INFLUENCE OF THE QUADRUPOLE-PAIRING FORCES ON THE NUCLEAR INERTIAL MASS PARAMETER

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The inertial mass parameter B for the collective quadrupole oscillations is calculated in the adiabatic approximation. It is assumed that the short range residual forces are of the monopole-plus-quadrupole-pairing type. The dependence of the mass parameter on the strength of the quadrupole-pairing interactions, G_2, is investigated in the pure harmonic oscillator model.

1. *Introduction*

Theoretical description of the collective phenomena in nuclear physics, such as the collective vibrations or nuclear fission, requires knowledge of the mass parameter B. This quantity represents the inertia of the whole system with respect to the given type of motion. The standard calculations of this parameter in connection with the spontaneous fission investigations ([1], [2]) included the monopole-pairing residual forces which couple two particles to the total angular momentum $J = 0$. It was found that the mass parameter depends very strongly on the pairing energy gap resulting from those interactions (e.g. [1]–[3]). On the other hand, inclusion of the higher multipole-pairing [4] modifies the pairing gap and influences other physical quantities characterising the nucleus (see Refs [5]–[10]), in particular the inertial mass parameter [3], [11].

The aim of this paper is to discuss the influence of the quadrupole-pairing forces, acting between two particles coupled to the total angular momentum $J = 2$, on the inertial mass parameter. Numerical calculations are performed within the pure anisotropic harmonic oscillator model. This model offers considerable computational facilities and allows some insight into the possible correlations between the changes in the mass parameter and the underlying single-particle structure of the nucleus.

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2. *Derivation of the mass parameter in the adiabatic approximation*

The inertial mass parameter B is extracted from the expression for the energy of the given collective motion, $\hbar\omega$. In the following the adiabaticity of this motion is assumed, i.e. the quantity $\hbar\omega$ is taken to be small, in particular when compared with the single--particle energies.

The Hamiltonian consists of the single-particle part and of the two-body interaction. The latter is separated into the long-range multipole-multipole part (in this case expressed as the quadrupole-quadrupole forces) and the short-range part, approximated by the multipole-pairing interactions, which contains in addition to the standard monopole--pairing — the quadrupole-pairing term.

The Bogoliubov-Valatin transformation to the quasi-particle creation and annihilation operators $\alpha_k^\dagger$, α_k is performed and the Hamiltonian is assumed in the following form:

$$H = H_0 + \frac{\kappa}{2}\hat{Q}^\dagger Q - G_0 T^\dagger T - G_2 T_q^\dagger T_q, \tag{1}$$

where

$$H_0 = \sum_k E_k \alpha_k^\dagger \alpha_k, \tag{2}$$

$$E_k = \sqrt{(e_k-\lambda)^2+\Delta_k^2}\,, \tag{2a}$$

$$\hat{Q}^\dagger = \sum_{k,l} q_{kl}\gamma_{kl}(\Gamma_{kl}^\dagger+\Gamma_{kl}), \tag{3}$$

$$T^\dagger = \tfrac{1}{2}\sum_{k,l}\delta_{kl}\beta_{kl}(\Gamma_{kl}^\dagger+\Gamma_{kl})+\tfrac{1}{2}\sum_{k,l}\delta_{kl}\alpha_{kl}(\Gamma_{kl}^\dagger-\Gamma_{kl}), \tag{4}$$

$$T_q^\dagger = \tfrac{1}{2}\sum_{k,l} q_{kl}\beta_{kl}(\Gamma_{kl}^\dagger+\Gamma_{kl})+\tfrac{1}{2}\sum_{k,l} q_{kl}\alpha_{kl}(\Gamma_{kl}^\dagger+\Gamma_{kl}) \tag{5}$$

with

$$\Gamma_{kl}^\dagger = \alpha_k^\dagger\alpha_{-l}^\dagger, \tag{6}$$

$$q_{kl} = \langle k|\sqrt{\frac{16\pi}{5}}\,r^2 Y_{20}|l\rangle, \tag{7}$$

$$\gamma_{kl} = u_k v_l + u_l v_k,$$

$$\alpha_{kl} = u_k u_l + v_k v_l,$$

$$\beta_{kl} = u_k u_l - v_k v_l. \tag{8}$$

In this approach the pairing energy gap becomes state dependent:

$$\Delta_k = \Delta_0 + q_{kk}\Delta_2. \tag{9}$$

The Fermi energy λ and the gap parameters Δ_0, Δ_2 are found from the BCS equations for the system of A particles:

$$\Delta_0 = \frac{G_0}{2}\left(\Delta_0 \sum_k \frac{1}{E_k} + \Delta_2 \sum_k \frac{q_{kk}}{E_k}\right),$$

$$\Delta_2 = \frac{G_2}{2}\left(\Delta_0 \sum_k \frac{q_{kk}}{E_k} + \Delta_2 \sum_k \frac{q_{kk}^2}{E_k}\right),$$

$$A = \sum_k 2v_k^2 = \sum_k \left(1 - \frac{e_k - \lambda}{E_k}\right). \tag{10}$$

Furthermore it is assumed that the n-th collective state of the system is described by the phonon creation operator:

$$\Gamma_n^\dagger = \sum_{k,l} a_{kl}^{(n)} \Gamma_{kl}^\dagger + \sum_{k,l} b_{kl}^{(n)} \Gamma_{kl} \tag{11}$$

so that

$$[H, \Gamma_n^\dagger] = \hbar\omega_n \Gamma_n^\dagger. \tag{12}$$

Moreover

$$[\Gamma_n, \Gamma_m^\dagger] = \delta_{nm}$$

so that

$$\sum_{k,l} [(a_{kl}^{(n)})^2 - (b_{kl}^{(n)})^2] = 1. \tag{13}$$

The collective energy of the quadrupole oscillations, $\hbar\omega$, may be expressed in terms of the relevant stiffness and mass parameters, C and B respectively, as:

$$\hbar\omega = (C/B)^{1/2}.$$

Similarly one has for the reduced probability of the electromagnetic E2 transition:

$$B(\mathrm{E2}) = (2\sqrt{B \cdot C})^{-1}.$$

In the following the Random Phase Approximation (RPA) is applied to derive both the above mentioned quantities; those in turn are used to obtain final expression for the mass parameter B.

In the RPA method one derives from Eqs (11) — (13) the so called dispersion relation for $\hbar\omega$ which describes the collective energy dependence on the quadrupole force strength κ.

In the case of pure quadrupole-quadrupole oscillations this relation is simply a single non-linear in $\hbar\omega$ equation. In the present case three different modes of excitation are mixed, namely the pairing vibrations of the monopole-pairing and quadrupole-pairing type are added to the quadrupole oscillations — see Eq. (1). The RPA equations obtained in this situation lead to the dispersion relation in the form of a 5×5 matrix determinant. When proton-neutron coupling is allowed (see [10]) one ends up with a 10×10 determinant.

The calculations presented here are made for the interaction between one kind of particle only, i.e. for the 5×5 case.

Because of the assumed adiabaticity of the collective motion one can expand the dispersion relation in powers of $\hbar\omega$ retaining the lowest terms: constant and those proportional to $(\hbar\omega)^2$. This gives an approximate formula for the collective excitation energy. The B(E2) probability is found from Eqs (12) and (13) when similar expansion is performed.

The inertial mass parameter emerging from this approximation can be schematically written as:

$$B_Q = \frac{2\Sigma_3+\delta(\Sigma_3)}{[2\Sigma_1+\delta(\Sigma_1)]^2} \tag{14}$$

where

$$\Sigma_i = \sum_{k,l} \frac{q_{kl}^2\gamma_{kl}^2}{(E_k+E_l)^i}, \quad i = 1, 3, \tag{14a}$$

and the complicated corrections of both sums reduce in the case of pure monopole-pairing forces (i.e. $G_2 = 0$) to the well-known formulae derived by Bès [12] in 1961.

3. *Details of the calculation*

The numerical calculations were performed in the pure anisotropic harmonic oscillator model:

$$V_{\text{h.o.}} = \frac{M}{2}\left[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2\right] \tag{15}$$

with the deformation described by the Nilsson parameter ε:

$$\begin{aligned} \omega_\perp &= \omega_0(\varepsilon)\,(1+\tfrac{1}{3}\,\varepsilon), \\ \omega_z &= \omega_0(\varepsilon)\,(1-\tfrac{2}{3}\,\varepsilon) \end{aligned} \tag{16}$$

and $\omega_0(\varepsilon)$ for each values of deformation is determined from the nuclear volume conservation condition.

The mass parameter B_ε describing the collective motion of the nucleus in the ε-parametrization is related to B_Q given by (14) in the following way:

$$B_\varepsilon = B_Q\left(\frac{dQ}{d\varepsilon}\right)^2, \tag{17}$$

where Q is the macroscopic mass quadrupole moment of the ε-deformed system. This additional factor depends only on the values of the deformation and that of the particle number A.

In the following only the systems of one kind of particles are considered. The particle number A corresponds to the last spherical harmonic oscillator shell $N = 6$ filled in the ratio $x = 1/4$, $1/2$ and $3/4$ ($A = 126$, 140 and 154 respectively).

The BCS equations (10) were solved for $2\sqrt{15 \cdot A}$ levels. For pure monopole-pairing interaction the constant G_0 was taken to be equal $G_0^{(0)} = 15.36$ MeV$/A$. The problem of the normalization of the generalized pairing strength with $G_2 \neq 0$ was worked out on the assumption that the lowest quasi-particle energy $(E_k)_{\min}$ has fixed value for any combination of G_0 and G_2. As the change of the Fermi level λ with G is very small, this energy depends mainly on the Δ_k changes. Single-particle energy gap (Eq. (9)) consists ot two parts; one of them — Δ_2 — is very sensitive to the quadrupole-pairing strength and increases very fast from zero upwards with increasing G_2. The other one, Δ_0, depends mainly on G_0. It means that in order to keep $(E_k)_{\min}$ constant for $G_2 = 0$ and for various $G_2 \neq 0$ one has to change the monopole-pairing strength G_0 from $G_0^{(0)}$ to some $G_0(G_2)$ value.

The specific change of G_0 for a given G_2 value is different for varying particle number value. It is evidently connected with the single-particle structure of the energy levels which are the closest to the Fermi surface. In particular $G_0(G_2)$ should increase with increasing G_2 if the lowest level has $q_{kk} < 0$ and decrease if $q_{kk} > 0$.

Numerical calculations were performed for two deformation points: $\varepsilon = 0.1$ and $\varepsilon = 0.2$. In this paper we are not concerned with the deformation dependence of the investigated effects, so that the problem of re-defining the quadrupole moments entering the definition of the quadrupole-pairing interaction (see Refs [9]–[11]) is not an important one.

4. *Results and discussion*

First we shall investigate a simplified problem with no dynamic corrections arising from the monopole-pairing forces. At the same time the static influence of those forces is retained; it is reflected in the assumption of superconductivity of the investigated system. It means putting $G_0 = 0$ everywhere but in the BCS equations (10) defining the energy gap and Fermi level. Many terms in the described before RPA approach vanish and this allows one to rewrite the formula (14) for the mass parameter in the following way:

$$B_Q = \frac{2\Sigma_3}{(2\Sigma_1)^2} + \Delta B(G_2) \tag{18}$$

the quantities Σ_i given, as before, by (14a).

The explicit dependence on the quadrupole-pairing force strength, G_2, is contained only in the part here referred to as $\Delta B(G_2)$. If a similar assumption as the one described above for the monopole-pairing forces is made for G_2 ($G_2 = 0$ in the basic equation (1) but not in (10)), the second term on the right-hand side of Eq. (18) disappears. In this case the quadrupole-pairing interactions can still change the mass parameter through the modifications of the BCS equations and — in consequence — those of the energy gap.

The first term on the right-hand side of (18) is influenced by G_2 only through the G_2-dependence of the single-particle energy gaps Δ_k (Eq. (9)). It is known from the previous calculations (e.g. [1]–[3]) that the quantity of this type is very sensitive to the energy gap: even a small decrease in Δ results in a relatively big increase in $\tilde{B} = 2\Sigma_3/(2\Sigma_1)^2$.

In the present case the energy gap parameter depends on the single-particle state (Eq. (9)). Because of the energy denominators, the significant contributions to the relevant sums in $\tilde{B}$ come from the levels lying in the vicinity of the Fermi surface λ. In that way

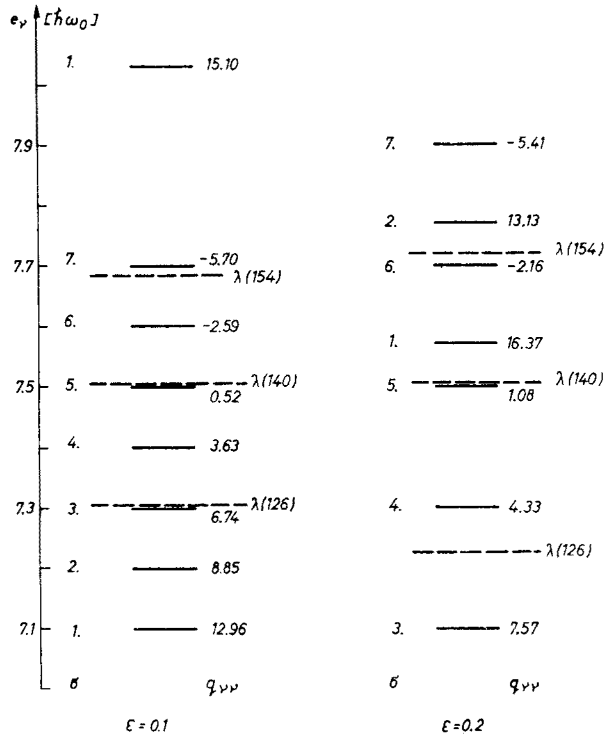

Fig. 1. Single-particle energy levels in $\hbar\omega_0(\varepsilon)$ in the deformed harmonic oscillator model for $\varepsilon = 0.1$ and $\varepsilon = 0.2$. The positions of the Fermi level $\lambda(A)$ for different values of the particle number A are shown as dashed lines. For each level the values of its degeneracy $\sigma = (N - n_z + 1)$ and single-particle quadrupole moment $q_{\nu\nu}$ are given

one can expect that for increasing G_2 values the term $\tilde{B}$ will decrease if the Δ_k values around the Fermi level become bigger. As the dynamic correction $\Delta B(G_2)$ is an increasing and monotonic function of G_2, one can predict the overall influence of the quadrupole-pairing forces on the mass parameter by analysing the properties of the single-particle states in the close neighborhood of the Fermi surface.

Fig. 1 shows the level sequence in the relevant energy region for the deformations $\varepsilon = 0.1$ and $\varepsilon = 0.2$. For each level e_ν the value of the single-particle quadrupole moment, $q_{\nu\nu}$, is given together with the level degeneracy $\sigma = (N - n_z + 1)$. The positions of the Fermi surfaces $\lambda(A)$ for different particle numbers A are indicated by dashed lines.

For $\varepsilon = 0.1$ in the neighborhood of the Fermi surface λ and for $A = 154$ the single--particle levels have negative values of q_{vv}; for $A = 140$ — small positive q_{vv} and for $A = 126$ — positive values of q_{vv}.

For $\varepsilon = 0.2$ one observes the appearance of the single-particle levels with big positive q_{vv} values near the Fermi surfaces for $A = 140$ and 154. Therefore one should expect in those two cases a drastic decrease in the $\tilde{B}$ values, connected with significantly larger values of Δ_k for those particular states when G_2 changes from zero upwards. On the other hand in the remaining cases where $\tilde{B}$ should not change very much with G_2, one expects that the total $B = \tilde{B}+\Delta B(G_2)$ will be more affected by the $\Delta B(G_2)$ dependence on G_2. In the first two cases ($A = 140$, 154 for $\varepsilon = 0.2$) the increase of ΔB is partially cancelled by the decrease of $\tilde{B}$ so that the net result should show weaker dependence on the quadrupole-pairing force strength, G_2.

It should be mentioned that the previously predicted changes of $G_0(G_2)$ for different G_2 and particle number values are qualitatively reproduced. For example, one expects that at $\varepsilon = 0.1$ $G_0(G_2)$ for $A = 154$ should increase ($q_{kk} = -5.7$) and for $A = 126$ — decrease ($q_{kk} = +6.7$) with G_2. The calculation yields ($G_2 = 0$, $G_0(G_2) = 15.36$ MeV/A for both $A = 126$ and 154):

$G_2 = 3 \cdot 10^{-5}$ MeV/fm^4; $G_0(G_2) = 14.72$ MeV/A for $A = 126$ and 15.48 for $A = 154$.

$G_2 = 6 \cdot 10^{-5}$ MeV/fm^4; $G_0(G_2) = 14.19$ MeV/A for $A = 126$ and 15.71 for $A = 154$.

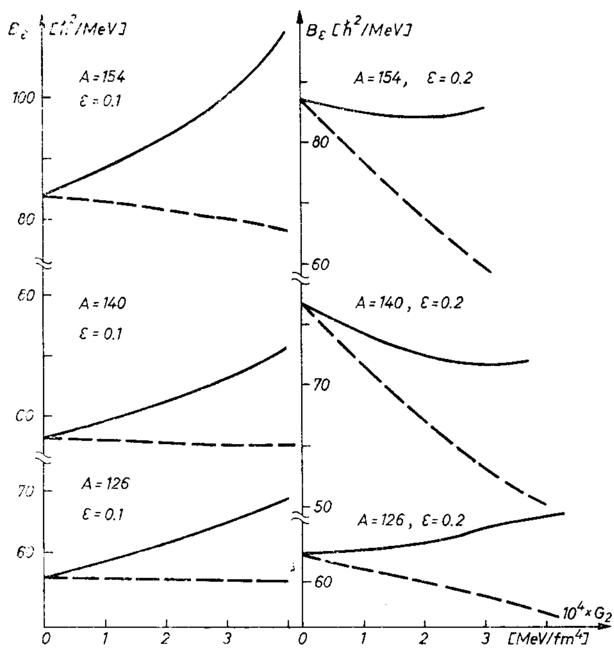

Fig. 2. Inertial mass parameter $B_\varepsilon[\hbar^2/\text{MeV}]$ as a function of quadrupole-pairing force strength G_2[MeV/fm^4], calculated without dynamic contributions from the monopole-pairing forces. The full lines correspond to the formula (18) and the dashed ones give $\tilde{B}$ (see text). Left-hand side is for $\varepsilon = 0.1$, right-hand side corresponds to $\varepsilon = 0.2$. In each case the value of the particle number A is given

It turns out that the computed values of the mass parameter B_ε agree very nicely with the above mentioned predictions. Fig. 2 shows the B_ε values in $\hbar^2$/MeV for different G_2, calculated with (solid line) and without (dashed line) dynamic correction $\Delta B(G_2)$. The left-hand side corresponds to the deformation $\varepsilon = 0.1$ and the right-hand side — to $\varepsilon = 0.2$. In each case the type of changes of the mass parameter is such as previously anticipated.

Unfortunately no such simple analysis is possible when taking into account both terms in the pairing forces. The resulting formula (14) is very complicated and involves many terms with different dependence on the quadrupole-pairing force constant G_2.

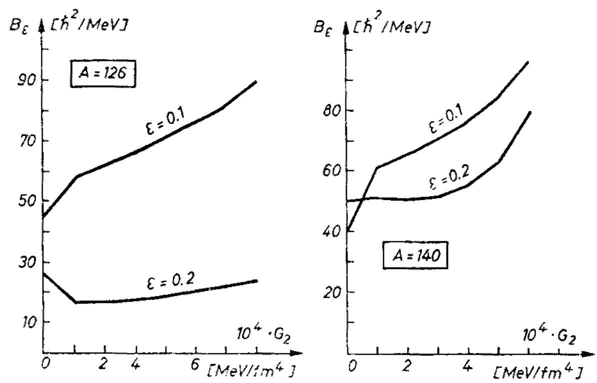

Fig. 3. Inertial mass parameter $B_\varepsilon[\hbar^2/\text{MeV}]$ as a function of $G_2[\text{MeV/fm}^4]$, calculated with full contribution from monopole- and quadrupole-pairing forces. In each case the particle number A and the deformation ε are specified

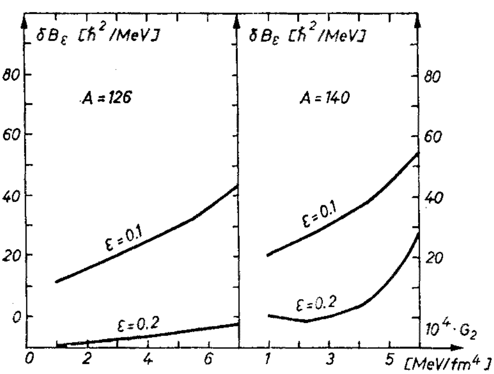

Fig. 4. Correction $\delta B_\varepsilon(G_2) = B_\varepsilon(G_0(G_2), G_2) - B_\varepsilon(G_0^{(0)}, G_2 = 0)$ to the mass parameter arising from the coupling of the quadrupole-pairing forces to other modes of excitation, calculated for different values of the G_2 constant

The numerical results obtained for the mass parameter as a function of G_2 are given in Fig. 3 for two particle number values: $A = 126$ and $A = 154$. As before, for each value of G_2 the monopole-pairing strength G_0 was chosen so as to reproduce the lowest quasi-particle energy obtained with pure monopole-pairing forces.

Fig. 4 shows the correction $\delta B = B_\varepsilon(G_0(G_2), G_2) - B_\varepsilon(G_0^{(0)}), G_2 = 0)$, arising from the coupling between the quadrupole-pairing and other modes of excitation. It is easily seen that these changes of B_ε are different for different systems. One may safely assume that those differences arise from the properties of the single-particle levels in the neighborhood of the Fermi surface. The inclusion of the quadrupole-pairing interactions may lead to smaller or larger values of B_ε as compared with the pure monopole-pairing case. The changes in both directions are too big to be neglected.

All this is in a qualitative agreement with the findings in Ref. [11] in the region of small deformations. In that paper a slightly different approach was used to obtain the mass parameter formula. The single-particle potential and the particle number of the investigated system were also different but those authors concluded, too, that the quadrupole-pairing forces may change the mass parameter B very significantly.

5. *Conclusions*

It was shown in a simple harmonic oscillator model that the modified short-range forces can lead to drastic changes of the microscopic inertial mass parameter values. Therefore, if for some reason the importance of the quadrupole-pairing interaction is established (see, however, Ref. [13]), it will be necessary to include their coupling with other modes of excitation when calculating mass parameters.

The magnitude of the corrections thus introduced depends decisively on the strength of the quadrupole-pairing force as well as on the details of the single-particle level scheme around the Fermi surface (i.e. on the particle number and the deformation of the system). One has to keep in mind that those corrections may change the final result by a factor of two or even more when compared with the standard solution with monopole-pairing forces only.

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