname{sign}(A_0^{0'})\, A_\nu^{\mu'}p^\nu$	total energy and momentum of an isolated system, electric current, electromagnetic potential
vectorial density of the second kind	$j^{\mu'} = \operatorname{sign}(A_0^{0'})\det(A_\beta^{\alpha'})\, A_\nu^{\mu'}j^\nu$	magnetic current (if it exists)
vectorial density of the third kind	$w^{\mu'} = \det(A_\beta^{\alpha'})\, A_\nu^{\mu'}w^\nu$	the Pauli-Lubański "vector" of an isolated system

A universe which contains a single particle does not have the time arrow since the time arrow can be defined only by a time-like *vector*.

It may be seen from Table I that the three kinds of densities occur naturally in physics.

3. *Two kinds of bivectors*

What has been said about vectors holds for other quantities as well. There are a priori four distinct transformation laws for bivectors:

$$F_{\mu'\nu'} = \left\{ \begin{array}{l} 1 \\ \text{sign}\,(A^0_{0'}) \\ \text{sign}\,(A^0_{0'})\,\det\,(A^\beta_{\alpha'}) \\ \det\,(A^\beta_{\alpha'}) \end{array} \right\} A^\varrho_{\mu'} A^\lambda_{\nu'} F_{\varrho\lambda}, \quad F_{\varrho\lambda} = -F_{\lambda\varrho}.$$

However, there exists an invariant duality relation

$$F_{\mu\nu} = \tfrac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta} G^{\alpha\beta},$$

$$G_{\mu\nu} = -\tfrac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta},$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric density of the third kind such that $\varepsilon_{0123} = -1$. Because of the duality relation antisymmetric tensor is equivalent to density of the third kind and density of the first kind is equivalent to density of the second kind. Thus *there are only two distinct bivectors.*

Two important physical quantities — the electromagnetic field $F_{\mu\nu}$ and the angular momentum and centre of mass integral $M_{\mu\nu}$ — are densities of the first kind:

$$F_{\mu'\nu'} = \text{sign}\,(A^0_{0'}) A^\alpha_{\mu'} A^\beta_{\nu'} F_{\alpha\beta},$$

$$M_{\mu'\nu'} = \text{sign}\,(A^0_{0'}) A^\alpha_{\mu'} A^\beta_{\nu'} M_{\alpha\beta}.$$

For this reason it will be convenient to consider density of the first kind and density of the third kind as fundamental objects; antisymmetric tensor and density of the second kind arise from the fundamental objects by means of the duality relation.

TABLE II

Two kinds of bivectors

Quantity	Time reflexion $x^{0'} = -x^0$, $x^{i'} = +x^i$, $i, k = 1, 2, 3$	Space reflexion $x^{0'} = +x^0$, $x^{i'} = -x^i$, $i, k = 1, 2, 3$	Total reflexion $x^{\mu'} = -x^\mu$, $\mu = 0, 1, 2, 3$
Bivectorial density of the first kind	$F_{0'i'} = F_{0i}$ $F_{i'k'} = -F_{ik}$	$F_{0'i'} = -F_{0i}$ $F_{i'k'} = F_{ik}$	$F_{\mu'\nu'} = -F_{\mu\nu}$
Bivectorial density of the third kind	$S_{0'i'} = S_{0i}$ $S_{i'k'} = -S_{ik}$	$S_{0'i'} = S_{0i}$ $S_{i'k'} = -S_{0i}$	$S_{\mu'\nu'} = S_{\mu\nu}$

Table II gives the results of time, space and total reflexions for the two distinct bivectors. It is seen that under total reflexion components of a density of the first kind do change sign while components of a density of the third kind do not. This resembles to some extent behaviour of Euclidean polar and axial vectors under space inversion.

4. *The Cartan–Whittaker bivector of a spinor*

Bivector $F_{\mu\nu}$ is called special or null if

$$F_{\mu\nu}F^{\mu\nu} = 2(F_{12}^2+F_{23}^2+F_{31}^2-F_{01}^2-F_{02}^2-F_{03}^2) = 0,$$

$$\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 8(F_{01}F_{23}+F_{02}F_{31}+F_{03}F_{12}) = 0.$$

The two equations can be written in the form

$$(iF^{01}-F^{23})^2+(iF^{02}-F^{31})^2+(iF^{03}-F^{12})^2 = 0.$$

Hence, there are two complex numbers u^0 and u^1 such that

$$iF^{01}-F^{23} = (u^0)^2-(u^1)^2,$$

$$iF^{02}-F^{31} = i[(u^0)^2+(u^1)^2],$$

$$iF^{03}-F^{12} = -2u^0u^1.$$

Cartan [2] and Whittaker [3] showed that u^A, $A = 0,1$, are components of a spinor.

It is clear that u^A determine $F_{\mu\nu}$ uniquely while $F_{\mu\nu}$ determine u^A up to an over-all sign (since the product u^0u^1 is determined). Thus the Cartan–Whittaker relation establishes a one to one correspondence between spinors and null bivectors.

We assume — in agreement with the original treatment of Cartan — that the Cartan–Whittaker relation is a *definition* of spinor. This assumption allows us to calculate components of spinor in a new basis. Indeed

$$(u^0)^2 = \tfrac{1}{2}[F^{02}-F^{23}+i(F^{01}-F^{13})],$$

$$(u^1)^2 = \tfrac{1}{2}[F^{02}+F^{23}-i(F^{01}+F^{13})],$$

$$u^0u^1 = \tfrac{1}{2}(F^{12}-iF^{03}),$$

and simultaneously

$$(u^{0'})^2 = \tfrac{1}{2}[F^{0'2'}-F^{2'3'}+i(F^{0'1'}-F^{1'3'})],$$

$$(u^{1'})^2 = \tfrac{1}{2}[F^{0'2'}+F^{2'3'}-i(F^{0'1'}+F^{1'3'})],$$

$$u^{0'}u^{1'} = \tfrac{1}{2}(F^{1'2'}-iF^{0'3'}).$$

Thus the connection between $u^{A'}$ and u^A can be calculated from the connection between $F^{\mu'\nu'}$ and $F^{\mu\nu}$ which (for pure reflexions) is given in Table II.

5. *Two kinds of spinors*

The Cartan–Whittaker bivector can be either a density of the first kind or a density of the third kind. Hence *there are two kinds of spinors*, one generated by density of the first kind and another one generated by density of the third kind. They behave identically under proper Lorentz transformations but differently under reflexions. Transformation laws for both kinds of spinors are given in Table III; they were calculated as indicated in the previous section. It should be remembered that components of a spinor are determined up to an over-all sign; in Table III in each case one sign is chosen arbitrarily but may be changed at will.

TABLE III

Two kinds of spinors

Quantity	Time reflexion $x^{0'} = -x^0$, $x^{i'} = +x^i$, $i = 1, 2, 3$	Space reflexion $x^{0'} = +x^0$, $x^{i'} = -x^i$, $i = 1, 2, 3$	Total reflexion $x^{\mu'} = -x^\mu$, $\mu = 0, 1, 2, 3$
Spinor of bivectorial density of the first kind	$u^{0'} = \bar{u}^1$, $u^{1'} = -\bar{u}^0$	$u^{0'} = i\bar{u}^1$, $u^{1'} = -i\bar{u}^0$	$u^{A'} = iu^A$, $A = 0, 1$
Spinor of bivectorial density of the third kind	$w^{0'} = \bar{w}^1$, $w^{1'} = -\bar{w}^0$	$w^{0'} = \bar{w}^1$, $w^{1'} = -\bar{w}^0$	$w^{A'} = -w^A$, $A = 0, 1$

The difference between two kinds of spinors comes out in a simple way if one treats reflexion as a motion i.e. creation of another spinor.

Let a be a vector and u a spinor with components u^A. Reflexion of the spinor u in the plane orthogonal to a is the spinor u' with components

$$u'^A = \frac{a_\mu}{\sqrt{|aa|}}\, \varepsilon^{AB}\overline{\sigma^{\mu}_{\dot{B}C}u^C}$$

if u is a spinor of bivectorial density of the first kind and

$$u'^A = \frac{a_\mu}{\sqrt{aa}}\, \varepsilon^{AB}\overline{\sigma^{\mu}_{\dot{B}C}u^C}$$

if u is a spinor of bivectorial density of the third kind. Here $\sigma^{\mu}_{\dot{A}B}$ are components of the Pauli matrices, ε^{AB} is the invariant antisymmetric spinor such that $\varepsilon^{01} = 1$. These formulae are equivalent to those from Table III, if one assumes that components of a spinor are equal to components of the reflected spinor in the reflected basis, which seems fairly obvious.

The vector a, which determines the plane of reflexion may be time-like or space-like but cannot be null. It is impossible to define reflexion in a null plane, just as it is impossible to see oneself in a mirror moving away with the velocity of light.

6. *Penrose's pictures of the two spinors*

To visualize the difference between the two kinds of spinors we shall construct Penrose's pictures of both spinors on the sphere of complex numbers.

Each null bivector can be written in the form

$$F_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}k^{\alpha}a^{\beta},$$

where

$$kk = 0, \quad ak = 0, \quad aa = -1.$$

If $F_{\mu\nu}$ is the Cartan–Whittaker bivector of the spinor u, then

$$k^0 = \overline{u^0}u^0 + \overline{u^1}u^1,$$

$$k^1 = \overline{u^0}u^1 + \overline{u^1}u^0,$$

$$k^2 = \frac{1}{i}(\overline{u^0}u^1 - \overline{u^1}u^0),$$

$$k^3 = \overline{u^0}u^0 - \overline{u^1}u^1.$$

Using the results from Table III one can show that, for both kinds of spinors, k is a density of the first kind:

$$k^{\mu'} = \operatorname{sign}(A^{0'}_{0})A^{\mu'}_{\nu}k^{\nu}.$$

This may be seen also from the fact that k^0 is necessarily positive.

Suppose that $F_{\mu\nu}$ is a density of the third kind; then a is a density of the first kind. Let us form the linear combination

$$k + \varepsilon a$$

which — for infinitesimal ε — is a null vector. Direction of k may be drawn on the sphere of complex numbers as the number

$$\frac{k^1 + ik^2}{k^0 + k^3}.$$

If we draw in the same way directions of all vectors $k + \varepsilon a$, we obtain a little segment (Fig. 1). *The segment is oriented* since ε is a scalar and we can tell the difference between $\varepsilon > 0$ and $\varepsilon < 0$ in each reference frame.

Suppose now that $F_{\mu\nu}$ is a density of the first kind (i.e. behaves like the electromagnetic field). a is now a density of the third kind and in the linear combination $k + \varepsilon a, \varepsilon$ must be a scalar density of the second kind. Therefore the sign of ε has no geometrical meaning and we cannot tell the difference between two ends of the little segment (Fig. 2).

Penrose's picture of a spinor of bivectorial density of the third kind is complete: it allows to construct the spinor up to a *real* factor. The picture of a spinor of bivectorial density of the first kind is not complete: it determines the phase of the spinor only up to $\pm\pi/2$.

Penrose's picture of a spinor leads almost inevitably to our definition of the reflected spinor.

Let a be a space-like vector which determines the plane of reflexion. The set of all null directions orthogonal to a forms a circle on the sphere of complex numbers. The

Fig. 1

Fig. 2

Fig. 1. Penrose's picture of a spinor of bivectorial density of the third kind: a point with an oriented direction on the sphere of complex numbers

Fig. 2. Penrose's picture of a spinor of bivectorial density of the first kind: a point with an unoriented direction on the sphere of complex numbers

circle is a picture of the direction of a; the picture is complete i.e. it allows to construct the direction of a. Let u be a spinor of bivectorial density of the third kind (Penrose's spinor), whose complete picture is a point with an oriented direction on the sphere of complex numbers. Consider the pictures of the direction of a and of the spinor u together. Reflexion of u in the plane orthogonal to a should be a spinor determined in a unique and Lorentz invariant way by u and a. There is only one such spinor, namely the spinor obtained from u by the transformation of reciprocal radii. This purely *geometrical* definition of the reflected spinor is identical with our previous analytical definition, based on the Cartan–Whittaker relation. Thus it is seen that the definition of reflected spinor which involves only a single spinor is both natural and inevitable from the geometrical point of view.

Bade and Jehle [4] define the reflected spinor in a way which is equivalent to the definition by means of reciprocal radii. But — as far as we can understand this paper — they consider only space reflexions and only one spinor, which seems to be the spinor of bivectorial density of the third kind.

7. *An axiom on complex multilinear forms*

The theory of reflexions developed so far may be based on a single axiom concerning behaviour of complex multilinear forms under reflexions.

Let a be again the vector which determines the plane of reflexion and let x be an arbitrary vector. The vector

$$x' = x - 2\frac{ax}{aa}a$$

is called reflexion of x in the plane orthogonal to a. Let $f(x)$ be a linear form i.e. a real function of vectorial argument such that for all real α, β and for all x, y

$$f(\alpha x+\beta y) = \alpha f(x)+\beta f(y).$$

There exists a vector f such that

$$f(x) = xf.$$

It is easy to show that

$$x'f' = xf$$

so that

$$f'(x) = f(x').$$

Suppose now that $f(x)$ is a complex linear form i.e. a complex function of vectorial argument such that for all *real* α, β and for all x, y

$$f(\alpha x+\beta y) = \alpha f(x)+\beta f(y).$$

It is not true anymore that $f(x)$ is a product of two vectors and therefore it is impossible to find the reflected form from what is known or assumed about vectors.

We *define* the reflected form by means of the formula

$$f'(x) = \overline{f(x')}.$$

This is an axiom. The axiom is certainly consistent with other axioms of space-time geometry since these axioms deal with events and vectors which form a linear space over the field of *real* numbers.

For a complex multilinear form $f(x, y, z, \ldots)$ we put

$$f'(x, y, z, \ldots) = \overline{f(x', y', z', \ldots)}.$$

Let us consider the complex Cartan–Whittaker bivector which is a complex bilinear form:

$$t^{01} = (u^0)^2-(u^1)^2 = it^{23},$$

$$t^{02} = i[(u^0)^2+(u^1)^2] = it^{31},$$

$$t^{03} = -2u^0u^1 = it^{12}.$$

If u is a spinor of bivectorial density of the first kind then the results from Table III may be summed up in the form

$$t^{\mu'\nu'} = \operatorname{sign}(A^{0'}_0)\det(A^{\alpha'}_\beta)A^{\mu'}_\mu A^{\nu'}_\nu \overline{t^{\mu\nu}}$$

i.e. the complex Cartan–Whittaker bivector is a *complex* density of the second kind. If u is a spinor of bivectorial density of the third kind then the results from Table III give

$$t^{\mu'\nu'} = A^{\mu'}_\mu A^{\nu'}_\nu \overline{t^{\mu\nu}}$$

i.e. the complex Cartan–Whittaker bivector is a *complex* tensor.

Conversely, from the axiom on complex multilinear forms we can deduce transformation law for spinors, as given in Table III.

Note added in proof

I am greatly indebted to Professor André Lichnerowicz and Professor Roger Penrose for the discussion on the subject of this paper.

Professor Lichnerowicz pointed out that, in his opinion, my treatment of reflexions of two-component spinors is equivalent to the classification of four-component spinors given in his mimeographed lectures. Unfortunately, I have not been able to see Professor Lichnerowicz's work and, as yet, I cannot make any comments on this point. I would like to point out, however, that the axiom on complex linear forms given at the end of this paper leads naturally to the idea that both time and space reflexions in quantum mechanics should be antilinear operations, which is not equivalent to the usual treatment of space reflexions.

REFERENCES

[1] R. Penrose, *Structure of Space-Time in Battelle Rencontres 1967*, ed. C. M. DeWitt and J. A. Wheeler, W. A. Benjamin, New York — Amsterdam 1968.
[2] É. Cartan, *Leçons sur la Théorie des Spineurs II*, Hermann et C^{ie}, Paris 1938, p. 66.
[3] E. T. Whittaker, *Proc. Roy. Soc.* **A158**, 38 (1937).
[4] W. L. Bade, H. Jehle, *Rev. Mod. Phys.* **25**, 714 (1953).
[5] J. Aharoni, *The Special Theory of Relativity*, Clarendon Press, Oxford 1959, p. 214.