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THE ANALYTICAL CONTINUATION OF THE DISCONTINUITY RELATION FOR THE THREE-PARTICLE SCATTERING AMPLITUDE IN SUBENERGY VARIABLES

By D. Krupa

Institute of Physics, Slovak Academy of Sciences, Bratislava*

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It is proved that the discontinuity relation for the three-particle scattering amplitude in the total energy variable can be analytically continued in the external subenergy variables arround the two-particle normal threshold branch points corresponding to these variables. Possible generalization to higher subsystem energies, i. e. towards higher number particle normal threshold cuts in the subenergy variables is outlined.

1. *Introduction*

In attempting to utilize dispersion relations for the scattering amplitude of more than two particles in initial and final states, one encounters the problem of how to determine the discontinuity of the scattering amplitude across the kinematical cut in a form suitable for dispersion-theoretical purposes.

If one is concerned only with the binary processes (i.e. for two particles in the initial state going into two particles in the final state), the discontinuity of the scattering amplitude across the kinematical cut is directly given by the two body unitarity relation. In that case, the discontinuity is proportional to the imaginary part of the scattering amplitude up to a certain phase space factor. In the case of the scattering of three or more particles, however, the discontinuity determined by the unitarity is still proportional to the imaginary part of the scattering amplitude; nevertheless, the unitarity equation does not determine the discontinuity relation suitable for dispersion relation integrals. The reason for this difficulty is that unitarity gives the total discontinuity, i.e. the discontinuity in all independent variables of the scattering amplitude simultaneously, while in order to state dispersion relations the discontinuity of the scattering amplitude in just one variable will be required. Such discontinuity, however, can be derived from the unitarity relation using the connectedness structure of the scattering amplitude and the discontinuity relations of the scattering amplitude in subenergy variables [1, 2]. The aim of the present paper is to prove that just

* Address: Institute of Physics, Slovak Academy of Sciences, 899 30 Bratislava, Czechoslovakia.

one variable discontinuity relation for the three-particle scattering amplitude can be analytically continued in subenergy variables. Such continuation is of special importance in the Isobar Model where one wants to continue on the second Riemann sheet to the pole position corresponding to the isobar formation [3].

In the following section it is briefly shown how the required discontinuity relation can be found. The problem of analytical continuation in subenergy variables is solved in Section 3. The last section is devoted to discussion and to conclusions.

In this paper, rather than adopting the "bubble" notation for the amplitudes, we write all equations in analytical form in order to show explicitly that each step of the argument is mathematically justified.

Leaving aside the whole complexity of the 3-body problem we shall restrict ourselves to only 3 to 3 particle amplitude where the 3 particle states are formed of free particles.

2. *The continuation of the discontinuity equation*

The only process allowed for physical energies below production threshold is elastic scattering. The discontinuity relation across the three-particle cut in total energy variable has the following form:

$$T(s_+, \sigma^{\mathrm{i}}_-, \sigma^{\mathrm{f}}_+, \ldots) - T(s_-, \sigma^{\mathrm{i}}_-, \sigma^{\mathrm{f}}_+, \ldots) = 2i \sum T(s_+, \sigma^{\mathrm{i}'}_+, \sigma^{\mathrm{f}}_+, \ldots) T(s_-, \sigma^{\mathrm{i}}_-, \sigma^{\mathrm{f}'}_-, \ldots). \quad (1)$$

Conventional notation is used here: T is the connected 3 to 3 particles amplitude[1]; s is the total energy squared in the three-body centre of mass system; σ are two particle subenergy variables taken jointly; labels $+$, $-$ identify the position of the particular variable above or below the corresponding cut; superscript i (f) stands for the initial (final) configuration; and the summation symbol stands for three-particle phase-space integrals. The dots stand for other variables which we shall mention explicitly only when necessary. This equation has been derived by several authors [1, 6] and, as it was pointed out in [5], the right hand side can be written in four different forms arising from the possibility of interchanging the total energy or internal subenergy variable boundary values denoted by primes; however, external subenergy variables σ^{i}_- and σ^{f}_+ boundary values are not changed.

One can ask what happens with Eq. (1) if we change, e.g. σ^{i}_1 subenergy variable which is the squared total energy of the particles 2 and 3 in their centre-of-mass system from σ^{i}_{1-} to σ^{i}_{1+}. If one wished to continue analytically the discontinuity Eq. (1) in the external subenergy variables σ^{i}_1, however, it is by no means clear that the discontinuity equation would not change. For instance, by adding the following term

$$2i \sum T(s_+, \sigma^{\mathrm{i}}_+, \sigma^{\mathrm{f}}_+, \ldots) T^{\mathrm{D}}(\sigma^{\mathrm{i}}_{1-}, \ldots),$$

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[1] T is defined by means of the S-matrix as follows:

$$S_{33} = \delta_{33} + i(2\pi)^4 \delta^4 \left(\sum_{\lambda=1}^{3} p^{\mathrm{f}}_\lambda - \sum_{\lambda=1}^{3} p^{\mathrm{i}}_\lambda\right) T_{33}.$$

As we are using symbol T only for the 3 to 3 particle amplitude we are omitting the subscripts on T.

where T^{D} is the amplitude for the process where the first particle does not interact with the rest of the system, and subtracting the term

$$2i \sum T(s_{-}, \sigma^{\mathrm{i}}_{+}, \sigma^{\mathrm{f}}_{+}, \ldots) T^{\mathrm{D}}(\sigma^{\mathrm{i}}_{1-}, \ldots)$$

and using the unitarity conditions in σ^{i}_{1} subenergy variable

$$T(s, \sigma^{\mathrm{i}}_{1+}, \sigma^{\mathrm{f}}, \ldots) - T(s, \sigma^{\mathrm{i}}_{1-}, \sigma^{\mathrm{f}}, \ldots) = 2i \sum T(s, \sigma^{\mathrm{i}}_{1+}, \sigma^{\mathrm{f}}, \ldots) T^{\mathrm{D}}(\sigma^{\mathrm{i}}_{1-}, \ldots) \tag{2}$$

one arrives at the discontinuity relation

$$\begin{aligned} &T(s_{+}, \sigma^{\mathrm{i}}_{1+}, \sigma^{\mathrm{f}}_{1+}, \ldots) - T(s_{-}, \sigma^{\mathrm{i}}_{1+}, \sigma^{\mathrm{f}}_{1+}, \ldots) \\ &= 2i \sum T(s_{+}, \sigma^{\mathrm{i}'}_{1+}, \sigma^{\mathrm{f}}_{1+}, \ldots) T(s_{-}, \sigma^{\mathrm{i}}_{1-}, \sigma^{\mathrm{f}'}_{1-}, \ldots) \\ &+ 2i \sum T(s_{+}, \sigma^{\mathrm{i}}_{1+}, \sigma^{\mathrm{f}}_{1+}, \ldots) T^{\mathrm{D}}(\sigma^{\mathrm{i}}_{1-}, \ldots) + 2i \sum T(s_{-}, \sigma^{\mathrm{i}}_{1+}, \sigma^{\mathrm{f}}_{1+}, \ldots) T^{\mathrm{D}}(\sigma^{\mathrm{i}}_{1-}, \ldots), \end{aligned} \tag{3}$$

which is the discontinuity of the three-particle amplitude, while both σ^{i}_{1} and σ^{f}_{1} are continued above the appropriate two particle subenergy variable cuts. Compared with Eq. (1), the right-hand side of the discontinuity relation has developed two additional terms involving an integration over the disconnected amplitudes, i.e. over the two-body phase space.

The purpose of this paper is to show that the last equation can actually be written in the same form as the discontinuity relation Eq. (1), with only the subenergy variable σ^{i}_{1} analytically continued from the boundary value σ^{i}_{1-} to the value σ^{i}_{1+}. In other words, it sets out to show that the discontinuity Eq. (1) can be analytically continued in the subenergy variables σ^{i}_{1} and σ^{f}_{1} without changing its relatively simple usual form.

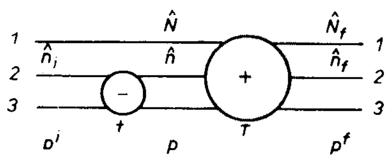

Fig. 1. Schematic representation of the last term of Eq. (3)

Before proceeding any further let us gain some insight into the nature of Eq. (3) by analysis of the last term on the right-hand side. Using the notation of Fig. 1 and substituting for the T^{D} amplitude

$$T^{\mathrm{D}} = 2p^{0}_{1}(2\pi)^{3}\delta^{3}(\vec{p}^{\,\mathrm{i}}_{1} - \vec{p}^{\,\mathrm{f}}_{1})t(\sigma_{1}, \vartheta),$$

where p^{0}_{1} and $\vec{p}_{1}$ are components of the four-momentum of the first particle and $t(\sigma_{1}, \vartheta)$ is the two-particle scattering amplitude for the scattering of particles 2 and 3, that term can be written as follows:

$$\sum T(s_{+}, \sigma^{\mathrm{i}}_{1+}, \sigma^{\mathrm{f}}_{1+}, \hat{n}, \hat{N}, \hat{n}_{\mathrm{f}}, \hat{N}_{\mathrm{f}}, \ldots) T^{\mathrm{D}}(\sigma^{\mathrm{i}}_{1-}, \hat{n} \cdot \hat{n}_{\mathrm{i}})$$

$$= \frac{1}{2} \int \frac{d^{3}\vec{p}_{1}}{(2\pi)^{3}2p^{0}_{1}} \frac{d^{3}\vec{p}_{2}}{(2\pi)^{3}2p^{0}_{2}} \frac{d^{3}\vec{p}_{3}}{(2\pi)^{3}2p^{0}_{3}} (2\pi)^{4}\delta^{4}\left(\sum p - \sum p^{\mathrm{i}}\right)$$

$$2p_1^0(2\pi)^3\delta^3(\vec{p}_1-\vec{p}_1^{\,i})t(\sigma_{1-}^i, \hat{n}\cdot\hat{n}_i)T(s_+, \sigma_{1+}^i, \sigma_{1+}^f, \hat{n}, \hat{N}, \hat{n}_f, \hat{N}_f, \ldots,$$

$$= \frac{1}{32\pi^2}\frac{k(\sigma_1^i)}{\sqrt{\sigma_1^i}}\int T(s_{1+}, \sigma_{1+}^i, \sigma_{1+}^f, \hat{n}, \ldots)t(\sigma_{1-}^i, \hat{n}\cdot\hat{n}_i)d\hat{n}. \tag{4}$$

Here $\hat{n}$ is a unit vector in the direction of the second particle momentum in a centre--of-mass frame of the 2–3 pair, k is the magnitude of the momentum of the second particle in the same reference system, $\hat{N}$ is a unit vector in the direction of $\vec{p}_1$ in the over-all centre--of-mass system. Because of the δ function dependence of T^D amplitude the three body phase space integral is reduced to a two body one. This simplification allows us to write the discontinuity relation (Eq. (3)) in the following form:

$$\text{disc } T(s, \sigma_{1+}^i, \sigma_{1+}^f, \hat{n}_i, \hat{n}_f, \ldots)$$

$$= \varrho(\sigma_1^i)\int[\text{disc } T(s, \sigma_{1+}^i, \sigma_{1+}^f, \hat{n}, \ldots)t(\sigma_{1-}^i, \hat{n}\cdot\hat{n}_i)d\hat{n} + g(s, \sigma_{1+}^i, \sigma_{1+}^f, \hat{n}_i, \hat{n}_f, \ldots) \tag{5}$$

where

$$\varrho(\sigma_1^i) = \frac{2i}{8(2\pi)^2}\frac{k(\sigma_1^i)}{\sqrt{\sigma_1^i}} \tag{6}$$

is the usual two-body phase space factor multiplied by $2i$ for convenience, and

$$g(s, \sigma_{1+}^i, \sigma_{1+}^f, \hat{n}_i, \hat{n}_f, \ldots)$$

$$= 2i\sum T(s_+, \sigma_{1+}^{i'}, \sigma_{1+}^f, \hat{n}_f, \ldots)T(s_-, \sigma_{1-}^i, \sigma_{1-}^{f'}, \hat{n}_i, \ldots). \tag{7}$$

Thus for the discontinuity $T(s, \sigma_{1+}^i, \sigma_{1+}^f, \ldots)$ we have an inhomogeneous integral equation of Fredholm type. The kernel of the integral equation is the two particle amplitude t^-. This is an important feature of the integral equation (5) owing to which the solution in a closed form can easily be found by employing the unitarity condition for the two-particle amplitude t^- ($t^- = t(\sigma_{1-}, \vartheta)$).

From the theory of Fredholm integral equations [7] it follows that if there exists a reciprocal kernel $\tau(n_i, n_f)$ satisfying the following integral relation

$$t^-(\hat{n}_i, \hat{n}_f)+\tau(\hat{n}_i, \hat{n}_f) = \varrho\int t^-(\hat{n}_i, \hat{n})\tau(\hat{n}, \hat{n}_f)d\hat{n} \tag{8}$$

then there exists an integral equation which is reciprocal to Eq. (5), i.e. the equation

$$g(\hat{n}_i, \hat{n}_f, \ldots) = \varrho(\sigma_1^i)\int g(\hat{n}, \hat{n}_f, \ldots)\tau(\hat{n}, \hat{n}_i)d\hat{n} + \text{disc } T(\hat{n}_i, \hat{n}_f, \ldots). \tag{9}$$

After comparing Eq. (8) with the unitarity condition for the two particle amplitude

$$t^+(\hat{n}_i, \hat{n}_f) - t^-(\hat{n}_i, \hat{n}_f) = \varrho\int t^-(\hat{n}_i, \hat{n})t^+(\hat{n}, \hat{n}_f)d\hat{n} \tag{10}$$

one can see that in this case the reciprocal kernel actually does exist and is equal to the two particle amplitude taken above the two particle kinematical cut in the complex σ_1 plane i.e. $\tau(\hat{n}_i, \hat{n}_f) \equiv -t(\sigma_{1+}^i, \hat{n}_i, \hat{n}_f)$. If we substitute for $g(s, \sigma_{1+}^i, \sigma_{1+}^f, \ldots)$ (Eq. (7)) we now obtain from Eq. (9):

$$\text{disc } T(s, \sigma_{1+}^i, \sigma_{1+}^f, \hat{n}_i, \hat{n}_f, \ldots) = 2i\sum T(s_+, \sigma_{1+}^{i'}, \sigma_{1+}^f, \hat{n}_f, \ldots)$$

$$\times[T(s_-, \sigma_{1-}^i, \sigma_{1-}^{f'}, \hat{n}_i, \ldots) + \varrho\int T(s_-, \sigma_{1-}^i, \sigma_{1-}^f, \hat{n}, \ldots)t^+(\hat{n}\cdot\hat{n}_i)d\hat{n}]. \tag{11}$$

The integral expression in brackets on the right-hand side is in fact the discontinuity of T in the subenergy variable σ_1^{i} (Eq. (2)). After substituting for the two particle phase space integral in Eq. (11) from Eq. (2), some terms cancel out and finally we are left with the following discontinuity relation

$$\operatorname{disc} T(s, \sigma_{1+}^{\mathrm{i}}, \sigma_{1+}^{\mathrm{f}}, \ldots) = 2i \sum T(s_+, \sigma_{1+}^{\mathrm{i}'}, \sigma_{1+}^{\mathrm{f}}, \ldots) T(s_-, \sigma_{1-}^{\mathrm{i}}, \sigma_{1-}^{\mathrm{f}'}, \ldots). \tag{12}$$

This is the same form of the discontinuity relation as that of Eq. (1), the only difference being that here σ_1^{i} is above the corresponding two particle cut in the complex σ_1^{i} plane. This equation has been rather intuitively stated in [5], while here we have presented an exact analytical proof of its validity and a justification for a simple analytical continuation of Eq. (1) into Eq. (12).

3. *Conclusion*

The fact that the form of the discontinuity relation across the three-particle cut in the variable s for the connected amplitude T with σ_{1+}^{i} and σ_{1+}^{f} is the same as the one with $\sigma_{1-}^{\mathrm{i}}, \sigma_{1+}^{\mathrm{f}}$ shows that the discontinuity relation Eq. (1) can easily be analytically continued in the variable σ_1^{i}. By analogy, the same can be proved to be true for the variable σ_1^{f}. Such continuation is particularly useful in the isobar formulation where both σ_1^{i} and σ_1^{f} are continued below the appropriate subenergy variable cuts to the point corresponding to two-body resonance energy [3], [8]. The original three-particle scattering amplitude is then reduced to an effective two-particle amplitude describing particle-resonance scattering, to which the conventional two-body dispersion relation technique can be applied [9].

The same argument which we have applied to σ_1 variables can also be applied to the discontinuity relation of the connected amplitude T in the subenergy variables σ_2 and σ_3 in initial and final states. Hence it follows that Eq. (1) can be arbitrarily continued in each of the variables $\sigma_\lambda^{\mathrm{i}}$ and $\sigma_\lambda^{\mathrm{f}} (\lambda = 1, 2, 3)$ around the two-particle cuts. Eq. (1), then, can be written as follows:

$$T(s_+, \sigma_\lambda^{\mathrm{i}}, \sigma_\lambda^{\mathrm{f}}, \ldots) - T(s_-, \sigma_\lambda^{\mathrm{i}}, \sigma_\lambda^{\mathrm{f}}, \ldots) = 2i \sum T(s_+, \sigma_\lambda^{\mathrm{i}'}, \sigma_\lambda^{\mathrm{f}}, \ldots) T(s_-, \sigma_\lambda^{\mathrm{i}}, \sigma_\lambda^{\mathrm{f}'}, \ldots) \tag{13}$$

where in all terms any subenergy variable $\sigma_\lambda^{\mathrm{i}}$ and $\sigma_\lambda^{\mathrm{f}}$ must be simultaneously, but otherwise arbitrarily, placed above or below the two-particle cuts.

As we have mentioned earlier, the right hand side of Eq. (12) can also be written in four different ways, arising from the possibility of interchanging energy or internal subenergy boundary values independently.

Up to this point our proof has applied only to the two particle threshold cuts in the σ variables. However, the proof can be generalized and it is possible to show that the result is valid also for higher particle threshold cuts in the σ variables. The unitarity condition in subenergy variables, Eq. (2), will then be generalized to include higher intermediate states, and the last two-terms of Eq. (3) will then contain a sum of terms where the integration will not be only over the two-particle phase space but also over three and higher number particle phase space. The proof would proceed in exactly the same way as in the present case.

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