ON THE DECAY LAWS OF UNSTABLE QUANTUM SYSTEMS

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In this paper we reexamine the problem of the decay law of unstable quantum systems. We show that purely exponential decay law should be measured in experiments in which it is known that a system stays in its initial state until it disintegrates. Here we call such experiments — first kind experiments. The second kind of experiments consist of those in which a system may undergo quantum transitions to other states before decay. Deviations from the exponential decay can be seen, in principle, in these experiments. Decay laws suitable for both kinds of experiments are explicitly derived. A review of the various approaches to the decay law problem is presented in the Introduction.

1. Introduction

When describing the decay phenomena of unstable quantum systems it is often erroneously assumed that the function

$$P(t, \psi) = \left| \langle \psi | \exp\left(- \frac{it}{\hbar} H \right) | \psi \rangle \right|^2 \tag{1}$$

gives the probability of a system surviving in the initial state $|\psi\rangle$ up to the time t, [1], (see also [5]). Hence this function is compared with a fraction of the undecayed unstable particles measured in the bubble chamber or other experiments in which one knows that a system is occupying the same initial state until its desintegration occurs. Results of this comparison are rather pitiful since the number $N(t)$ of undecayed particles measured in bubble chamber experiments drops off exponentially

$$N(t) = N(0) \exp(-t\Gamma), \tag{2}$$

while the function $P(t, \psi)$ has a different time behaviour, as was shown explicitly in the solvable examples [2–4] and also generally under some assumptions [5]. This discrepancy is usually interpreted in favor of the function $P(t, \psi)$ saying that deviations from the

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exponential decay law should certainly be measured in such experiments for greater times [5, 6]. However, according to Quantum Mechanics, this function gives the probability of finding a system at the time t back again in the initial state, without any condition concerning its behaviour at intermediate times. Hence, this function should be compared with the results of measurements in which one does not observe a system. Therefore, we shall stress that in analysis of experiments with unstable systems one has to distinguish between two different kinds of experiments of this type. The first kind there are those experiments in which one is observing a system constantly and the second kind those in which a system is not observed during a time t. In some of experiments (first kind) with unstable particles one knows that a particle prepared at $t = 0$ stays in its initial quantum state until it decays. This information should be taken into account if one wants to derive a decay law reproducing correctly the data from such experiments. The first attempt toward this goal was undertaken by Coester [7] who expressed the view according to which the exponential decay law is due to the fact that the system in "under continuous observation". Later Ekstein and Siegert [8] remarked that each bubble of a track produced by a particle in the bubble chamber or in the photographic emulsion should be treated as an experiment showing whether a particle is decayed or not. Thus repeated reductions of the wave function at random instants occur exactly in the same manner as in Heisenberg's theory of tracks produced by the Wilson camera [9]. This leads to a function for the decay law which is different from $P(t, \psi)$ and, according to the authors, should be asymptotically exponential. Fonda, Ghirardi, Rimini and Weber have further developed this idea and derived the almost exponential decay law [10]

$$N(t) = \alpha \exp[-\bar{\nu}(t+T)] + \beta(t) \exp[-\lambda(t+T)] \tag{3}$$

with $|\beta(t)|$ bounded by a constant. Here T stands for the time of flight of particles from their source to the bubble chamber and λ stands for the frequency of random measurements suffered by each decaying particle. The second term is argued to be small in comparison with the first one since λT is big in practice. The decay parameter $\bar{\nu}$ is then determined implicitly by the relation

$$\lambda \int_0^\infty dt P(t, \psi) \exp[(\bar{\nu}-\lambda)t] = 1. \tag{4}$$

This relation suggests a rather strange dependence of the decay parameter $\bar{\nu}$ of a particle on λ which is connected with the details of the experimental set-up used in measurement [11].

The theory was further refined and generalized by Yoshihuku [12] who pointed out the difficulties in description of a neutral particle decay caused by the lack of track when the particle is undecayed. He introduced the two frequencies λ_1, λ_2 of random measurements performed on a system before and after its decay in order to introduce the neutral particle decay into the scheme and arrived at the following purely exponential decay law in the limit of the constant observation of the system ($\lambda_1, \lambda_2 \to \infty$)

$$N(t) = N(0) \exp[P'(0, \psi)t]. \tag{5}$$

Note, however, that for a selfadjoint Hamiltonian H the last exponent is simply equal to one and the decay is absent at all (Turing's paradox!).

We shall present here a simple derivation of the decay law of a system which stays in its initially prepared state until it decays (first kind experiment). The paper is an extension and refinement of our first attempt at the problem described in the preprint [13].

2. Derivation of the exponential decay law — first kind experiment

We will now consider a stationary time evolution of an unstable quantum system. Instability of a system means that the Hamiltonian which governs this evolution is not Hermitean

$$H = M - \frac{i\hbar}{2}\Gamma. \tag{6}$$

Here M and Γ are Hermitean operators. The matrix elements of the evolution operator

$$a(t) = \exp\left(-\frac{it}{\hbar}H\right). \tag{7}$$

satisfy the Schrödinger equation

$$i\hbar\dot{a}(t) = Ha(t) \tag{8}$$

and the initial condition

$$a(0) = \mathbf{1}. \tag{9}$$

They may be converted into an equivalent Feller's type integral equation

$$a(t) = \exp\left(-\frac{it}{\hbar}H_{\mathrm{d}}\right) - \frac{i}{\hbar}\int\limits_0^t ds \exp\left[-\frac{i}{\hbar}(t-s)H_{\mathrm{d}}\right]H_{\mathrm{off}}a(s) \tag{10}$$

as it may be verified simply by differentiation with respect to the time [14]. Here H_{d} is the diagonal part of the Hamiltonian H specified for the given state $|\psi\rangle$ and H_{off} is its off-diagonal part

$$H_{\mathrm{d}} = |\psi\rangle\langle\psi|H|\psi\rangle\langle\psi|, \tag{11}$$

$$H_{\mathrm{off}} = H - H_{\mathrm{d}}. \tag{12}$$

Hence we may write for the diagonal element of the evolution operator

$$\langle\psi|\exp\left(-\frac{it}{\hbar}H\right)|\psi\rangle = \exp\left(-\frac{it}{\hbar}\langle\psi|H|\psi\rangle\right)$$

$$-\frac{i}{\hbar}\sum_{\alpha\neq 1}\int\limits_0^t ds\langle\psi|H|\psi_\alpha\rangle\langle\psi_\alpha|\exp\left(-\frac{is}{\hbar}H\right)|\psi\rangle\exp\left(-\frac{i}{\hbar}(t-s)\langle\psi|H|\psi\rangle\right)$$

$$= f(t,\psi)\exp\left(-\frac{it}{\hbar}\langle\psi|H|\psi\rangle\right), \tag{13}$$

where the function $f(t, \psi)$ has an expansion

$$f(t, \psi) = 1+\sum_{\alpha\neq 1} \langle\psi|H|\psi_\alpha\rangle \langle\psi_\alpha|H|\psi\rangle \left(\frac{1}{i\hbar}\right)^2$$

$$\times\left[\frac{\exp-\frac{it}{\hbar}(\langle\psi|H|\psi\rangle-\langle\psi_\alpha|H|\psi_\alpha\rangle)-1}{(\langle\psi|H|\psi\rangle-\langle\psi_\alpha|H|\psi_\alpha\rangle)^2}(i\hbar)^2+\frac{i\hbar t}{\langle\psi_\alpha|H|\psi_\alpha\rangle-\langle\psi|H|\psi\rangle}\right]+\ldots \quad (14)$$

where the summation runs over the vectors $|\psi_\alpha\rangle$ forming together with $|\psi\rangle = |\psi_1\rangle$ an orthogonal base in the Hilbert space of our system. Each element of the right-hand-side of this equation corresponds to a different way of passing from the initial state $|\psi\rangle$ at $t = 0$ to the same state at time t (different Feynman paths). The first one

$$a(t, \psi) = \exp\left(-\frac{it}{\hbar}\langle\psi|H|\psi\rangle\right) \quad (15)$$

gives the probability amplitude of preserving the initial state $|\psi\rangle$ up to the time t and so it is appropriate for the first kind experiment. Such an interpretation follows from a definition of the Hamiltonian and Markovian character of the time evolution of a quantum system.

Indeed one has for a small time interval

$$\langle\psi_\alpha|a(\Delta t)|\psi_{\alpha'}\rangle = \delta_{\alpha\alpha'}-\frac{i}{\hbar}\langle\psi_\alpha|H|\psi_{\alpha'}\rangle\Delta t+o(\Delta t). \quad (16)$$

It is clear from this formula that the diagonal elements of the energy matrix describe the tendency of a system to preserve its state while the off diagonal elements describe the tendency for changing a state. Thus for an infinitesimal time interval the diagonal elements of the evolution matrix give the probability amplitudes for finding a system in the same initial state without changing the state at intermediate times. It is not true for a finite time interval but in that case one may utilize the Markovian character of the evolution in order to include this additional condition. We split the time interval into pieces and pass from the state $|\psi\rangle$ to the same one at $t = 0$ step by step using formula (16). We will obtain for the conditional probability amplitude $a(t, \psi)$ of this kind of process the formula

$$a(t, \psi) = \lim_{n\to\infty}\left[1-\frac{i}{\hbar}\langle\psi|H|\psi\rangle\frac{t}{n}+o\left(\frac{t}{n}\right)\right]^n = \exp\left(-\frac{it}{\hbar}\langle\psi|H|\psi\rangle\right) \quad (17)$$

which agrees with (15). One sees from Eqs (13) and (14) that this function differs from the diagonal element of the evolution matrix unless the sum of the other elements vanishes. This would be the case however if the passages to the state $|\psi\rangle$ would be forbidden, i.e., if the matrix elements $\langle\psi|H|\psi_\alpha\rangle$ would vanish for $\alpha \neq 1$.

For the decay law of a system prepared at the initial time $t = 0$ in the state $|\psi\rangle$ we obtain now from formula (17)

$$|a(t, \psi)|^2 = \exp(-t\Gamma_\psi), \tag{18}$$

where

$$\Gamma_\psi = \langle\psi|\Gamma|\psi\rangle = -P'(0, \psi). \tag{19}$$

Clearly if Γ would be a positive number then all states would decay at the same universal rate which might not be the case in general.

We shall remark that Ekstein and Siegert consider the formula similar in spirit to our formula (17) but the time interval $\frac{t}{n}$ is kept final there and physical arguments are given against passing to the limit $n \to \infty$ (see. [8], formula (4)). We did not find these arguments convincing since the information which we have about the system in the experiments under consideration amount physically to its constant observation. Hence the limit $n \to \infty$ is legitimate and yields the exponential decay law.

3. *Concluding remarks*

We have shown that if an unstable quantum system stays all the time in its initial state until it decays (first kind experiment), then the exponential decay law is to be expected.

The question then arises of how to decide in practice whether a system preserves its initially prepared state or not. One may refer to the bubble chamber or other experiments where it may be approximately checked by observing tracks of charged particles. For neutral particles, like e.g. K^0-ons, there may be an experimental situation when any change of state, caused, e.g., by the weak interactions, is immediately detected. Thus, in such a case we can also assume that the particle was occupying an initial state until its decay. The theory applies to both these cases.

The question of constructing the Hamiltonian for unstable systems remains untouched here. We shall mention only the Weisskopf–Wigner approximate method [16] and also the Królikowski–Rzewuski rigorous method for deriving the equation of motion for projection of the state vector onto a subspace of the Hilbert space of states [17]. It turns out that this equation contains a "potential" which is not Hermitean and describes passages between the given subspace, which may be identified with the set of decaying states, and its orthogonal complement which is the space of states of the decay products. Results concerning this type of approach to the decay problem may be found in the literature [18–25].

Concluding, the purely exponential decay law results from the nonhermicity of the Hamiltonian and from the assumption that the initially prepared state of a system is preserved in time up to the moment of decay. Deviations from the exponential decay law could be seen only in experiments in which the condition of constant observation of a system is violated (e.g. second kind experiments).

In this case the regeneration effect, represented by higher terms in Eq. (13), may spoil the pure exponential time behaviour of the number $N(t)$ of "undecayed systems". We add the quotation marks since this number now includes e.g. systems also recombined from products of their decay. For the decay law in this case one obtains the function which according to formula (14), may be written in the form

$$P(t, \psi) = |f(t, \psi)|^2 \exp(-t\Gamma_\psi), \tag{20}$$

where the function $f(t, \psi)$ measures the deviations from the exponential decay law.

We shall discuss the question of nonhermicity of a Hamiltonian in connection with the regeneration problem in a subsequent publication.

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