QUANTIZATION OF SOLITARY WAVES IN NONLINEAR FIELD THEORIES

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The field equations of some self interacting systems with polynomial interaction Lagrangians possess particular solutions similar to solitary waves of classical field theories. These particular solutions can be interpreted as analogues of the operators $a_k^\dagger e^{+ik\cdot x}$ and $a_k e^{-ik\cdot x}(k \cdot x = k_0x^0 - \boldsymbol{k} \cdot \boldsymbol{x})$, where a_k and $a_k^\dagger$ are annihilation and creation operators of free field theories. A solitary wave propagator can be constructed using the superposition principle of quantum theory rather than the mathematical superposition of solutions of a differential equation. The propagator has poles at integral multiples of the mass of the associated linear theory and has zeros which depend upon the coupling constants.

The study of self interacting field theories with polynomial interaction Lagrangians has been undertaken previously from the point of view of contemporary perturbation theory — that is — the self interaction is treated on the same basis as interactions among different fields. The motivation for such studies has been either to renormalize interactions among different fields or to model the more complex interacting field theories [1, 2]. However, the appearance of resonances in multi-meson and meson-baryon systems provides direct experimental motivation for studying self interacting theories and makes the self interaction interesting in itself. This, in turn, implies the need for a re-examination of the methods employed to describe the self interacting field theory — in particular — to allow for persistent interactions. There is no longer any necessity for asking that the fields reduce to free fields in the limit of large times and, in fact, it is not physically plausible that they should.

Some clasical theories with self interactions have, in addition to the perturbation (linearized) solutions, remarkable traveling wave solutions known variously as solitary waves, solitons and conoidal waves [3]. The common property which these waves possess is that they all have constant phase velocity. This is a surprising result in view of the fact that the systems have both dispersion and nonlinearity present. Although it differs slightly

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from convention, all of these exicitations in nonlinear systems will be referred to throughout this paper as solitary waves [4].

In addition to having constant phase velocity, the solitary wave solutions discussed here satisfy the interacting field equations for all times. Consequently, they seem appropriate to describe the properties of persistent self interactions. Furthermore, the solitary wave fields contain either positive or negative frequencies, but not both simultaneously (for non-localized fields). Consequently, non-localized fields with the space independent coefficients of $e^{\pm ik\cdot x}(k \cdot x = k_0 x^0 - \boldsymbol{k} \cdot \boldsymbol{x})$ replaced by creation or annihilation operators of the linear theory will also be solutions of the field equations.

The specific field theories considered here have field equations of the form

$$(\partial_\mu \partial^\mu + m^2)\phi + \alpha\phi^{2p+1} + \lambda\phi^{4p+1} = 0, \quad (p \neq 0, -\tfrac{1}{2}, -1), \tag{1}$$

where

$$\partial_\mu \partial^\mu = \partial_t^2 - \nabla^2 \quad (\hbar = c = 1), \tag{2}$$

m is the mass of the associated linear field theory and α and λ are coupling constants. For $\alpha = 0$, $p = 1/2$ or $\lambda = 0$, $p = 1$ this is the field equation of the $\lambda\phi^4$ theory while for $\alpha \neq 0$, $p = 1/2$ it is a spinless analogue of the massive Yang–Mills theory. Clearly, dispersion is provided by the m^2 term while the interaction terms furnish the nonlinearity. Nonetheless, solitary wave solutions of Eq. (1) exist. These are particular solutions having constant phase velocity. The non-localized solutions separate into positive or negative frequency solutions. Explicit forms for these solutions may be found either by the method of base solutions [5] or by direct integration [6]. Since the latter method is more transparent it will be used here.

In order to exhibit the solutions assume

$$\phi = \phi(\chi), \tag{3}$$

$$\chi = \pm k \cdot x, \tag{4}$$

$$k^2 \neq 0, \tag{5}$$

where k is an arbitrary (constant) four vector. Eq. (1) becomes

$$k^2 \frac{d^2\phi}{d\chi^2} + m^2\phi + \alpha\phi^{2p+1} + \lambda\phi^{4p+1} = 0. \tag{6}$$

Multiplication by $d\phi/d\chi$ and integration gives

$$\tfrac{1}{2}\left(\frac{d\phi}{d\chi}\right)^2 + \frac{m^2\phi^2}{2k^2} + \frac{\alpha\phi^{2p+2}}{2(p+1)k^2} + \frac{\lambda\phi^{4p+2}}{2(2p+1)k^2} = \frac{B}{2}, \tag{7}$$

where B is constant. Separation leads to the second integral

$$\chi + \chi' = \int d\phi \left(B - \frac{m^2\phi^2}{k^2} - \frac{\alpha\phi^{2p+2}}{(p+1)k^2} - \frac{\lambda\phi^{4p+2}}{(2p+1)k^2}\right)^{-1/2} \quad (\chi' \text{ constant}). \tag{8}$$

As it stands this integral is rather complicated — it defines hyperelliptic or automorphic functions [7], depending on the values of B, m, k, α, λ and p. However, for the special case $B = 0$ simplifications are immediate and several types of solutions can be found in terms of elementary functions. For this case the integral is

$$\chi+\chi' = \int \frac{d\phi}{\phi}\left(-\frac{m^2}{k^2}-\frac{\alpha\phi^{2p}}{(p+1)k^2}-\frac{\lambda\phi^{4p}}{(2p+1)k^2}\right)^{-1/2}. \tag{9}$$

Let

$$\psi = \phi^{2p}, \tag{10}$$

$$\frac{1}{2p}\frac{d\psi}{\psi} = \frac{d\phi}{\phi}, \tag{11}$$

so

$$2p(\chi+\chi') = \int \frac{d\psi}{\psi}\left(-\frac{m^2}{k^2}-\frac{\alpha\psi}{(p+1)k^2}-\frac{\lambda\psi^2}{(2p+1)k^2}\right)^{-1/2}. \tag{12}$$

This integral is elementary. There are several interesting special cases.

Case 1. $\alpha < 0$ and $k^2 < 0$ (k spacelike)

$$\phi = \left[-\frac{\alpha}{2(p+1)m^2}+\left(\left\{\frac{\alpha}{2(p+1)m^2}\right\}^2-\frac{\lambda}{(2p+1)m^2}\right)^{1/2}\cosh\left(2p\sqrt{\frac{m^2}{-k^2}}(\chi+\chi')\right)\right]^{-\frac{1}{2p}}. \tag{13}$$

This type of solution is a soliton — it clearly describes a system localized in space-time, propagating with constant phase velocity $k_0/|k|$. It has the interesting property that it is singular at $\alpha = \lambda = 0$ [8, 9]. For $2p = 1$, $\alpha = 0$ this function reduces, at the origin, to $2^{1/2}$ times the solution given by Goldstone [10].

Case 2. $\alpha < 0$, $k^2 < 0$. Similar to case 1, but with the constant of integration chosen differently. The solution is

$$\phi = \left[-\frac{\alpha}{2(p+1)m^2}+\frac{\mathscr{E}e^{2pm\chi/|k|}}{2}+\left(\left\{\frac{\alpha}{2(p+1)m^2}\right\}^2-\frac{\lambda}{(2p+1)m^2}\right)\frac{e^{-2pm\chi/|k|}}{2\mathscr{E}}\right]^{-\frac{1}{2p}}, \tag{14}$$

$\mathscr{E}$ arbitrary. This solution is regular for vanishing coupling constant, but it has the interesting property that it is asymmetric in x.

Although these solutions have not been quantized, they are included as interesting examples of the combined effects of nonlinearity and dispersion. They are particular solutions and, in the absence of physical interpretation, may be viewed as curiosities.

Finally, the case of immediate interest for quantization has $k^2 = m^2$. Particular solutions are, for a system of volume V(6),

$$\phi_{pk}^{(\pm)} = \left[\left(1 - \frac{\alpha}{4(p+1)m^2(V\omega_k)^p} A_k^{(\pm)2p} e^{\mp 2ipk\cdot x}\right)^2 - \frac{\lambda A_k^{(\pm)4p}}{4(2p+1)m^2(V\omega_k)^{2p}} e^{\pm 4ipk\cdot x}\right]^{-\frac{1}{2p}} \times \frac{A_k^{(\pm)} e^{\mp ik\cdot x}}{\sqrt{V\omega_k}}, \tag{15}$$

where $\omega_k = (\boldsymbol{k}^2 + m^2)^{1/2}$. One objection which comes to mind immediately is that these are complex or non-hermitian solutions. However, it is only the *general* solution of the field equation which must be real or hermitian. Particular solutions, e.g., $a_k e^{-ik\cdot x}$ for the Klein–Gordon equation, are acceptable.

For $\alpha = \lambda = 0$ these solutions reduce to particular solutions of the Klein–Gordon equation. It is evident that they contain either positive or negative frequency terms separately. This suggests the identification of $A_k^{(\pm)}$ with the annihilation or creation operators of the linear theory. The condition that the classical solutions be nonsingular is similar

$$\lambda A^2/8m^2\omega V \neq 1 \tag{16}$$

to appropriate for the $\lambda\phi^4$ theory.

A second important characteristic of these solutions is that they contain the coupling constants for all times. Thus, they satisfy the interacting field equations for all times, rather than reducing to solutions of the Klein–Gordon equation for large times. This is the property which we intuitively expect for self interacting systems.

Finally, the classical solutions are not localized. This is also consistent with the analogy to the solutions $ae^{-ik\cdot x}$ of the Klein–Gordon equation.

Turning to specific matters in the quantization of these solutions, take the commutator of the annihilation and creation operators to be

$$[A_k^{(+)}, A_q^{(-)}] = \delta_{n_k, n_q}, \tag{17}$$

where, in the box

$$\boldsymbol{k} = \frac{2\pi}{L}(n_{k_1}e_1 + n_{k_2}e_2 + n_{k_3}e_3) \text{ etc.} \tag{18}$$

With these commutation relations the norm of $\phi^{(-)}|0\rangle$ is

$$\langle 0|\phi^{(+)}\phi^{(-)}|0\rangle = (V^2\omega_k\omega_q)^{-1/2} \sum_{\substack{n=0\\ s=0}}^{\infty} C_n^{\frac{1}{2p}}(\mathscr{E}) C_s^{\frac{1}{2p}}(\mathscr{E}) \left[(V\omega_k)^{-p} r\right]^n \left[(V\omega_q)^{-p} r\right]^s \times \langle 0|\,(A_k^{(+)})^{2pn+1}(A_q^{(-)})^{2ps+1}|0\rangle e^{-i(2pn+1)k\cdot x + i(2ps+1)q\cdot x}, \tag{19}$$

where

$$\mathscr{E} = \alpha/(4m^2 r(p+1)), \tag{20}$$

$$r = [\alpha^2/(16m^4(p+1)^2) - \lambda/(4m^2(2p+1)]^{1/2} \tag{21}$$

and C_n^ν is a Gegenbauer polynomial [7]. With the commutator given above

$$\langle 0| A_k^{(+)2pn+1} A_q^{(-)2ps+1} |0\rangle = \delta_{s,n}\delta_{n_k,n_q}(2pn+1)!. \tag{22}$$

This expression is only valid for $2p$ an integer, although a generalization can probably be found. However, restricting the discussion to this case, the norm of $\phi^- |0\rangle$ is

$$\langle 0|\phi^{(+)}\phi^{(-)}|0\rangle = (\omega_k V)^{-1} \sum_{n=0}^{\infty} (2pn+1)![C_n^{\frac{1}{2p}}(\mathscr{E})]^2[(V\omega_k)^{-p}r]^{2n} \tag{23}$$

independent of x. Thus, the state created by $\phi^{(-)}$ is non-localized, once more in keeping with the analogy to the operator $a^\dagger e^{ik\cdot x}$ of the linear theory.

With the particular solutions of the interacting field equations in hand and with some understanding of the physical interpretation of the states created by these operators we must next inquire about their further use. One of the central elements of the linear theory is the propagator. How can the solutions of the nonlinear theory, since they are only particular solutions, be used to calculate a propagator? In the linear theory the usual procedure is to construct the general solution of the field equation by a superposition of the particular solutions. Explicit use is made of the mathematical principle of superposition of solutions of a linear differential equation to obtain the general solution. However, this principle is worthless for solutions of a nonlinear differential equation — the superpositions are no longer solutions of the nonlinear differential equation. Consequently, it is necessary to re-examine the construction of the propagator for the linear theory to see if physics can direct us to the construction of a propagator without employing the general solution of the differential equation.

The propagator for the linear theory can be constructed in the following way. The operator $a_k^\dagger e^{ik\cdot x}$ is a creation operator: it creates a particle with four momentum k at a space-time point x. The norm of the state $a_k^\dagger e^{ik\cdot x} |0\rangle$ is clearly independent of x, in keeping with the uncertainty principle. If $|\varepsilon\rangle$ is an arbitrary ket, the conditional amplitude for a particle to be created at x with momentum k and be found in $|\varepsilon\rangle$ is $\langle\varepsilon|\psi_k^{(-)}(x)|0\rangle$, where $\psi_k^-(x) = a_k^\dagger e^{ik\cdot x}/\sqrt{\omega_k V}$. Similarly, the conditional amplitude for a particle of momentum q to be annihilated from $|\varepsilon\rangle$ at position y is $\langle 0|\psi_q^{(+)}(y)|\varepsilon\rangle$. Hence, the conditional amplitude for the sequence is the product of amplitudes, e.g.,

$$P_{\varepsilon kq}(x, y) = \langle 0|\psi_q^{(+)}(y)|\varepsilon\rangle \langle\varepsilon|\psi_k^{(-)}(x)|0\rangle. \tag{24}$$

The time reversed sequence for negative energy particles is completely indistinguishable from the direct process so we must take the arithmetic average of the two amplitudes.

If the intermediate momenta k and q are not observed and the states $|\varepsilon\rangle$ are not observed, we sum on k, q and ε. Finally, if the states $|\varepsilon\rangle$ are closed the total propagator is

$$P(x, y) = \tfrac{1}{2} \sum_{\varepsilon k, q,} \{\langle 0|\psi_q^{(+)}(y)|\varepsilon\rangle \langle\varepsilon|\psi_k^{(-)}(x)|0\rangle\theta(y_0 - x_0)$$
$$+ \langle 0|\psi_q^{(+)}(x)|\varepsilon\rangle \langle\varepsilon|\psi_k^{(-)}(y)|0\rangle\theta(x_0 - y_0)\}. \tag{25}$$

Making specific use of the form for $\psi_k^{(\pm)}$ and the commutation relations this is

$$P(x-y) = \tfrac{1}{2} \sum_{k,q} \langle 0|e^{-iq\cdot y+ik\cdot x} \frac{a_q a_k^\dagger \theta(y_0 - x_0)}{\sqrt{\omega_k\omega_q V^2}} + e^{-iq\cdot x+ik\cdot y} \frac{a_q a_k^\dagger \theta(x_0 - y_0)}{\sqrt{\omega_k\omega_q V^2}} |0\rangle$$

$$= \tfrac{1}{2} \sum_k (\omega_k V)^{-1}\{e^{-ik\cdot(y-x)}\theta(y_0 - x_0) + e^{-ik\cdot(x-y)}\theta(x_0 - y_0)\} = i\Delta_{\mathrm{F}}(x-y). \tag{26}$$

Thus, the propagator can be constructed without using the general solution of the field equation but, instead, using the quantum theoretical rules for the superposition of conditional probability amplitudes. The mathematical step of construction of general solutions by superposition of particular solutions is unnecessary. For the linear theory the propagator is the total propagator since the momentum states form a complete set.

To construct a propagator in the nonlinear theory we proceed in the same way, using the creation and annihilation operators $\phi_{pk}^{(-)}$ and $\phi_{pk}^{(+)}$. However, there are some important restrictions which must be recognized and this propagator will be referred to as the solitary wave propagator. First, there is no a priori reason to interpret the system described by $\phi_k^{(-)}$ as a single particle and k need be the momentum of the system. Second, there are other states through which the process may proceed. These states will be created by other solutions of the field equations. With these qualifications in mind the solitary wave propagator is

$$P^{\mathrm{sol}}(x, y) = \tfrac{1}{2} \sum_{k,q} \langle 0|\phi_{pk}^{(+)}(y)\phi_{pq}^{(-)}(x)\theta(y_0 - x_0) + \phi_{pk}^{(+)}(x)\phi_{pq}^{(-)}(y)\theta(x_0 - y_0)|0\rangle. \tag{27}$$

As in calculating the norm of $\phi^-|0\rangle$, we make use of the expansion of the operators and the commutation relations to obtain

$$P^{\mathrm{sol}}(x-y) = \tfrac{1}{2} \sum_{n,k} (\omega_k V)^{-1}\left[C_n^{\frac{1}{2p}}(\mathscr{E})\right]^2 [(V\omega_k)^{-p} r]^{2n}(2pn+1)!$$
$$\times \{e^{-i(2pn+1)k\cdot(y-x)}\theta(y_0 - x_0) + e^{-i(2pn+1)k\cdot(x-y)}\theta(x_0 - y_0)\}. \tag{28}$$

This expression can be put in a more transparent form by using the representation for the step function

$$\theta(\pm Z) = \pm \frac{1}{2\pi i} \int \frac{dh}{h \pm i\varepsilon} e^{-iahZ} \quad (a > 0). \tag{29}$$

The n-th term of the series is

$$P_n = \int dh \tfrac{1}{2} \sum_k \frac{(2pn+1)!}{2\pi i\omega_k V} \left[C_n^{\frac{1}{2p}}\right]^2 [(V\omega_k)^{-p} r]^n$$

$$\times \left\{\frac{1}{h-i\varepsilon} \exp\{-i(2pn+1)[(h-\omega_k)(y_0-x_0)+\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})]\}\right.$$

$$\left. - \frac{1}{h+i\varepsilon} \exp\{-i(2pn+1)[(h+\omega_k)(y_0-x_0)-\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})]\}\right\}. \quad (30)$$

Replacing the sum of $\boldsymbol{k}$ by an integral (strictly speaking, this requires having $V\to\infty$, which also requires a re-definition of the coupling constants; however, it is useful to leave the V dependence explicit) one has

$$P_n = \int \frac{d^3k\, dh (2pn+1)! \left[C_n^{\frac{1}{2p}}\right]^2 r^{2n}}{2i\omega_k^{2pn+1} V^{2pn} (2\pi)^4} \left[\frac{1}{h-i\varepsilon} \exp\{-i(2pn+1)[(h-\omega_k)(y_0-x_0)\right.$$

$$\left. +\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})]\} - \frac{1}{h+i\varepsilon} \exp\{-i(2pn+1)[(h+\omega_k)(y_0-x_0)-\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})]\}\right]. \quad (31)$$

In the first term, let $h-\omega_k = k_0$ and in the second $h+\omega_k = k_0$ to get

$$\int \frac{d^4k (2pn+1)! \left[C_n^{\frac{1}{2p}}\right]^2 r^{2n}}{2i(2\pi)^4 \omega_k^{2pn+1} V^{2pn}} \left[\frac{1}{k_0+\omega_k-i\varepsilon} - \frac{1}{k_0-\omega_k+i\varepsilon}\right] \exp[-i(2pn+1)k\cdot(y-x)]$$

$$= \frac{ir^{2n}(2pn+1)!}{(2\pi)^4 V^{2pn}} \left[C_n^{\frac{1}{2p}}\right]^2 \int \frac{d^4k}{\omega_k^{2pn}} \frac{\exp[-i(2pn+1)k\cdot(y-x)]}{k^2-m^2+i\varepsilon}. \quad (32)$$

Finally, let $(2pn+1)k = k'$ to obtain (drop primes)

$$P_n = \frac{ir^{2n}(2pn+1)!}{(2\pi)^4 V^{2pn}} \left[C_n^{\frac{1}{2p}}\right]^2 \int \frac{d^4k (2pn+1)^{2pn-2}}{(\boldsymbol{k}^2+(2pn+1)^2 m^2)^{pn}} \frac{e^{-ik\cdot(y-x)}}{k^2-(2pn+1)^2 m^2+i\varepsilon}$$

$$= \frac{ir^{2n}(2pn+1)!(2pn+1)^{2pn-2}}{V^{2pn}[(2pn+1)^2 m^2-\nabla^2]^{pn}} \Delta_F(y-x; (2pn+1)^2 m^2). \quad (33)$$

The solitary wave propagator is then

$$P^{\text{sol}}(y-x) = \sum_{n=0}^{\infty} \frac{ir^{2n}(2pn+1)!(2pn+1)^{2pn-2}}{V^{2pn}[(2pn+1)^2 m^2-\nabla^2]^{pn}} \left[C_n^{\frac{1}{2p}}\right]^2 \Delta_F(y-x; (2pn+1)^2 m^2). \quad (34)$$

The factor $V\omega_k$ is an invariant so the solitary wave propagator is invariant. Thus, the amplitude for propagation from x to y through the solitary wave state is Lorentz invariant.

From the presence of the term $\Delta_E(y-z)$ it is evident that the series has singularities on the light cone similar to the linear theory ($n = 0$). The operator $[(2pn+1)^2m^2-\nabla^2]^{-pn}$ modifies these singularities and for some n the expressions on the light cone become finite. At points off the light cone the series itself is divergent due to the factor $(2pn+1)!$ This term arises from the functional dependence of the solutions on the creation and annihilation operators. It is straightforward to show that this divergent series is asymptotic (see references [11] ans [12] for the $\lambda\phi^4$ theory). Finally, it is easy to see that the solitary wave propagator does not satisfy the field equation.

In momentum space the solitary wave propagator (the coefficient of $e^{-ik\cdot(y-x)}$ in Eq. (33)) is

$$P^{\rm sol}(k) = \sum_{n=0}^{\infty} \frac{ir^{2n}(2pn+1)!(2pn+1)^{2pn-2}\left[C_n^{\frac{1}{2p}}\right]^2}{(2\pi)^4 V^{2pn}[(2pn+1)^2m^2+\boldsymbol{k}^2]^{pn}} \frac{1}{k^2-(2pn+1)^2m^2+i\varepsilon}. \tag{35}$$

From inspection we see that this series has poles at $|k| = (2pn+1)m$, so we expect that the fields $\phi_{pk}^{(\pm)}$ describe many-particle systems with masses of the particles equal to integer multiples $(2pn+1)$ of the mass of the particle of the linear theory. This is something of a surprise — the masses of the particles in the interacting theory are independent of the coupling constant. Furthermore, since the value of the n-th term approaches $-\infty$ at $(2pn+1)m-\varepsilon = |k|$ and $+\infty$ at $(2pn+1)m+\varepsilon$ the series must have an infinite number of zeroes. These are dependent on the coupling constant and for the $\lambda\phi^4$ theory the first one occurs approximately at

$$|k| \approx m\{1+2/(1+3!\lambda^2/128m^4)\}. \tag{36}$$

This behavior leads to oscillating cross-sections for baryon-antibaryon scattering [13]. In momentum space the series is also asymptotic, the n-th term vanishing for large $|k|$, while for fixed $|k|$ and increasing n the terms diverge.

Finally, if we apply the theory to neutral, pseudoscalar mesons, evaluating p by taking $m = m_{\pi^0} = 135$ MeV and $m = m_{\eta^0}$ for $n = 1$, the theory predicts a sequence of neutral, pseudoscalar mesons at $m_n = (3n+1)135 \approx 945, 1350\ldots$. The propability of occurrence of the higher mass states, proportional to the residue of the propagator, depends upon the coupling constants through the Gegenbauer polynomials and the factor r.

In conclusion, the nonlinear field theories described in this paper possess solitary wave solutions which can be interpreted as analogues of the creation and annihilation operators $a_k^\dagger e^{ik\cdot x}$ and $a_k e^{-ik\cdot x}$ of the linear theory obtained by setting the coupling constants to zero. The propagator of the linear theory can be constructed independent of the mathematical principle of superposition of solutions of a differential equation by use of the quantum superposition principle for conditional probability amplitudes. The resulting theory describes persistent self interactions since the fields contain the coupling constants for all times and always satisfy interacting field equations. The theory is a many-particle theory with a spectrum of masses independent of the coupling constants and equal to $(2pn+1)m$, where m is the mass of the associated linear theory.

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