TWO-PARTICLE CORRELATIONS IN NUCLEI*

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We present a general discussion of many-body densities in nuclei, including two-body and one-body densities. The short-range part of the two-body density is calculated in the independent-pair approximation for nuclear matter, and we study effects of Pauli correlations, correlations induced by a hard-core, and correlations coming from the Moszkowski-Scott interaction. These model correlation functions are used as a basis for a comparison of two classes of measurements of the two-body density where one has relatively well-known probes: inelastic sum rules (electron and nucleon) and elastic proton-nucleus scattering at high energies. We carry out calculations for ^{12}C and ^{16}O and emphasize that the low-momentum-transfer part of the sum rules depends critically on the finite extent of the nucleus and the low-lying collective modes. The most promising of these tools for measuring short-range correlations appears to be high-momentum transfer inelastic nucleon sum rules, although there are problems associated with the mesonic degrees of freedom in the nucleus. Other possibilities for measuring two-body densities are discussed very briefly.

1. Introduction

The problem of calculating and measuring the density distribution of nucleons in the nucleus is important and fundamental to nuclear physics. One-body densities are directly calculated, for example, in an independent-particle Hartree-Fock basis and are directly measured by elastic electron scattering. The problem of getting at two-body densities, or two-body correlation functions, is much more vexing in nuclei.[1] While the two-body density is an essential ingredient in all calculations of nuclear energies, the two-body density in nuclei has never been unambigously measured.

The purpose of this paper is to present a general discussion of many-body densities in nuclei, including two-body and one-body densities. We first define these densities and

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[1] For a good review of this topic, see Refs [1, 2].

exhibit some of their general properties. We emphasize that the two-body density contains a great deal of nuclear information. We also show that the two-body density can be decomposed into a sum over one-body transition densities. We demonstrate how the short-range part of the two-body density is calculated in the independent-pair approximation for nuclear matter, and we discuss three model calculations of the two-body density containing: Pauli correlations, correlations induced by the hard-core interaction, and correlations coming from a Moszkowski–Scott interaction, which has an exponential attraction outside of the hard-core repulsion.

We use these model correlation functions as a basis for a discussion of two classes of measurements of the correlation function where one has relatively well-known probes. First, we discuss inelastic sum rules for both electron scattering and very high-energy proton scattering from nuclei. We calculate the inelastic sum rules for nuclear matter and also apply this analysis to the finite nuclei ^{12}C and ^{16}O. We emphasize that the low-momentum-transfer part of the sum rules depends critically on the finite extent of the nucleus and the low-lying collective modes, and hence on long-range correlations. There are significant differences, however, between various types of short-range correlations in the high-momentum-transfer part of the sum rules. The magnitude of the short-range correlation effects are small in the Coulomb sum rule, as first emphasized in the pioneering work of McVoy and van Hove [3]. The short-range correlation effects are three times as large in the nucleon sum rule, and of course the elementary cross sections are much larger in this case. One may be able to see measurable effects with high energy protons. We emphasize that meson production may place a fundamental restriction on our ability to measure the sum rules, and indeed to describe nuclei in terms of elementary nucleons.

The second measurement process we discuss is that of high energy *elastic* proton scattering from nuclei as described in Glauber's multiple scattering approach [4]. Here the two-body density and two-body correlations affect the double and higher order multiple scattering processes. The overall effect of short-range correlations on the elastic differential cross section are small, however, as has been noted by other authors in this field.

In fact, many of the results presented in this paper have been discussed by other authors in one context or another. Here we try to take a unified approach to the problem of correlations in nuclei, presenting their general properties, discussing their role in nuclear structure, calculating some model two-body densities, and then analyzing in some detail two possible measurements of these densities where one has relatively well-known probes.

Section 2 contains the general discussion of nuclear densities. In Section 3 we discuss the theory and measurement of one-body densities very briefly. In Section 4, we discuss the theory of two-body densities and carry out the calculations of the model two-body densities for nuclear matter discussed above. Section 5 contains a discussion of measurement of two-body densities, both through the inelastic sum rules and through elastic hadron-nucleus scattering at high energies. Section 6 is a very brief discussion of some other possibilities for measuring the two-body densities, and Section 7 is a short summary.

2. *Nuclear densities*

We proceed from the assumption that the nucleon coordinates provide a complete set of variables for a description of the nucleus, and thus neglect any explicit dependence on the mesonic degrees of freedom. We further assume that there is a ground-state wave function $\Phi_0(\boldsymbol{x}_1, \dots, \boldsymbol{x}_A)_{\alpha_1 \dots \alpha_A}$ for the nucleus where the subscripts α denote appropriate spin and isospin components for the identical nucleons, and we define the A-body nuclear density as the square of this wave function

$$\varrho^{(A)}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_A) = |\Phi_0(\boldsymbol{x}_1, \dots, \boldsymbol{x}_A)|^2. \tag{2.1}$$

The appropriate spin sums are implied. This is the probability for finding the nucleus in a certain spatial configuration, and is a symmetric function of the coordinates[2]. Often one does not need all of the nuclear information contained in this expression, and it is useful to define inclusive densities

$$\varrho^{(n)}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \equiv \int d\boldsymbol{x}_{n+1} \dots d\boldsymbol{x}_A \varrho^{(A)}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_A) \tag{2.2}$$

where A-n of the nucleon coordinates are integrated out. For example, the two-body density is defined by integrating over all-but-two of the coordinates

$$\varrho^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \int d\boldsymbol{x}_3 \dots d\boldsymbol{x}_A |\Phi_0(\boldsymbol{x}_1, \dots, \boldsymbol{x}_A)|^2 \tag{2.3}$$

and the one-body densities by integrating over all-but-one coordinate

$$\varrho^{(1)}(\boldsymbol{x}_1) = \int \varrho^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) d\boldsymbol{x}_2. \tag{2.4}$$

Note that here

$$\int \varrho^{(1)}(\boldsymbol{x}) d\boldsymbol{x} = 1. \tag{2.5}$$

The one-body density gives the expectation value of any one-body operator depending on position only according to

$$\langle \Phi_0 | \sum_{i=1}^{A} O(i) | \Phi_0 \rangle = A \int \varrho^{(1)}(\boldsymbol{x}) O(\boldsymbol{x}) d\boldsymbol{x} \tag{2.6}$$

and similarly the two-body density provides the expectation value of any two-body operator

$$\langle \Phi_0 | \sum_{i<j=1}^{A} O(i, j) | \Phi_0 \rangle = \frac{A(A-1)}{2} \int \varrho^{(2)}(\boldsymbol{x}, \boldsymbol{y}) O(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y}. \tag{2.7}$$

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[2] We do not deal here with the full A-body density matrix where the coordinates of the nucleons in the bilinear expression in Eq. (2.1) are evaluated at different points. The full density matrix is needed, for example, in the evaluation of the kinetic energy operator which involves derivatives. The definition in Eq. (2.1) is sufficiently general for the present discussion.

The n-body densities in Eq. (2.2) possess the expansion

$$\begin{aligned}\varrho^{(n)}(1, 2, \ldots, n) = {} & \varrho^{(1)}(1)\varrho^{(1)}(2) \ldots \varrho^{(1)}(n) \\ & + \sum_{\text{all possible pairs of contractions}} [\varrho(1)\varrho(2) \ldots \varrho(n)] \\ & + \sum_{\text{all possible three-body contractions}} [\varrho(1)\varrho(2) \ldots \varrho(n)] \\ & + \ldots \\ & + \sum_{\text{all possible } n\text{-particle contractions.}} [\varrho(1)\varrho(2) \ldots \varrho(n)] \end{aligned} \quad (2.8)$$

The justification for this expansion given by Foldy and Walecka [5] lies in the fact that Eq. (2.8) fulfills all requirements of symmetry and normalization. This expansion is, in fact, only a recursive definition of the n-particle contraction $C(1, 2, \ldots, n)$. It is identically satisfied for $n = 1$

$$\varrho^{(1)}(x_1) = \varrho^{(1)}(x_1) \quad (2.9)$$

and for $n = 2$, Eq. (2.8) defines the pair contraction

$$\varrho^{(2)}(x_1, x_2) = \varrho^{(1)}(x_1)\varrho^{(1)}(x_2) + C^{(2)}(x_1, x_2). \quad (2.10)$$

Note by definition, the pair contraction is symmetric and must vanish when integrated over either of its arguments. For $n = 3$, we have

$$\begin{aligned}\varrho^{(3)}(x_1, x_2, x_3) = {} & \varrho^{(1)}(x_1)\varrho^{(1)}(x_2)\varrho^{(1)}(x_3) + C^{(2)}(x_1, x_2)\varrho^{(1)}(x_3) + C^{(2)}(x_2, x_3)\varrho^{(1)}(x_1) \\ & + C^{(2)}(x_3, x_1)\varrho^{(1)}(x_2) + C^{(3)}(x_1, x_2, x_3) \end{aligned} \quad (2.11)$$

where again $C(1, 2, 3)$ is symmetric and must vanish when integrated over any of its arguments. The extension to Eq. (2.3) is evident. It clearly fulfills the normalization condition.

$$\int \varrho^{(n)}(x_1, x_2, \ldots, x_n)dx_n = \varrho^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) \quad (2.12)$$

since Eq. (2.5) holds by definition and

$$\int C^{(n)}(x_1, x_2, \ldots, x_n)dx_n = 0. \quad (2.13)$$

Note that this last relation also implies that the expansion in Eq. (2.8) correctly reproduces the expectation value of any one-body operator according to Eq. (2.6), and any two-body operator according to Eq. (2.7).

If spin (including isospin) is explicitly included to allow for the possibility of computing matrix elements of spin (isospin)-dependent operators, then the one- and two-body densities become matrices in the spin indices

$$\begin{aligned}\varrho^{(2)}(x_1, x_2)_{\alpha_1\alpha_2;\alpha_1'\alpha_2'} = {} & \int dx_3 \ldots dx_A \Phi_0(x_1, \ldots, x_A)_{\alpha_1\ldots\alpha_A} \\ & \times \Phi_0^*(x_1, \ldots, x_A)_{\alpha_1'\alpha_2'\alpha_3\ldots\alpha_A}, \end{aligned} \quad (2.14)$$

$$\varrho^{(1)}(x_1)_{\alpha_1;\alpha_1'} = \int \varrho^{(2)}(x_1, x_2)_{\alpha_1\alpha_2;\alpha_1'\alpha_2}dx_2. \quad (2.15)$$

(Repeated indices are summed.) The expansion in Eq. (2.8) is immediately generalized to this case [5]

It is convenient to also have a definition of the same densities in second quantization [6].

$$A(A-1)\varrho^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)_{\alpha_1\alpha_2;\alpha_1'\alpha_2'} = \langle \Psi_0 | \hat{\psi}^\dagger_{\alpha_1'}(\boldsymbol{x}_1)\hat{\psi}^\dagger_{\alpha_2'}(\boldsymbol{x}_2)\hat{\psi}_{\alpha_2}(\boldsymbol{x}_2)\hat{\psi}_{\alpha_1}(\boldsymbol{x}_1) | \Psi_0 \rangle, \quad (2.16)$$

$$A\varrho^{(1)}(\boldsymbol{x}_1)_{\alpha_1;\alpha_1'} = \langle \Psi_0 | \hat{\psi}_{\alpha_1'}(\boldsymbol{x}_1)\hat{\psi}_{\alpha_1}(\boldsymbol{x}_1) | \Psi_0 \rangle. \quad (2.17)$$

These expressions provide the appropriate expectation values of one-body and two-body operators. Here $|\Psi_0\rangle$ is the exact ground state of the target in second quantization.

By taking a Fourier transform, we can discuss densities in momentum space. One- and two-body *form factors* are just expectation values of special one- and two-body operators

$$\tilde{\varrho}^{(1)}(\boldsymbol{q}) \equiv \int e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\varrho^{(1)}(\boldsymbol{x})d\boldsymbol{x} = \tilde{\varrho}^{(1)}(-\boldsymbol{q})^*, \quad (2.18)$$

$$\tilde{\varrho}^{(2)}(\boldsymbol{q}_1, \boldsymbol{q}_2) \equiv \int e^{i\boldsymbol{q}_1\cdot\boldsymbol{x}_1}\varrho^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)e^{i\boldsymbol{q}_2\cdot\boldsymbol{x}_2}d\boldsymbol{x}_1 d\boldsymbol{x}_2 = \tilde{\varrho}^{(2)}(-\boldsymbol{q}_1, -\boldsymbol{q}_2)^*. \quad (2.19)$$

In momentum space, the normalization conditions, Eqs (2.4), (2.5), become

$$\tilde{\varrho}^{(1)}(0) = 1, \quad (2.20)$$

$$\tilde{\varrho}^{(2)}(\boldsymbol{q}, 0) = \tilde{\varrho}^{(1)}(\boldsymbol{q}). \quad (2.21)$$

It is important to note that all many-body densities can be expressed in terms of one--body static and transition densities. Therefore, if we know all one-body transition densities, we know in principle all many-body densities as well. We may immediately establish this relationship by using the canonical anticommutation relations

$$\{\hat{\psi}_\alpha(\boldsymbol{x}), \hat{\psi}^\dagger_\beta(\boldsymbol{y})\} = \delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \quad \{\hat{\psi}_\alpha(\boldsymbol{x}), \hat{\psi}_\beta(\boldsymbol{y})\} = \{\hat{\psi}^\dagger_\alpha(\boldsymbol{x}), \hat{\psi}^\dagger_\beta(\boldsymbol{y})\} = 0 \quad (2.22)$$

in Eq. (2.16) and then inserting a complete set of states

$$\sum_n |\Psi_n\rangle\langle\Psi_n| = \hat{1}. \quad (2.23)$$

This leads to the expression

$$A(A-1)\varrho^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = A^2 \sum_n \varrho^{(1)}_{0n}(\boldsymbol{x}_1)\varrho^{(1)}_{n0}(\boldsymbol{x}_2) - A\delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\varrho^{(1)}(\boldsymbol{x}_1) \quad (2.24)$$

where the one-body transition densities are defined by

$$A\varrho^{(1)}_{n0}(\boldsymbol{x}) \equiv \langle \psi_n | \hat{\psi}^\dagger_\alpha(\boldsymbol{x})\psi_\alpha(\boldsymbol{x}) | \psi_0 \rangle. \quad (2.25)$$

With a Fourier transform, we can establish the remarkable relation in momentum space

$$A(A-1)\tilde{\varrho}^{(2)}(\boldsymbol{q}_1, \boldsymbol{q}_2) = A^2 \sum_n \tilde{\varrho}^{(1)}_{0n}(\boldsymbol{q}_1)\tilde{\varrho}^{(1)}_{n0}(\boldsymbol{q}_2) - A\tilde{\varrho}^{(1)}(\boldsymbol{q}_1+\boldsymbol{q}_2). \quad (2.26)$$

Thus "exhaustive" knowledge of the one-body transition form factors provides us immediately with all the two-body form factors. Equations (2.26) can be rewritten as

$$A(A-1)\tilde{\varrho}^{(2)}(\boldsymbol{q}_1, \boldsymbol{q}_2) = A(A-1)\tilde{\varrho}^{(1)}(\boldsymbol{q}_1)\tilde{\varrho}^{(1)}(\boldsymbol{q}_2) + A[\tilde{\varrho}^{(1)}(\boldsymbol{q}_1)\tilde{\varrho}^{(1)}(\boldsymbol{q}_2) - \tilde{\varrho}^{(1)}(\boldsymbol{q}_1+\boldsymbol{q}_2)] + A^2 \sum_{n\neq 0} \tilde{\varrho}^{(1)}_{0n}(\boldsymbol{q}_1)\tilde{\varrho}^{(1)}_{n0}(\boldsymbol{q}_2) \quad (2.27)$$

and in coordinate space

$$A(A-1)\varrho^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = A(A-1)\varrho^{(1)}(\boldsymbol{x}_1)\varrho^{(1)}(\boldsymbol{x}_2) + A[\varrho^{(1)}(\boldsymbol{x}_1)\varrho^{(1)}(\boldsymbol{x}_2) - \delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\varrho^{(1)}(\boldsymbol{x}_1)] + A^2 \sum_{n\neq 0} \varrho^{(1)}_{0n}(\boldsymbol{x}_1)\varrho^{(1)}_{n0}(\boldsymbol{x}_2). \quad (2.28)$$

Here the two-body form factor and ground-state density are separated in a sum of three terms

(*i*) a "no-correlation" term in which the two-body form factor (density) factors into a product of one-body form factors (densities) of the ground state,

(*ii*) a "self-correlation" term which is proportional to A and contains only ground-state information,

(*iii*) a true correlation term which is proportional to A^2 and exhibits only information about transitions to excited states.

The last two terms in Eqs (2.27), (2.28) are called the two-particle correlation form factor and density respectively (compare Eq. (2.10))

$$A(A-1)\tilde{C}^{(2)}(\boldsymbol{q}_1, \boldsymbol{q}_2) = A[\tilde{\varrho}^{(1)}(\boldsymbol{q}_1)\tilde{\varrho}^{(1)}(\boldsymbol{q}_2) - \tilde{\varrho}^{(1)}(\boldsymbol{q}_1+\boldsymbol{q}_2)] + A^2 \sum_{n\neq 0} \tilde{\varrho}^{(1)}_{0n}(\boldsymbol{q}_1)\tilde{\varrho}^{(1)}_{n0}(\boldsymbol{q}_2), \quad (2.29)$$

$$A(A-1)C^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = A[\varrho^{(1)}(\boldsymbol{x}_1)\varrho^{(1)}(\boldsymbol{x}_2) - \delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\varrho^{(1)}(\boldsymbol{x}_1)] + A^2 \sum_{n\neq 0} \varrho^{(1)}_{0n}(\boldsymbol{x}_1)\varrho^{(1)}_{n0}(\boldsymbol{x}_2). \quad (2.30)$$

These equations are just Fourier transforms of each other.

Equation (2.30) can also be used to identify "short-range" and "long-range" correlations in nuclei. We rewrite the last term in Eq. (2.30) for an angular momentum $J_0 = 0$ nucleus in a form where the angular dependence of the transition densities appears explicitly

$$A^2 \sum_{n\neq 0} \varrho^{(1)}_{0n}(\boldsymbol{x}_1)\varrho^{(1)}_{n0}(\boldsymbol{x}_2) = 4\pi A^2 \sum_{\substack{\nu\neq 0 \\ E_\nu, J_\nu, M_\nu}} \varrho^{(1)}_{0\nu}(x_1)\varrho^{(1)}_{\nu 0}(x_2) Y^*_{J_\nu M_\nu}(\hat{\boldsymbol{x}}_1) Y_{J_\nu M_\nu}(\hat{\boldsymbol{x}}_2) \quad (2.31)$$

$$= A^2 \sum_{\substack{\nu\neq 0 \\ E_\nu, J_\nu}} (2J_\nu+1)\varrho^{(1)}_{0\nu}(x_1)\varrho^{(1)}_{\nu 0}(x_2) P_{J_\nu}(\cos\theta_{12}) \quad (2.32)$$

where $n = \{\nu, E_\nu, J_\nu, M_\nu, \ldots\}$ is a complete set of quantum numbers characterizing the excited state and θ_{12} is the angle between $\hat{\boldsymbol{x}}_1$ and $\hat{\boldsymbol{x}}_2$. The low-lying ($E_\nu < E_{\min}$) transition densities $\varrho^{(1)}_{0\nu}(x_1)$ with $J_\nu < J_{\min}$ are strongly peaked at the nuclear surface and have

$x_1 \cong x_2 \cong R$. They will give rise to the longest-range correlations in nuclei, which run over the whole nuclear surface. We are therefore led to the (still exact) separation

$$A(A-1)C^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = A(A-1)\left[C^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{\text{long}} + C^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{\text{short}}\right] \tag{2.33}$$

where the long-range part

$$A(A-1)C^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{\text{long}} \equiv A\varrho^{(1)}(\boldsymbol{x}_1)\varrho^{(1)}(\boldsymbol{x}_2) + A^2 \sum_{\substack{\nu \neq 0 \\ J_\nu \leqslant J_{\min}; E_\nu \leqslant E_{\min}}} \varrho_{0n}^{(1)}(\boldsymbol{x}_1)\varrho_{n0}^{(1)}(\boldsymbol{x}_2) \tag{2.34}$$

has to be determined from experiment or estimated in a collective nuclear model, and the remaining short-range part

$$A(A-1)C^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2)^{\text{short}} \equiv -A\delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\varrho^{(1)}(\boldsymbol{x}_1) + A^2 \bar{\sum} \varrho_{0n}^{(1)}(\boldsymbol{x}_1)\varrho_{n0}^{(1)}(\boldsymbol{x}_2) \tag{2.35}$$

(Here $\bar{\Sigma}$ is sum over the complement of the region in Eq. (2.34)) is reasonably well-known from nuclear-matter calculations.

3. One-body density: theory and measurement

If we have a model or theory of the nucleus, we can compute the one-body density through Eqs (2.3), (2.4), or (2.17). In an independent-particle model of the nucleus, the one-body density is just average of the square of the single particle wave functions of the occupied states. Knowledge of the one-body density from any source is enough to provide the expectation value of any one-body operator. The one-body *charge* density, for example, can be found from elastic electron scattering which directly measures the form factor

$$F(q) = \int j_0(qx)\varrho_c^{(1)}(\boldsymbol{x})d\boldsymbol{x}. \tag{3.1}$$

The one-body density evidently does not determine the nuclear wavefunction. In fact, since all the other $A-1$ coordinates are integrated over, the one-body density just determines an average one-body property of the nucleus. It is presumably always possible to construct a one-body potential which, in an independent-particle model, will yield any appropriate one-body density. For this reason, the analysis of any process involving only one-body densities can never provide a definitive basis for discussing two-particle correlations in nuclei[3].

4. Two-body density: theory

A. Pauli Correlations

We first briefly review the well-known result for correlations arising from the Pauli exclusion principle in a uniform system of non-interacting fermions with a spin-degeneracy of g (note for nuclear matter with both spin-1/2 protons and neutrons we would have $g = 4$). We use plane waves in a large volume Ω satisfying periodic boundary conditions

[3] In this connection, see Ref. [7].

for the single-particle wave functions. In this simple model, the structure of the terms in Eq. (2.30) can be studied explicitly. The particle density is given by

$$\frac{A}{\Omega} = \frac{gk_F^3}{6\pi^2} \tag{4.1}$$

and we find

$$\begin{aligned} -\delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_2)A\varrho^{(1)}(\boldsymbol{x}_1) &= -\delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\langle\Psi_0|\hat{\psi}^\dagger(\boldsymbol{x}_1)\hat{\psi}(\boldsymbol{x}_1)\,|\Psi_0\rangle \\ &= -\delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\,(gk_F^3/6\pi^2) \end{aligned} \tag{4.2}$$

while

$$\begin{aligned} A^2\sum_{n\neq 0}\varrho_{0n}^{(1)}(\boldsymbol{x}_1)\varrho_{n0}^{(1)}(\boldsymbol{x}_2) &= \sum_{n\neq 0}\langle\Psi_0|\hat{\psi}^\dagger(\boldsymbol{x}_1)\hat{\psi}(\boldsymbol{x}_1)\,|\Psi_n\rangle\,\langle\Psi_n|\hat{\psi}^\dagger(\boldsymbol{x}_2)\hat{\psi}(\boldsymbol{x}_2)\,|\Psi_0\rangle \\ &= \delta^{(3)}(\boldsymbol{x}_1-\boldsymbol{x}_2)\left(\frac{gk_F^3}{6\pi^2}\right) - g\left[\left(\frac{k_F^3}{6\pi^2}\right)\frac{3j_1(k_F|\boldsymbol{x}_1-\boldsymbol{x}_2|)}{k_F|\boldsymbol{x}_1-\boldsymbol{x}_2|}\right]^2. \end{aligned}$$

Inserting these results in Eq. (2.30), the delta functions cancel yielding for the two-body correlation density the familiar result

$$A(A-1)C^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = -\frac{1}{g}\left(\frac{A}{\Omega}\right)^2\left[\frac{3j_1(k_F|\boldsymbol{x}_1-\boldsymbol{x}_2|)}{k_F|\boldsymbol{x}_1-\boldsymbol{x}_2|}\right]^2 + \frac{A}{\Omega^2} \tag{4.4}$$

$$\equiv \frac{A(A-1)}{\Omega^2}\,[g_{FG}(|\boldsymbol{x}_1-\boldsymbol{x}_2|)-1] \tag{4.5}$$

where we have introduced a correlation function

$$g_{FG}(\xi) \equiv \frac{A}{A-1}\left[1-\frac{1}{g}\left(\frac{3j_1(k_F\xi)}{k_F\xi}\right)^2\right]. \tag{4.6}$$

The Fourier transform of this correlation function can be evaluated explicitly with the result, which we shall need shortly

$$A(A-1)\tilde{C}^{(2)}(\boldsymbol{q}, -\boldsymbol{q}) = A\delta_{\boldsymbol{q},0} - A\left[1-\tfrac{3}{2}\left(\frac{q}{2k_F}\right)+\tfrac{1}{2}\left(\frac{q}{2k_F}\right)^3\right]\theta\left(1-\frac{q}{2k_F}\right). \tag{4.7}$$

B. Independent pair approximation

The main source of our knowledge of the two-body density in nuclei comes from the study of the properties of nuclear matter, an idealization, approximately achieved in the interior of large nuclei. If we take $N = Z$, turn off all Coulomb effects, and then let $A \to \infty$ in the semi-empirical mass formula, we describe a material with the properties: $E/A = -15.75$ MeV; $k_F \cong 1.42\,\text{fm}^{-1}$; and $\varrho \cong 0.19$ nucleons/fm^3 where the nuclear matter density has been estimated from elastic electron scattering results. A great deal of effort over the past two decades has gone into the theoretical understanding of these properties. In order to calculate the binding energy of nuclear matter, we need the two-body density, since we must compute the expectation value of the potential, a two-body operator. The starting point for the study of nuclear matter is the independent-pair approximation representing the theory developed by Brueckner, Bethe, Goldstone and others. The basic idea is the following [6]: The nuclear potential is very strong at small interparticle separations. It is essential to have the correct two-body density at these small separations. To calculate these densities, one takes the interaction between the pair of particles into account exactly and solves the Schrödinger equation for the pair in question. The effect of all the other particles in the medium is taken into account in an average fashion: first, through the Pauli principle which prevents the interacting pair from going into the occupied levels and second, through the single-particle spectrum which represents the average interaction with the other particles. The mathematical expression of this "independent-pair approximation" is the Bethe-Goldstone equation

$$\Psi_{K_1K_2}(1,2) = \phi_{K_1}(1)\phi_{K_2}(2) + \sum_{K_1'>k_F}\sum_{K_2'>k_F} \frac{\phi_{K_1'}(1)\phi_{K_2'}(2)\,\langle K_1'K_2'|V|\Psi_{K_1K_2}\rangle}{E-E_{K_1'K_2'}}, \tag{4.8}$$

$$E-E_{K_1K_2} = \langle K_1K_2|V|\Psi_{K_1K_2}\rangle. \tag{4.9}$$

Here $\phi_K = (1\sqrt{\Omega})\, e^{iK\cdot x}\eta_\lambda$ are the single-particle solutions for a uniform medium. η_λ is the appropriate spin (isospin) wave function. Equations (4.8)–(4.9) are simply the two-particle Schrödinger equation written in integral form, where the sum over virtual states omits the contribution of the states occupied by the other nucleons inside the Fermi sea. Once the Bethe-Goldstone wavefunction is determined from Eq. (4.8), the energy shift of a pair of particles due to the interaction V can be calculated from Eq. (4.9). Since the interaction potential depends only on the relative coordinate, the center-of-mass momentum remains a good quantum number for the interacting pair and a two-particle wavefunction takes the form (assuming V is spin-independent)

$$\Psi_{K_1K_2}(1,2) = \frac{1}{\sqrt{\Omega}}\, e^{iP\cdot X}\, \frac{1}{\sqrt{\Omega}}\, \psi_K(x)\eta_{\lambda_1}(1)\eta_{\lambda_2}(2) \tag{4.10}$$

where

$$P = K_1+K_2, \qquad X = \tfrac{1}{2}(x_1+x_2),$$
$$K = \tfrac{1}{2}(K_1-K_2), \qquad x = x_1-x_2. \tag{4.11}$$

As an example, we show some solutions to the Bethe-Goldstone equation. Figure 1 shows the S-wave part of the wavefunction for a pair with $P = 0$ and $K = 0$. If there were no interaction, this wavefunction would be $j_0(Kr)$. Here the pair is assumed to interact only through a pure hard-core potential which extends out to a distance $c = k_F b \equiv 0.6$ where b is the hard-core diameter. Because all other degenerate levels are already occupied by other nucleons, there can be no phase shift in the relative wavefunction at large distances and the Bethe-Goldstone wavefunction must "heal" to the unperturbed wavefunction over a distance of the order of k_F^{-1}. It is clear from Fig. 1 that the wavefunction vanishes at the hard-core and goes over to the unperturbed wavefunction at a "healing distance" of about $k_F r = 1.9$. Also shown on this figure is the dimensionless interparticle distance

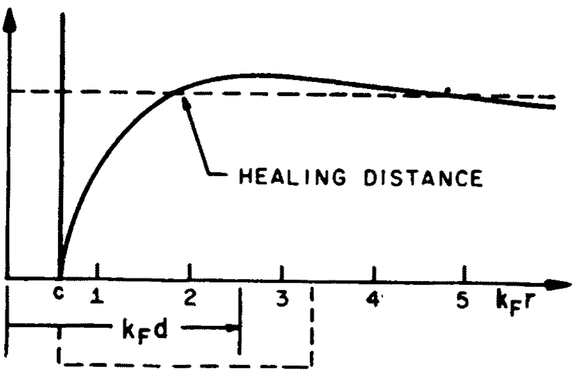

Fig. 1. The S-wave part of the Bethe–Goldstone wave-function for a pair with $\boldsymbol{P} = 0$, $\boldsymbol{K} = 0$ and hard-core interaction in nuclear matter [6]. Here $k_F d \equiv (3\pi^2/2)^{1/3}$ is the average interparticle spacing

defined by the relation $A/\Omega = 1/d^3 = 2k_F^3/3\pi^2$. Figure 2 shows the deviation of the true wavefunction $(k_F/K)u(Kr)$ where $\psi_K^{l=0}(r) \equiv u(Kr)/Kr$ from the unperturbed wavefunction $(k_F/K)\sin Kr$ for several other values of center-of-mass and relative momentum (given in units of k_F) of the pair. These curves are due to Gomes et al. [8]. Note the similarity

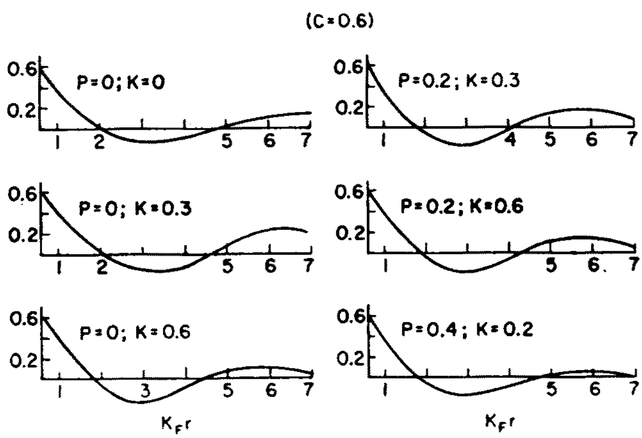

Fig. 2. The deviation of the exact S-wave wave-function $(k_F/K)u(Kr)$ from the unperturbed wavefunction $(k_F/K)\sin Kr$ for a pair of particles with hard-core interaction in nuclear matter for several values of the center-of-mass and relative momentum (given in units of k_F; here $\boldsymbol{P} \equiv 1/2(\boldsymbol{K}_1+\boldsymbol{K}_2)$) [8]

in the healing distance and the bulge of the hard-core wavefunction over the unperturbed value. Figure 3 is taken from a paper by Moniz and Nixon [9] and compares the hard-core wavefunction with their own two-particle wavefunction generated from the "standard hard-core attractive-exponential potential" of Moszkowski and Scott [10] for different values of K. These curves were calculated in the reference spectrum approximation of Bethe et al. [11] whereby particles above the Fermi sea are assumed to have the free particle spectrum while holes below the Fermi sea are assumed to move with an effective mass in an attractive potential. (Here a value $k_F = 1.36\ \text{fm}^{-1}$ is used for nuclear matter). Also

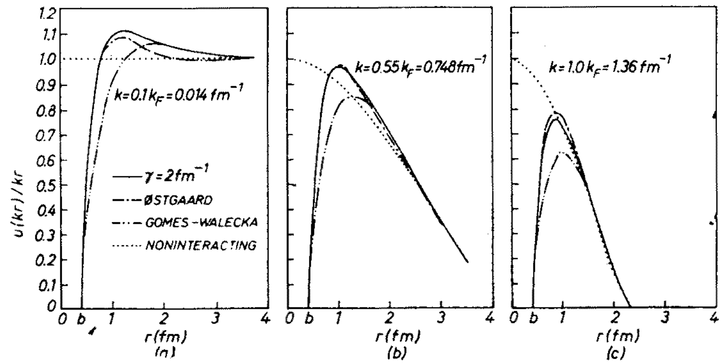

Fig. 3. S-wave relative wavefunctions for a pair of nucleons with $\boldsymbol{P} = 0$ in nuclear matter under various assumptions about the interparticle interaction (see text) [9]

shown are the solutions of Østgaard et al. who carried out a detailed solution to the Bethe--Goldstone equation with realistic inter-nucleon potentials [12].

Knowing the Bethe-Goldstone wavefunction, we can immediately construct the two--body density in the independent-pair approximation. We assume a spin-isospin degeneracy g for each momentum state ($g = 4$ for nuclear matter). Because of the short-range of the singular part of the nuclear force, the strongest correlations will be in S-states, as we have seen. For all unlike pairs of particles, we shall thus use the S-wave correlated Bethe-Goldstone wavefunctions. For like particles, for example, p↑ and p↑, relative S-states are excluded by the Pauli principle and we shall simply assume antisymmetrical plane waves for the pair. Thus, the two-body density takes the form

$$A(A-1)\varrho^{(2)}(1,2) = g \sum_{\boldsymbol{K}_1 \neq \boldsymbol{K}_2}^{k_F} \sum^{k_F} |\phi^{\mathscr{A}}_{\boldsymbol{K}_1\boldsymbol{K}_2}(1,2)|^2 + g(g-1) \sum_{\boldsymbol{K}_1}^{k_F} \sum_{\boldsymbol{K}_2}^{k_F} \frac{|\Psi^{\mathrm{BG}}_{\boldsymbol{K}_1\boldsymbol{K}_2}(1,2)|^2}{N_{\boldsymbol{K}_1\boldsymbol{K}_2}}. \tag{4.12}$$

The degeneracy factors in front are the number of like and unlike pairs for given values of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$. Here, $N_{\boldsymbol{K}_1\boldsymbol{K}_2}$ is a normalization constant defined by

$$\int dx_2 |\psi^{\mathrm{BG}}_{\boldsymbol{K}_1\boldsymbol{K}_2}(1,2)|^2 \equiv \frac{N_{\boldsymbol{K}_1\boldsymbol{K}_2}}{\Omega} \equiv \frac{1}{\Omega}\left(1 + \frac{\eta_{\boldsymbol{K}_1\boldsymbol{K}_2}}{\Omega k_F^3}\right) \tag{4.13}$$

and is inserted so that the two-body density obeys the normalization condition of Eq. (2.4). Note that for nuclear matter we have from translational invariance

$$\varrho^{(1)} = \frac{1}{\Omega} \equiv \varrho. \tag{4.14}$$

We may define a two-body correlation function as in Eq. (2.10)

$$\varrho^{(2)}(1,2)-\varrho^{(1)}(1)\varrho^{(1)}(2) = C(1,2). \tag{4.15}$$

This has the property that it vanishes identically if integrated over either one of the coordinates. For a uniform medium, this correlation function must take a form which depends only on the relative coordinate, and we define

$$C(1,2) \equiv \varrho^2[g(|\boldsymbol{x}_1-\boldsymbol{x}_2|)-1]. \tag{4.16}$$

Using the identity

$$A(A-1) \equiv g\sum_{\boldsymbol{K}_1\neq\boldsymbol{K}_2}^{k_F}\sum^{k_F} + g(g-1)\sum_{\boldsymbol{K}_1}^{k_F}\sum_{\boldsymbol{K}_2}^{k_F} \tag{4.17}$$

we may write the two-body correlation function for nuclear matter in the independent-pair approximation with S-wave correlations in the following form:

$$A(A-1)C(1,2) = \frac{g}{\Omega^2}\sum_{\boldsymbol{K}_1\neq\boldsymbol{K}_2}^{k_F}\sum^{k_F}(|\Omega\phi^{\mathscr{A}}_{\boldsymbol{K}_1\boldsymbol{K}_2}(1,2)|^2-1)$$

$$+\frac{g(g-1)}{\Omega^2}\sum_{\boldsymbol{K}_1}^{k_F}\sum_{\boldsymbol{K}_2}^{k_F}\left(\frac{|\Omega\Psi^{\mathrm{BG}}_{\boldsymbol{K}_1\boldsymbol{K}_2}(1,2)|^2}{N_{\boldsymbol{K}_1\boldsymbol{K}_2}}-1\right) \tag{4.18}$$

$$\equiv A(A-1)\,[C_{\mathrm{FG}}+C_{\mathrm{BG}}]. \tag{4.19}$$

The first term is the correlation function one would get for a pure, non-interacting Fermi gas and comes from the correlation of like pairs. It is precisely the result given in Eq. (4.4). The second term is the result of S-wave correlations between unlike pairs. We assume only S-wave correlations in the relative wavefunction (Eq. (4.10)), that is

$$\psi^{\mathrm{BG}}_{\boldsymbol{K}}(\boldsymbol{r}) \cong \psi^{l=0}_{\boldsymbol{K}}(\boldsymbol{r})+\sum_{l\neq 0}^{\infty} j_l(Kr)i^l(2l+1)P_l(\cos\theta_{\hat{K}\hat{r}}) \tag{4.20}$$

and, further, that the relative wavefunction is independent of the total momentum of the pair (this is a good approximation — see Fig. 2). Using the following relation

$$\int_0^1\frac{d\boldsymbol{K}_1}{4\pi/3}\int_0^1\frac{d\boldsymbol{K}_2}{4\pi/3} = 24\int_0^1 K^2(1-\tfrac{3}{2}K+\tfrac{1}{2}K^3)dK\,\frac{d\Omega_K}{4\pi} \tag{4.21}$$

which is the distribution of relative momenta $\boldsymbol{K}=\frac{1}{2}(\boldsymbol{K}_1-\boldsymbol{K}_2)$ in a Fermi gas, we can write the contribution of the unlike pairs in the following manner

$$A(A-1)C_{\mathrm{BG}} = \left(\frac{g-1}{g}\right)\left(\frac{A}{\Omega}\right)^2\Bigg[24\int_0^1 K^2(1-\tfrac{3}{2}K+\tfrac{1}{2}K^3)dK\,\frac{d\Omega_K}{4\pi}$$

$$\times(|\psi^{l=0}_K(x)|^2-|j_0(Kx)|^2)\Bigg]+\left(\frac{g-1}{g}\right)\left(\frac{A}{\Omega}\right)^2\left[-\frac{\eta}{\Omega k_F^3}\right] \tag{4.22}$$

here $x = k_F r$ and η is the average value of the normalization constant

$$\eta \equiv \langle \eta_{K_1K_2} \rangle \equiv \int_0^1 \frac{dK_1}{4\pi/3} \int_0^1 \frac{dK_2}{4\pi/3} \eta_{K_1K_2}. \tag{4.23}$$

Note that the final terms in Eqs (4.4) and (4.22) are of order $1/\Omega$ with respect to the first terms. These terms are only important when taking the $q = 0$ Fourier component of these expressions. We have calculated the pair correlation function from these expressions for two different nucleon-nucleon interactions. First, we have taken a pure hard-core interaction. A convenient parametrization of the hard-core Bethe-Goldstone wave functions (Fig. 2) is given in Ref. [8] as

$$\psi_K^{l=0}(r) = \frac{\sin(Kr)}{Kr} - \frac{\sin(Kb)}{Kr} \frac{\left[1 - \frac{2}{\pi}\,\mathrm{Si}(\beta r)\right]}{\left[1 - \frac{2}{\pi}\,\mathrm{Si}(\beta b)\right]} \tag{4.24}$$

where b is the hard-core radius, $\beta = 1.10\, k_F$, and $\mathrm{Si}(x)$ is the sine-integral function. This gives a remarkably good fit to the healing length and to the first bulge past the healing length, for all values of the center-of-mass and relative momentum of the interacting pair.

Second, we have used the "standard hard-core potential" of Moszkowski and Scott [10]

$$V(r) = \begin{cases} \infty & r < b \\ -v_0 e^{-\mu(r-b)} & r > b \end{cases} \tag{4.25}$$

where $v_0 = 260$ MeV, $b = .4$ fm, and $\mu = 2.08\ \mathrm{fm}^{-1}$. This potential gives an average over the $^1S_0$ and $^3S_1$ effective ranges and a bound state at zero energy. The wavefunction calculated by Moniz and Nixon [9] with this potential in first approximation using the reference spectrum method is

$$Kr\psi_K^{l=0}(r) = \sin(Kr) - \sin(Kb)e^{-\gamma(r-b)} + \frac{m^* M v_0}{2\gamma\hbar^2}\left[e^{-\mu(r-b)}\zeta(r) - e^{-\gamma(r-b)}\zeta(b)\right] \tag{4.26}$$

where

$$\zeta(r) = \frac{(\gamma-\mu)\sin Kr - K\cos(Kr)}{(\mu-\gamma)^2 + K^2} + e^{-\gamma(r-b)}\sin(Kb)\frac{2\gamma}{\mu(\mu+2\gamma)} + \frac{(\mu+\gamma)\sin(Kr) + K\cos(Kr)}{(\mu+\gamma)^2 + K^2} \tag{4.27}$$

with $m^* = 1$ and $\gamma = 2\ \mathrm{fm}^{-1}$. The pair correlation function calculated from these expressions is shown in Fig. 4. Since the Moszkowski-Scott [10] potential contains an attraction outside the hard core, the wave function is pulled in, the healing distance becomes smaller, and the bulge over the unperturbed wavefunction occurs at a smaller interparticle separation than with just the hard-core interaction. The calculations have been performed for two values of the Fermi momentum $k_F = 1.42\ \text{fm}^{-1}$ (nuclear matter) and $k_F = 1.12\ \text{fm}^{-1}$ (^{12}C). To get the effective correlation function between two nucleons we have to take the

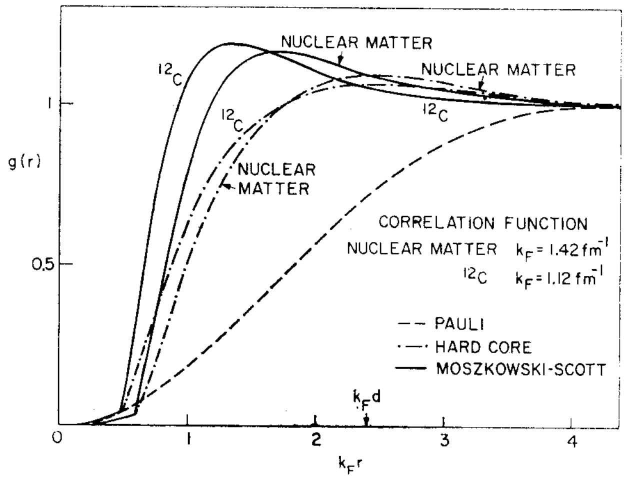

Fig. 4. The interacting-pair and Pauli correlation functions (see Eqs (4.4), (4.16), (4.19), (4.22), (4.28)) for several types of correlations and nuclear matter at differential densities. The final $g(r)$ is the statistical average of these curves (Eq. (4.28))

statistical average of Pauli and interacting pair correlation function, as indicated by Eqs (4.16)-(4.19), i. e.

$$g(r) = \left[\frac{g-1}{g}\right] g(r)_{\text{interacting pair}} + \left[\frac{1}{g}\right] g(r)_{\text{Pauli}} \tag{4.28}$$

where $g = 2$ for p–p correlations and $g = 4$ for N–N correlations. Since the bulge in the interacting pair part of the correlation function occurs just where the Pauli contribution is rising most steeply, the net correlation function for nuclear matter is a smooth, monotonically increasing function.

Using these densities and a hard-core attractive-square-well potential adjusted to fit the singlet effective range and scattering length, the binding energy per particle in nuclear matter can be computed as a function of the density. This calculation is discussed in detail in Chapter 11 of Fetter and Walecka [6]. This is not meant to be a definitive calculation of the properties of nuclear matter, but merely illustrates that the two-body density which we have computed is capable of explaining the saturation properties of nuclear matter. With a more sophisticated and detailed calculation of the two-body densities, one can get very close to the equilibrium properties of nuclear matter. Thus, we do have a basic theoretical understanding of the two-body density at small interparticle separations in nuclei.

The small healing distance provides a justification of our independent-pair approximation, for by the time the two-particle separation has reached the average interparticle spacing the two-particle wavefunction has healed and the particles move as if in plane-wave states. This small healing distance thus suppresses the contributions of higher clusters to the energy. The small healing distance also provides a basis for understanding the success of the single-particle shell model in nuclei. Most of the time the particle moves through the nucleus as if in a plane-wave state, and the only effect of the strong short-range interactions between the particles is to modify the two-particle density at small interparticle separations. For example, the one-body density in nuclear matter (Eq. (4.14)) is a smooth well-behaved function.

The short-range anti-correlation of nucleons in the nucleus has other basic theoretical implications. In fact, in a sense, it can be said that the short-range anti-correlation between nucleons, and the small healing distance in the nucleus are responsible for most of nuclear physics as we know it. Not only do we understand the equilibrium properties of nuclear matter in the independent-pair approximation, and the success of the shell model, but also the suppression of many-body *forces* in nuclei, since many nucleons can never get close enough together so that one nucleon will modify the meson force fields between the others[4]. For a similar reason, we can understand why nuclei behave to an amazing extent as collections of individual particles with the properties of free nucleons, for the anti-correlations keep the nucleons apart and suppress the contribution of meson exchange currents to weak and electromagnetic processes[5].

Before leaving this section, we would like to re-emphasize that the two-body density in Eq. (2.3) is actually a very complex object. We have here concentrated on the behaviour of this density at small interparticle separations, for it is the two-body density in this region that is necessary for computing the energy. There are also contributions to the two-body density from long-range separations in nuclei. For example, in a deformed nucleus, if one nucleon is at one bulge of the nucleus, then there will be a positive probability for finding the other nucleon at the bulge on the other side of the nucleus and a vanishing probability for finding it in the depleted regions of nuclear density at the nuclear surface. In nuclear matter, there will be contributions of long-range correlations to the two-body density coming from the collective modes of excitation of the medium. In graphical language, there are modes of excitation in the medium which must be discussed by summing bubble diagrams, for example, rather than just the ladder diagrams we have concentrated on to get the binding energies. Thus, we can expect that our independent-pair approximation describes the two-body density correctly at small interparticle separations and presumably describes the oscillations in this density which occur with spatial variations over distances of the interparticle separation or smaller. There will also be long-range spatial correlations in the two-body density which we will not be able to describe in the independent-pair approximation but for which we need some models of the collective oscillations of the nucleus.

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[4] See, e.g. Ref. [13].

[5] In fact, the anti-correlation means that the only important exchange current is that coming from one-pion exchange, and calculations indicate that this has small electromagnetic effects.

5. *Two-body density: measurement*

Let us turn now to the question of how we can obtain a direct experimental measurement of the two-body density. We will first discuss the theory of two phenomena which directly involve the two-particle density: first, the general analysis of inelastic sum rules and second, the analysis of elastic hadron-nucleus scattering at high energy.

A. Inelastic sum rule

Here we have two specific processes in mind. The first is inelastic electron scattering (e, e′) through the Coulomb interaction. This is a clean analysis since the Coulomb contribution to the sum rule can be in principle separated by a Rosenbluth plot, although to our knowledge this has never been done. The second application is inelastic nucleon scattering (N, N′) at high energies in the impulse approximation (cf. Refs [14–16]). Here we make the simplifying assumption that the projectile interacts with a spin-isospin independent amplitude with the nucleons inside the nucleus, and we neglect the distortion of the incident projectile. In fact, this distortion can probably be included in the analysis without essentially changing the result as we shall see. This analysis may be applicable to experiments at LAMPF with several-hundred MeV protons.

In both these cases, the double differential cross section for scattering from the nucleus takes the form [6]

$$\frac{1}{\sigma_{\mathrm{p}}}\frac{d^2\sigma}{d\Omega' d\varepsilon'} \equiv S(\boldsymbol{q}, \omega) = \sum_n |\langle \Psi_n | \hat{\varrho}(-\boldsymbol{q}) | \Psi_0 \rangle|^2 \delta(\omega - (\varepsilon_n - \varepsilon_0)), \tag{5.1}$$

$$\hat{\varrho}(-\boldsymbol{q}) \equiv \int e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \hat{\varrho}(\boldsymbol{x}) d\boldsymbol{x}. \tag{5.2}$$

Here σ_{p} is the point cross section for the scattering of the projectile on a free target particle, $\boldsymbol{q}$ is the momentum transfer, and ω is the energy transfer to the target by the projectile. $\hat{\varrho}(\boldsymbol{x})$ is the nuclear density operator. For Coulomb scattering of electrons (we refer to the response function in this case as $C(q)$), $\hat{\varrho}$ is the charge density $\hat{\psi}^\dagger \frac{1}{2}(1+\tau_3)\hat{\psi}$. In the case of the high-energy scattering of nucleons or other hadrons in the impulse approximation with a coherent elementary amplitude (we shall refer to the response function in this case as $S(q)$), $\hat{\varrho}$ is the matter density $\hat{\psi}^\dagger\hat{\psi}$. If we restrict the sum over n in Eq. (5.1) to all states *except* the ground state, we have the inelastic response surface and shall denote this with a subscript "in". We can also eliminate the contribution of the ground state in Eq. (5.1) by using the fluctuation density defined as

$$\tilde{\varrho} \equiv \hat{\varrho} - \langle \psi_0 | \hat{\varrho} | \psi_0 \rangle \tag{5.3}$$

maintaining the sum over a complete set of states.

By integrating Eq. (5.1) over energy loss we arrive at the inelastic sum rule[6]

$$\int_0^\infty d\omega \left[\frac{1}{\sigma_{\mathrm{p}}}\frac{d^2\sigma}{d\Omega' d\varepsilon'}\right]_{\mathrm{in}} = \int_0^\infty S_{\mathrm{in}}(\boldsymbol{q}, \omega) d\omega \equiv S_{\mathrm{in}}(q) = \sum_n |\langle \Psi_n | \tilde{\varrho}(-\boldsymbol{q}) | \Psi_0 \rangle|^2. \tag{5.4}$$

[6] Note that experimentally, for example in (ee′), we are limited to the region $\omega \leqslant q$. We must assume that all the important target excitations are contained in this region to make use of these results.

We may now use closure on the sum to derive the expression

$$S_{\text{in}}(q) = \int e^{-iq\cdot x}\int e^{+iq\cdot y}dxdy[\langle\Psi_0|\hat\psi^\dagger(x)\hat\psi(x)\hat\psi^\dagger(y)\hat\psi(y)|\Psi_0\rangle$$
$$-\langle\Psi_0|\hat\psi^\dagger(x)\hat\psi(x)|\Psi_0\rangle\langle\Psi_0|\hat\psi^\dagger(y)\hat\psi(y)|\Psi_0\rangle] \tag{5.5}$$

where the matter density has been written in second quantization. Making use of the canonical anticommutation relations (Eqs (2.22)), we can write the inelastic sum rule as

$$S_{\text{in}}(q) = A[1-|F_{\text{el}}(q)|^2]+A(A-1)\int e^{-iq\cdot x}C(x, y)e^{iq\cdot y}dxdy \tag{5.6}$$

where C is the two-body correlation function and $F_{\text{el}}(q)$ is the Fourier transform of the ground-state matter density

$$F_{\text{el}}(q) = \int e^{iq\cdot x}\varrho^{(1)}(x)dx. \tag{5.7}$$

For a uniform system, these equations take the simplified form

$$S_{\text{in}}(q) = A[1-\delta_{q,0}]+\Omega\int e^{-iq\cdot z}[A(A-1)C(z)]dz. \tag{5.8}$$

If we insert the correlation functions from Eqs (4.4), (4.19), (4.22), then the result is

$$S_{\text{in}}(q) = A[1-(1-\tfrac{3}{2}\zeta+\tfrac{1}{2}\zeta^3)\theta(1-\zeta)]+S_{\text{in}}^{\text{BG}}(q), \tag{5.9}$$

$$S_{\text{in}}^{\text{BG}}(q) \equiv \Omega\int e^{-iq\cdot z}[A(A-1)C_{\text{BG}}(z)]dz \tag{5.10}$$

where $\zeta \equiv q/2k_{\text{F}}$.

The first term on the right hand side of (5.9) is the result for non-interacting Fermi gases. Making use of Eq. (4.22), we can simplify this result to the form

$$\frac{1}{A}S_{\text{in}}^{\text{BG}}(q) = \left(\frac{g-1}{g}\right)\left(\frac{A}{\Omega k_{\text{F}}^3}\right)\int e^{-iQ\cdot x}dx\left[24\int_0^1 K^2(1-\tfrac{3}{2}K+\tfrac{1}{2}K^3)dK\,\frac{d\Omega_K}{4\pi}\right.$$
$$\left.\times(|\psi_K^{l=0}(x)|^2-|j_0(Kx)|^2)\right]-\left(\frac{g-1}{g}\right)\left(\frac{A}{\Omega k_{\text{F}}^3}\right)\eta\delta_{q0}. \tag{5.11}$$

Here $Q \equiv q/k_{\text{F}}$ and $x \equiv k_{\text{F}}r$.

Note that the last term coming from our normalization condition contains a Kronecker delta and serves to guarantee the detailed normalization condition [17]

$$S_{\text{in}}(0) = 0 \tag{5.12}$$

and follows from the defining relation, Eq. (5.6), since the volume integral of the correlation function vanishes identically.

The corresponding Coulomb sum rules for inelastic electron scattering are given by

$$C_{\text{in}}(q) = C_{\text{in}}^{\text{FG}}(q)+C_{\text{in}}^{\text{BG}}(q), \tag{5.13}$$

$$\frac{C_{\text{in}}^{\text{FG}}(q)}{Z} = 1-(1-\tfrac{3}{2}\zeta+\tfrac{1}{2}\zeta^3)\theta(1-\zeta) = \frac{S_{\text{in}}^{\text{FG}}(q)}{A}; \quad \zeta \equiv \frac{q}{2k_{\text{F}}}, \tag{5.14}$$

$$\frac{C_{\text{in}}^{\text{BG}}(q)}{Z} = \tfrac{1}{3}\,\frac{S_{\text{in}}^{\text{BG}}(q)}{A}. \tag{5.15}$$

The important point here is that the charge density involves only the protons, and it is only the proton correlation function which enters into the sum rule. There are *equal numbers* of like and unlike pairs of protons in the nucleus, whereas when we refer to the matter density, there are *3 times* as many unlike pairs as like pairs of nucleons. This is the reason for the extra factor 1/3 in Eq. (5.15).

It is also possible to say some things about the behavior of the inelastic sum rules at long wavelength in a realistic finite nuclear system. For example, we see in the long wavelength the operator in Eq. (5.2) is simply the volume integral of the density. If this is the baryon density, the volume integral is the total number of baryons; if it is the charge density, it is the total number of protons. In either case, this number is a constant of the motion and it cannot cause transitions. Thus the inelastic sum rule in Eq. (5.4) must vanish in the low-q limit. Furthermore, if the system is of finite extent, then the particles are bound, and the two-body densities must fall off exponentially at large separations. This implies that $S_{\text{in}}(q)$ and $C_{\text{in}}(q)$ must be analytic and even functions of q in a certain strip in the q plane (we assume the ground-state has $J = 0$). Thus, at long wavelengths the inelastic sum rules must have the power series expansion

$$\frac{S_{\text{in}}(q)}{A} = \beta_{\text{M}} q^4 + \gamma_{\text{M}} q^6 + \ldots, \tag{5.16}$$

$$\frac{C_{\text{in}}(q)}{Z} = \alpha_{\text{c}} q^2 + \beta_{\text{c}} q^4 + \gamma_{\text{c}} q^6 + \ldots . \tag{5.17}$$

Note that the first term in the power series of the exponential Eq. (5.2) is just the dipole moment of the appropriate density. If it is the mass density, the dipole moment is the position of the center-of-mass, and this operator cannot give rise to transitions between different intrinsic nuclear states. For this reason the power series in (5.16) must start as q^4. If we are talking about the charge density, the dipole moment is just the dipole operator which also governs photoabsorption in nuclei. Thus, the first coefficient in Eq. (5.17) can be related to the total photoabsorption cross section in nuclei.

$$\alpha_{\text{c}} = \frac{1}{4\pi^2 \alpha Z} \int_0^\infty \frac{\sigma_\gamma(E)}{E}\, dE. \tag{5.18}$$

This result is due to Foldy and Walecka [18]. Using a simple empirical expression for this bremsstrahlung-weighted cross section given by Levinger, this coefficient can be evaluated [18]

$$\alpha_{\text{c}} q^2 \simeq 0.846 \left(\frac{q}{k_{\text{F}}}\right)^2 . \tag{5.19}$$

Let us now turn to some examples with the inelastic sum rules. First we concentrate on infinite nuclear systems. The quantities $|1 - S_{\text{in}}(q)/A|$ (the nucleon sum rule) and $|1 - C_{\text{in}}(q)/Z|$ (the Coulomb sum rule) are plotted as a function of q/k_{F} for nuclear matter with $k_F = 1.42\ \text{fm}^{-1}$ in Figs 5 and 6. The correlation effect at high q is a factor of 3 bigger for the nucleon sum rule (Fig. 5) than for the Coulomb sum rule (Fig. 6) as we have discussed. The dash-dot curve is a calculation with pure hard-core correlations, starting from Eq. (4.24),

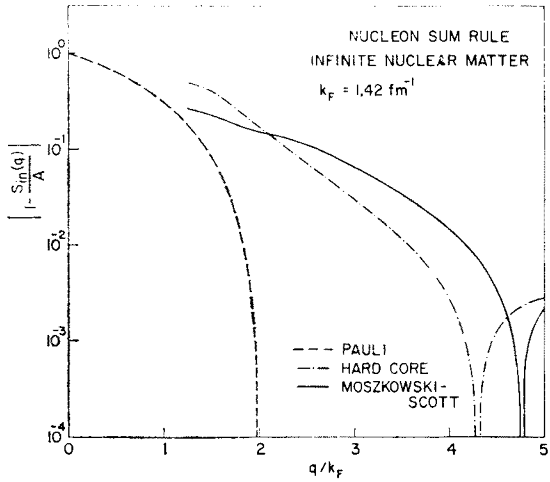

Fig. 5. The quantity $|1-S_{in}(q)/A|$ (nucleon sum rule) as a function of q/k_F for infinite nuclear matter

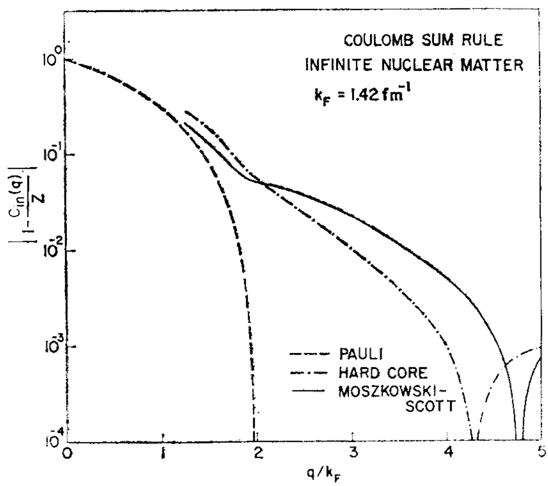

Fig. 6. The quantity $|1-C_{in}(q)/A|$ (Coulomb sum rule) as a function of q/k_F for infinite nuclear matter

and the solid curve is calculated using the wave functions of Eqs (4.26) and (4.27) derived from the Moszkowski-Scott interaction. The dashed curve is obtained by using Pauli correlations alone (Eq. (5.14)). Here in the intermediate- to high-momentum-transfer region, there is a clear distinction between Fermi gas correlations, pure hard-core correlations, and the more realistic correlations, including both the effects of the repulsion and the attraction in the nucleon-nucleon interaction. Similar conclusions have been reached by Małecki et al. [19] although we believe with less justification for the particular correlation functions used. We have not extrapolated these calculated curves into the long-wave-length region for the reasons which we have discussed previously.

To understand what is happening here, the quantity $x^2[g(x)-1]$ is plotted in Fig. 7. This is the density which actually determines the contribution of the unlike pairs to the

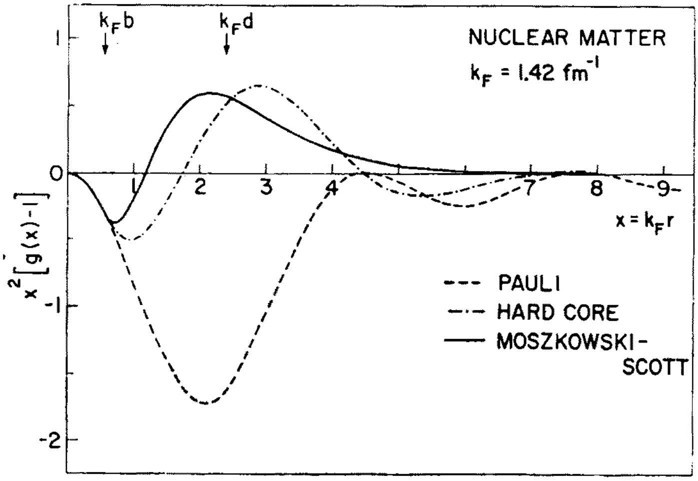

Fig. 7. The quantity $x^2[g(x)-1]$ as a function of $x = k_F r$. (See Fig. 4 and caption)

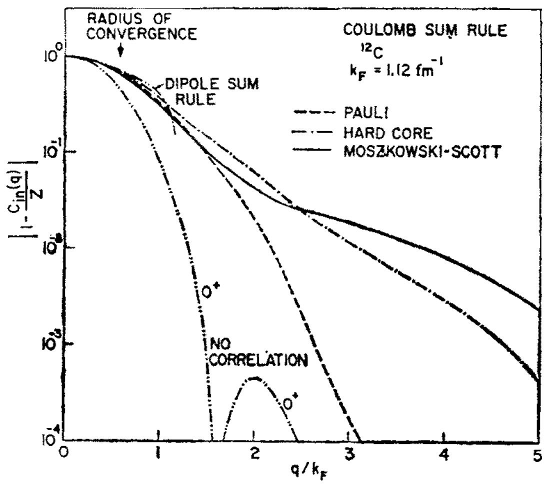

Fig. 8. The quantity $|1-C_{in}(q)/Z|$ (Coulomb sum rule) as a function of q/k_F for $e+{}^{12}C$ scattering computed using Eq. (5.20) and various types of correlations (see text). Also shown is the correct long-wavelength behavior (Eqs (5.17)–(5.19))

sum rule in Eq. (5.10). This must be weighted by $j_0(Qx)$ in order to find the Fourier component at a particular momentum $Q = q/k_F$.

In Fig. 8 we make a calculation for the finite nucleus ^{12}C. Here we take the two-body correlation function (and hence the two-body density by Eq. (2.10)) to be of the form

$$C(\boldsymbol{x}_1, \boldsymbol{x}_2) = \varrho^{(1)}(\boldsymbol{x}_1)\varrho^{(1)}(\boldsymbol{x}_2)\,[g(|\boldsymbol{x}_1-\boldsymbol{x}_2|)-1]. \tag{5.20}$$

The one-body densities are taken from elastic electron scattering and the two-body correlation function is calculated for nuclear matter with a Fermi wave number determined by Moniz et al. [20] from the width of the quasielastic peak in ^{12}C. At long wavelengths (small values of q), the finite extent of the system is important and the Coulomb sum rule in Fig. 8 must start as q^2. In fact, the coefficient (Eqs (5.17), (5.18)) is the bremsstrahlung-weighted integrated photoabsorption cross section which is dominated by the giant resonance collective mode in nuclei. Also indicated in Fig. 8 is the result obtained from the first term in the expansion (Eq. (5.18)) and the arrow indicates an estimate of the radius of convergence of this expansion based on the binding energy of the last nucleon. The results for both hard-core and Moszkowski-Scott correlations are quite close to those obtained from Pauli correlations alone for momentum transfers out to $q/k_F \lesssim 2$. There is a decided difference at larger values of q, although the overall magnitude of the quantity plotted is small. That the correlations play a very small role in the Coulomb sum rule is an old conclusion due to McVoy and van Hove [3]. In fact the solid curve gives a very good match to the calculations of McVoy and van Hove in the region $0 \leqslant q/k_F \lesssim 2$. These curves also show the clear inadequacy of any Fermi gas calculation for a finite system at long wavelengths.

The main conclusion from this figure is that one can only expect 1–10% effects on the Coulomb sum rule coming from hard-core, or more realistic, correlations between unlike protons. These effects, while small, should be measured. There are real differences between the curves calculated with Pauli, hard-core correlations, and with more realistic correla-

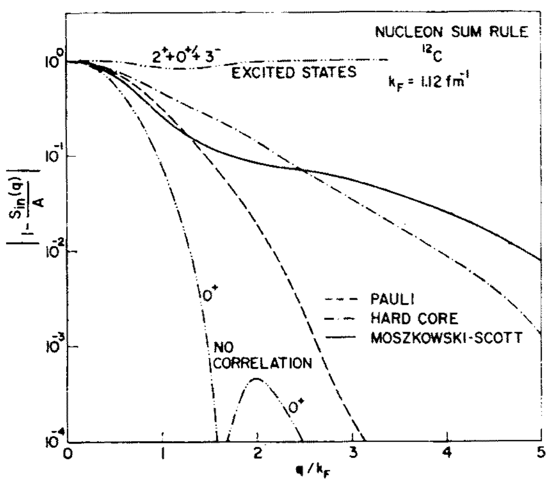

Fig. 9. The quantity $|1-S_{\text{in}}(q)/A|$ (nucleon sum rule) as a function of q/k_F for $p+{}^{12}C$ scattering computed using Eq. (5.20) and various types of correlations (see text). Also shown is the long-wavelength behavior (Eq. (5.16)) obtained from explicitly summing the contribution of the first three excited states in ^{12}C

tions. The effect is clean and should be measured by separating the Coulomb from the transverse scattering.

Figure 9 shows the inelastic sum rule for high-energy nucleon, or other hadron scattering from ^{12}C. Here the effects of correlations are larger as we have indicated. In the intermediate- to high-momentum-transfer region there is a clear distinction between Fermi gas correlations, pure hard-core correlations, and more realistic correlations, including both the effects of the repulsion and attraction in the nucleon-nucleon interaction. The low-momentum transfer behaviour obtained by explicitly summing the contributions of the first three excited states to p-^{12}C scattering is also indicated in this figure.

To try and indicate that this is not all nonsense, Fig. 10 shows the similar calculation of the sum rule for ^{3}He where the inelastic sum rule has been measured by inelastic X-ray

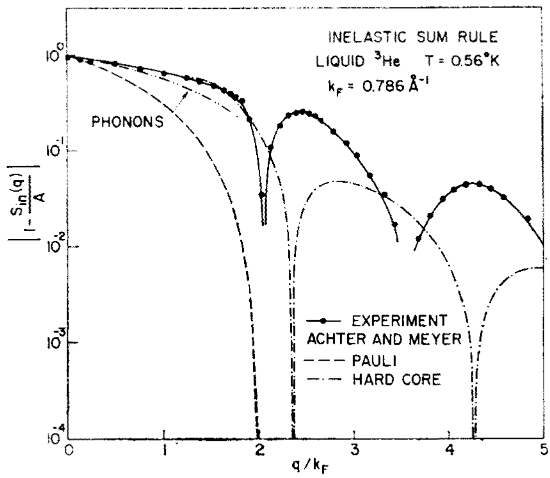

Fig. 10. The inelastic sum rule $|1-S_{\text{in}}(q)/A|$ as a function of q/k_F for liquid ^{3}He calculated using S-wave hard core correlations with $c = k_F b = 1.3$. The experimental data is from Ref. [21]. Also shown is the calculated long-wavelength behavior coming from phonon excitation

scattering from ^{3}He liquid [21]. The experimental points are shown on the figure, together with a calculation using the same hard-core correlation function which we showed previously, and the hard-core radius of $c = k_F b = 1.3$[7]. This was taken from a paper by Burkhardt [22] who used a hard-core plus attractive square well potential chosen to generate the same set of phase shifts as the Leonard-Jones potential for ^{3}He. It is clear that this very simple model does give a qualitative explanation of the experimental observation. Again, we have not taken the curves down to small values of q because it is an experimental fact that the inelastic sum rule in this region is given by the phonon contribution [23] and these long-range correlations, or collective effects, have not been built into two-body correlation functions calculated in the independent-pair approximation developed to reproduce the small-distance behavior of the two-particle correlation function. This inadequacy is clearly understood in Fig. 11, which shows the function $g(r)$ used for helium in the simple hard-core model and the function $g(r)$ extracted from a Fourier transform of the experimental data by Achter and Meyer [21]. The present calculation is in no sense meant to be a definitive description of the inelastic sum rule in ^{3}He. It is meant only to

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[7] Note for ^{3}He, $A/\Omega \equiv 1/d^3 = k_F^3/3\pi^2$, so that $k_F d = (3\pi^2)^{1/3} = 3.09$.

illustrate that there are effects at high-momentum-transfer coming from the short-range correlations and that we can hope to calculate these effects in the independent-pair approximation.

There is still a problem of distortion of the projectile wavefunction in the case of high-energy hadron scattering from nuclei. From Eq. (5.6), we see that the effect of distortion will be to replace the plane wave in the Fourier transform by the distorted wave in the

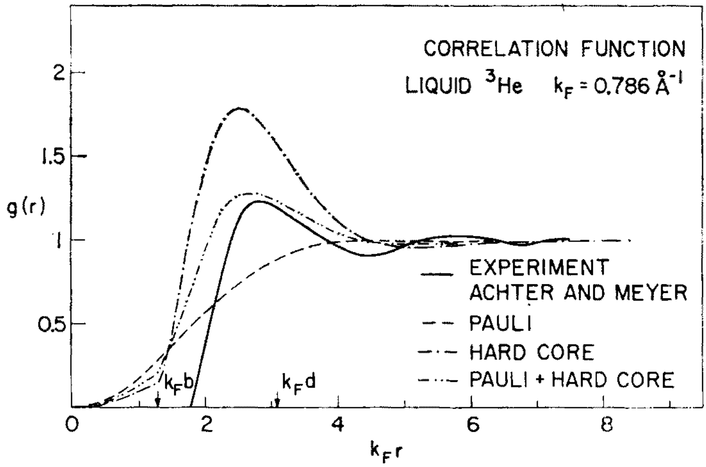

Fig. 11. The correlation function $g(r)$ for liquid ^{3}He. Shown are the values obtained from S-wave hard core correlations with $c = k_F b = 1.3$ as well as the value extracted from their data (Fig. 10) by Achter and Meyer [21]. Here $k_F d = (3\pi^2)^{1/3}$ is the average interparticle separation

optical potential of the nucleus. To the extent that the one-body density varies only slowly over the region of correlation, it is clear that the main effect of distortion in Eq. (5.6) is to provide an A_{eff} in the denominator in Eq. (5.15), but need not change the general shape of the curve in Fig. 9. This point clearly needs further study and is currently under examination.

We believe there is enough here to say, however, that the investigation of the inelastic sum rule in high-energy nucleon scattering, for example at Los Alamos, may be able to provide some information on the two-nucleon correlation function.

There is still the basic problem for both sum rules in Figs 8 and 9 of being able to separate nuclear physics effects described in terms of elementary nucleons, which is the basis of all of our discussion of nuclear densities and sum rules, from the effects of the meson degrees of freedom. These are not included in our calculations and are evidenced by contributions of meson production to the scattering processes. One may never be able to separate these processes in a single-arm experiment where only the scattering projectile is detected; however, it may be possible to subtract these effects by doing coincidence experiments. In any event, pion production places the most severe limitation on being able to make accurate measurements of the sum rules. This may have, in fact, more profound implications for nuclear physics and may indicate that one really cannot consistently do nuclear physics in terms of elementary nucleons.

B. Elastic proton-nucleus scattering at high energies

The elastic scattering of high-energy hadrons by nuclei is very well described through the analysis developed by Glauber [4] and others. In this analysis the incident hadron undergoes repeated forward scattering from the nucleons in the nucleus. The nuclear amplitude is then computed by averaging over the positions of the target particles using the ground-state density (Eq. (2.1)). Since we now have a reaction mechanism which is understood to a high degree, and which is sensitive to the location of all the target particles, we may hope to use this to study two-particle densities in the nucleus. Since the two-body correlations defined in Eq. (2.10) only produce *corrections* to the double and higher-order scattering terms, correlation effects in this process will turn out to be very small. In the following we will use the nuclear two-body densities we have developed and the Glauber formalism to study elastic p-^{12}C and p-^{16}O scattering and we will exhibit the effects of correlations caused by the Pauli exclusion principle, by a pure hard-core interaction, and by the Moszkowski-Scott potential.

1. The Glauber formalism

In the high-energy Glauber multiple scattering formalism [4] one makes the following assumptions and approximations:
(*i*) The eikonal approximation which limits us to high energies and small scattering angles.
(*ii*) The closure, or "frozen nucleus" approximation which neglects the Fermi motion of the target nucleons.
(*iii*) The dynamical assumption of the additivity of the phase functions which in potential scattering is equivalent to the assumption of non-overlapping elementary potentials.

Corrections to these approximations have been studied by many authors (e. g. Wallace [24]) and have been shown to be surprisingly small. Thus it appears that we can have some confidence in our understanding of this reaction mechanism.

We proceed to apply this multiple-scattering analysis to elastic proton-nucleus scattering at high energy. The scattering amplitude in Glauber's approximation is given by

$$F(\boldsymbol{q}) = \frac{ik}{2\pi}\int e^{i\boldsymbol{q}\cdot\boldsymbol{b}}\Gamma(\boldsymbol{b})d^{(2)}\boldsymbol{b}. \tag{5.21}$$

Here $\hbar\boldsymbol{q}$ is the momentum transfer

$$\hbar\boldsymbol{q} = \hbar(\boldsymbol{k}_i - \boldsymbol{k}_f) \tag{5.22}$$

and $\boldsymbol{b}$ is the impact parameter. All the nuclear physics is contained in the nuclear profile function which is given (neglecting Coulomb effects) as an integral over the nucleon coordinates

$$\Gamma(\boldsymbol{b}) = \int \Phi_0^*(\boldsymbol{x}_1, \dots, \boldsymbol{x}_A)\,\{1-\prod_{j=1}^{A}[1-\gamma_j(\boldsymbol{b}-\boldsymbol{s}_j)]\}\delta^{(3)}(\boldsymbol{x}-A^{-1}\sum_{k=1}^{A}\boldsymbol{x}_k)$$
$$\times\,\Phi_0(\boldsymbol{x}_1, \dots, \boldsymbol{x}_A)d\boldsymbol{x}_1 \dots d\boldsymbol{x}_A. \tag{5.23}$$

The elementary profile functions are just the two-dimensional Fourier transforms of the elementary nucleon-nucleon amplitudes $f_j(\boldsymbol{q})$

$$\gamma_j(\boldsymbol{b}-\boldsymbol{s}_j) = \frac{1}{2\pi i k}\int e^{-i\boldsymbol{q}\cdot(\boldsymbol{b}-\boldsymbol{s}_j)} f_j(\boldsymbol{q}) d^{(2)}\boldsymbol{q}. \tag{5.24}$$

The vector $\boldsymbol{s}_j$ denotes the projection of the nuclear coordinate $\boldsymbol{x}_j$ onto the plane perpendicular to the incident direction. In general, the delta function in Eq. (5.23) must be included in the definition of the nuclear density in Eq. (2.1) to ensure that we are properly dealing with $3A$-3 independent internal coordinates and to provide a correct treatment of the motion of the center-of-mass of the nucleus [5]. If the ground state of the nucleus is well described by harmonic oscillator wavefunctions, however, we can dispose of the delta function in Eq. (5.23) by multiplying the amplitudes by the center-of-mass correction factor $f_{CM}(q) = \exp[q^2/4A\alpha^2]$ where α^{-1} is the harmonic oscillator parameter. This should be a good approximation for light and medium weight nuclei, whereas for heavy nuclei, $f_{CM}(q)$ rapidly approaches unit value and can thus be neglected. A more extensive discussion of the correct treatment of center-of-mass motion in defining and relating nuclear densities is contained in Ref. [5].

The nuclear profile function $\Gamma(\boldsymbol{b})$ can be expanded in terms of products of elementary profile functions $\gamma_j(\boldsymbol{b}-\boldsymbol{s}_j)$

$$\begin{aligned}\Gamma(\boldsymbol{b}) = \langle\Phi_0| &\sum_j \gamma_j(\boldsymbol{b}-\boldsymbol{s}_j) - \sum_{j<k}\gamma_j(\boldsymbol{b}-\boldsymbol{s}_j)\gamma_k(\boldsymbol{b}-\boldsymbol{s}_k) \\ &+ \sum_{j<k<l}\gamma_j(\boldsymbol{b}-\boldsymbol{s}_j)\gamma_k(\boldsymbol{b}-\boldsymbol{s}_k)\gamma_l(\boldsymbol{b}-\boldsymbol{s}_l) - \ldots |\Phi_0\rangle,\end{aligned} \tag{5.25}$$

and since the nucleus consists of A identical nucleons, we have

$$\begin{aligned}\Gamma(\boldsymbol{b}) = \langle\Phi_0| &A\gamma_1(\boldsymbol{b}-\boldsymbol{s}_1) - \frac{A(A-1)}{2}\gamma_1(\boldsymbol{b}-\boldsymbol{s}_1)\gamma_2(\boldsymbol{b}-\boldsymbol{s}_2) \\ &+ \frac{A(A-1)(A-2)}{6}\gamma_1(\boldsymbol{b}-\boldsymbol{s}_1)\gamma_2(\boldsymbol{b}-\boldsymbol{s}_2)\gamma_3(\boldsymbol{b}-\boldsymbol{s}_3) - \ldots |\Phi_0\rangle.\end{aligned} \tag{5.26}$$

The first term in this multiple scattering expansion describes single scattering, or the impulse approximation. In general the $\gamma_j(\boldsymbol{b}-\boldsymbol{s}_j)$ are operators in spin and isospin space, but we assume here for simplicity that we can deal with a spin and isospin averaged nucleon--nucleon amplitude

$$f_j(q) \equiv \alpha(q) = \alpha_0 e^{-aq^2} \tag{5.27}$$

with

$$\alpha_0 = \frac{ik\sigma_{tot}}{3\pi}(1-i\varrho). \tag{5.28}$$

The parameters in α_0 have been assumed to take the values $\sigma_{tot} = 43.6$ mb, $\varrho = -.252$, and $a = .102\ \text{fm}^2$ [25]. The inclusion of spin and isospin dependences of the elementary amplitude is straightforward, and is presented, for example, in Ref. [25].

We now apply this formalism to elastic p-^{12}C and p-^{16}O scattering. The form factor of ^{12}C can be well represented [26] by

$$\tilde{\varrho}^{(1)}(q) = (1-cq^2)e^{-\tilde{d}q^2} \tag{5.29}$$

with c = .296 fm^2 and $\tilde{d}$ = .681 fm^2. For ^{16}O we get a reasonable fit to the form factor (except for the second minimum [26]) with c = .410 fm^2 and $\tilde{d}$ = .819 fm^2. This form factor has to be corrected for the finite size of the proton's charge distribution and center--of-mass motion. This can be done approximately, within the harmonic-oscillator framework, by replacing $\tilde{d}$ with d where

$$d = (\tilde{d} - \tfrac{1}{6}\langle r_{\mathrm{p}}^2\rangle)\frac{A}{A-1} \tag{5.30}$$

where the root-mean-square radius of the proton is $\langle r_{\mathrm{p}}^2\rangle^{1/2}$ = .8 fm. The corresponding density is

$$\varrho^{(1)}(x) = \varrho_0(1+\beta x^2)e^{-\gamma x^2} \tag{5.31}$$

with

$$\varrho_0 = \frac{1-3c/2d}{8(\pi d)^{3/2}}; \qquad \beta = \frac{c}{4d^2(1-3c/2d)}; \qquad \gamma = 1/4d. \tag{5.32}$$

Thus, we can immediately write down the proton-nucleus single-scattering amplitude

$$F_1(q) = \alpha_0(1-cq^2)e^{-(d+a)q^2} \tag{5.33}$$

and the corresponding single-scattering profile function

$$\Gamma_1(b) = \frac{\alpha_0}{2ik(d+a)}\left[\left(1-\frac{c}{d+a}\right)+\left(\frac{c}{d+a}\right)\frac{b^2}{4(d+a)}\right]e^{-\frac{b^2}{4(d+a)}}. \tag{5.34}$$

2. No correlations

We first consider multiple-scattering for uncorrelated target nucleons, retaining just the first term in Eq. (2.8). Then the profile function for double scattering is given by the square of $\Gamma_1(b)$

$$\Gamma_2(b) = [\Gamma_1(b)]^2 \tag{5.35}$$

and the corresponding amplitude by

$$F_2(q) = -\frac{i\alpha_0^2}{k}\frac{1}{4(d+a)}\left[1-\frac{c}{a+d}+\frac{c^2}{2(a+d)^2}-\frac{c}{2}q^2+\frac{c^2}{16}q^4\right]e^{-\frac{(a+d)}{2}q^2}. \tag{5.36}$$

In general, the n^{th}-order profile function with uncorrelated target particles is

$$\Gamma_n(b) = [\Gamma_1(b)]^n = \left[\frac{\alpha_0}{2ik(a+d)}\right]^n \sum_{\nu=0}^{n} \phi_{n,\nu} b^{2\nu} e^{-\frac{nb^2}{4(a+d)}} \tag{5.37}$$

with

$$\phi_{n,\nu} = \binom{n}{\nu}\left(1-\frac{c}{a+d}\right)^{n-\nu}\left(\frac{c}{4(a+d)^2}\right)^{\nu}. \tag{5.38}$$

For the n^{th}-order scattering term we have

$$F_n(q) = \frac{2ik(a+d)}{n}\left[\frac{\alpha_0}{2ik(a+d)}\right]^n p_n(q^2)e^{-\frac{(a+d)}{n}q^2}. \tag{5.39}$$

Here $p_n(q^2)$ denotes a polynomial of order n in q^2

$$p_n(q^2) = \sum_{\nu=0}^{n} p_{n,\nu}(q^2)^{\nu} \tag{5.40}$$

with the coefficients

$$p_{n,\nu} = \left[-\frac{(a+d)}{n}\right]^{\nu}\sum_{\varrho=\nu}^{n}\phi_{n,\varrho}\left[\frac{4(a+d)}{n}\right]^{\varrho}\binom{\varrho}{\nu}^2(\varrho-\nu)!$$

$$= \frac{n!}{\nu!}\left[-\frac{(a+d)}{n}\right]^{\nu}\sum_{\mu=0}^{n-\nu}\frac{1}{\mu!}\left(1-\frac{c}{a+d}\right)^{\mu}\left[\frac{c}{n(a+d)}\right]^{n-\mu}\binom{n-\mu}{\nu}. \tag{5.41}$$

The multiple scattering amplitude for elastic scattering corrected for center-of-mass motion is

$$F(q) = e^{\frac{d}{A}q^2}\sum_{n=1}^{A}\binom{A}{n}(-1)^{n+1}F_n(q). \tag{5.42}$$

For $d = 1/4\alpha^2$, $c = 0$, $a \equiv \bar{\beta}^2/2$, and $A = 4$ we get

$$F(q) = ike^{q^2/16\alpha^2}\left(\frac{1+2\alpha^2\bar{\beta}^2}{2\alpha^2}\right)\sum_{m=1}^{4}\binom{4}{m}\frac{(-1)^{m+1}}{m}\left[\frac{\sigma\alpha^2(1-i\varrho)}{2\pi(1+2\alpha^2\bar{\beta}^2)}\right]^m$$

$$\times\exp\left[-\left(\frac{1+2\alpha^2\bar{\beta}^2}{4m\alpha^2}\right)q^2\right] \tag{5.43}$$

which is the result originally derived for ^{4}He by Czyż and Leśniak [27].

3. Double scattering and correlations

We now want to calculate the double scattering term including correlations. The contribution to the double-scattering term from the correlation part of the two-body density in Eq. (2.10) can be written with the aid of Eqs (5.21), (5.24), (5.26) as

$$F_2^{\text{corr}}(q) = \frac{ik}{2\pi(2\pi ik)^2}\int d^2\boldsymbol{b}\,d^2\boldsymbol{q}_1\,d^2\boldsymbol{q}_2\,e^{i(\boldsymbol{q}-\boldsymbol{q}_1-\boldsymbol{q}_2)\cdot\boldsymbol{b}}f_1(\boldsymbol{q}_1)f_2(\boldsymbol{q}_2)\tilde{C}(\boldsymbol{q}_1,\boldsymbol{q}_2). \tag{5.44}$$

With the substitutions $\boldsymbol{K} = \frac{1}{2}(\boldsymbol{q}_1+\boldsymbol{q}_2)$, $\boldsymbol{\kappa} = \frac{1}{2}(\boldsymbol{q}_1-\boldsymbol{q}_2)$ and the parametrization of the elementary amplitude in Eqs (5.27) and (5.28) we obtain

$$F_2^{\text{corr}}(q) = -\frac{i\alpha_0^2}{2\pi k}e^{-\frac{1}{2}aq^2}\int d^2\boldsymbol{\kappa}e^{-2a\kappa^2}\tilde{C}(\tfrac{1}{2}\boldsymbol{q}+\boldsymbol{\kappa}, \tfrac{1}{2}\boldsymbol{q}-\boldsymbol{\kappa}). \tag{5.45}$$

If we write (cf. Eq. (5.20))

$$C(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \varrho^{(1)}(\boldsymbol{x}_1)\varrho^{(1)}(\boldsymbol{x}_2)\left[g(|\boldsymbol{x}_1-\boldsymbol{x}_2|)-1\right] \tag{5.46}$$

and use Eq. (5.31), then after some algebra we arrive at the result

$$F_2^{\text{corr}}(q) = -\frac{i\alpha_0^2}{4k(a+d)}e^{-\frac{1}{2}(a+d)q^2}(p_{20}^{\text{corr}}+p_{21}^{\text{corr}}q^2+p_{22}^{\text{corr}}q^4) \tag{5.47}$$

where

$$\begin{aligned} p_{20}^{\text{corr}} &= \tfrac{1}{2}(a+d)\varrho_0^2(8\pi d)^{3/2}\{g_{00}(1+15\beta^2d^2+6\beta d)+g_{20}\tfrac{1}{2}\beta(1+\beta d)+g_{40}\tfrac{1}{16}\beta^2\},\\ p_{21}^{\text{corr}} &= -\tfrac{1}{2}(a+d)\varrho_0^2(8\pi d)^{3/2}\{2\beta d^2g_{00}(1+5\beta d)+g_{20}\tfrac{1}{6}\beta^2d^2+g_{22}\tfrac{1}{6}\beta^2d^2\},\\ p_{22}^{\text{corr}} &= \tfrac{1}{2}(a+d)\varrho_0^2(8\pi d)^{3/2}g_{00}\beta^2d^4, \end{aligned} \tag{5.48}$$

and we have defined the parameters

$$g_{nl} = 4\pi\int_0^\infty \kappa d\kappa e^{-2a\kappa^2}\int_0^\infty e^{-\frac{1}{2}\gamma\xi^2}[g(\xi)-1]j_l(\kappa\xi)\xi^{n+2}d\xi. \tag{5.49}$$

Thus we immediately arrive at the correlation correction to the double scattering profile function

$$\Gamma_2^{\text{corr}}(b) = -\frac{\alpha_0^2}{4k^2(a+d)^2}[\phi_{20}^{\text{corr}}+\phi_{21}^{\text{corr}}b^2+\phi_{22}^{\text{corr}}b^4]e^{-b^2/2(a+d)} \tag{5.50}$$

with

$$\phi_{22}^{\text{corr}} = \frac{p_{22}^{\text{corr}}}{(a+d)^4}, \quad \phi_{21}^{\text{corr}} = -\frac{p_{21}^{\text{corr}}}{(a+d)^2}-8\frac{p_{22}^{\text{corr}}}{(a+d)^3},$$

$$\phi_{20}^{\text{corr}} = p_{20}^{\text{corr}}+2\frac{p_{21}^{\text{corr}}}{(a+d)}+8\frac{p_{22}^{\text{corr}}}{(a+d)^2}. \tag{5.51}$$

4. Multiple scattering and correlations

The multiple scattering profile function can be easily calculated if we retain the first two terms in the expansion (2.8), i.e. up through two-particle correlations in the ground--state density. We obtain

$$\Gamma(b) = \sum_{n=1}^{A}\binom{A}{n}(-1)^{n+1}\bar{\Gamma}_n(b) \tag{5.52}$$

and

$$\bar{\Gamma}_n(b) = \sum_{m=0}^{[n/2]} \binom{n}{2m} (2m-1)!![\Gamma_1(b)]^{n-2m}[\Gamma_2^{\text{corr}}(b)]^m \tag{5.53}$$

which is identical to the expression

$$\Gamma(b) = 1 - \sum_{m=0}^{[A/2]} \frac{A!}{(A-2m)!m!2^m} [1-\Gamma_1(b)]^{A-2m}[\Gamma_2^{\text{corr}}(b)]^m \tag{5.54}$$

used by Moniz and Nixon [9]. It is evident that also in this case, we have a representation for $\bar{\Gamma}_n(b)$ identical to that given for $\Gamma_n(b)$ in Eqs (5.37), (5.39), (5.40), and (5.41) only with new coefficients $\bar{\phi}_{n,v}$.

5. Numerical calculations

We have calculated $p+{}^{12}C$ and $p+{}^{16}O$ elastic scattering in the no-correlation approximation and for three different choices of the correlation function: (*i*) Pauli correlations, (*ii*) hard-core correlations, and (*iii*) Moszkowski–Scott correlations. In Figs 12, 13, we

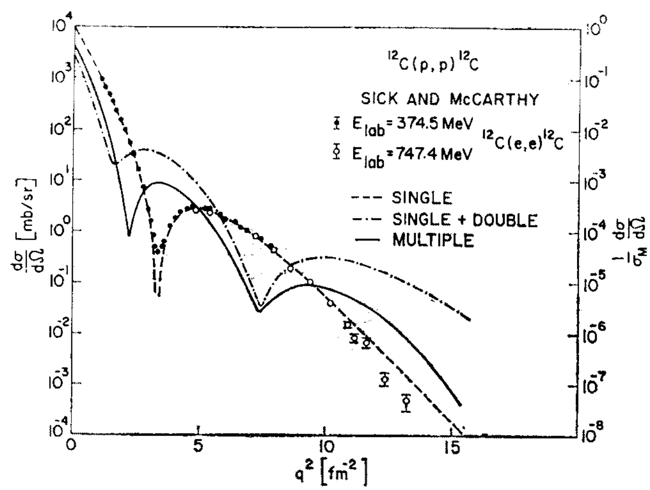

Fig. 12. The contribution of different multiple scattering terms to elastic $p+{}^{12}C$ scattering. The single scattering term is compared to the form factor as measured in electron scattering [26]

show the contribution of the different multiple scattering terms to the total no-correlation amplitude in $p+{}^{12}C$ and $p+{}^{16}O$ respectively. Also shown is the experimental form factor as measured with electron scattering by Sick and McCarthy [26]. Figures 14, 15 show the effects of different correlations on $p+{}^{12}C$ (Fig. 14) and $p+{}^{16}O$ (Fig. 15) scattering. The experimental data is from Refs [28–31]. There is little difference between the no-correlation approximation and calculations involving two-body correlations in the nucleus. This makes it extremely hard to determine correlations with high-energy proton-nucleus scatter-

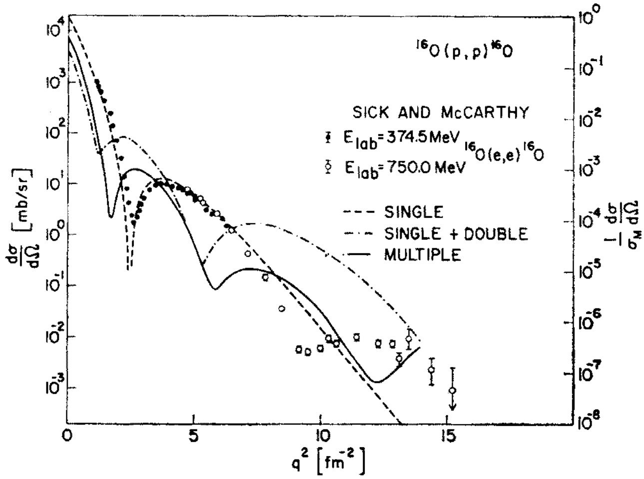

Fig. 13. The contribution of different multiple scattering terms to elastic $p+{}^{16}O$ scattering. The single scattering term is compared to the form factor as measured in electron scattering [26]

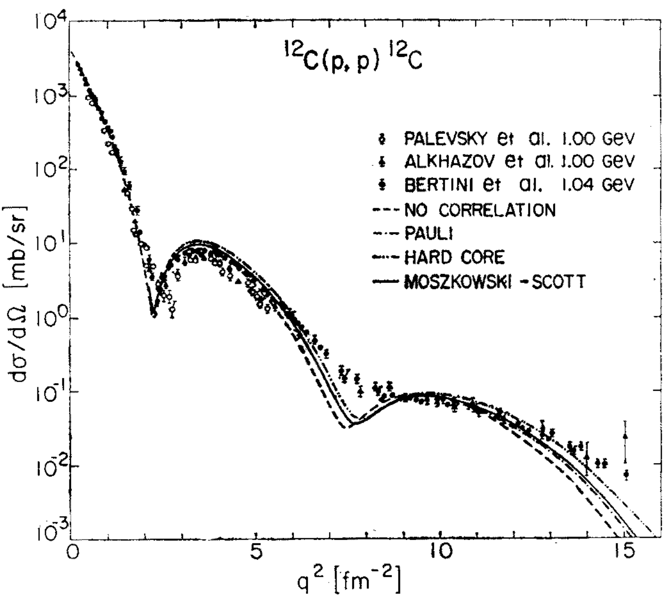

Fig. 14. The effects of different correlations on elastic $p+{}^{12}C$ scattering calculated using Eqs (5.46), (5.52). The experimental data is from Refs [28–31]

ing. It can also be seen in Figs 14, 15 that most of the correlation effects in the Moszkowski-Scott case are very similar to the essentially uninteresting Pauli correlation, and in this case it would be almost impossible to find a difference between Pauli and Moszkowski-Scott correlations in an experiment. If the real correlations would be closer to hard-core

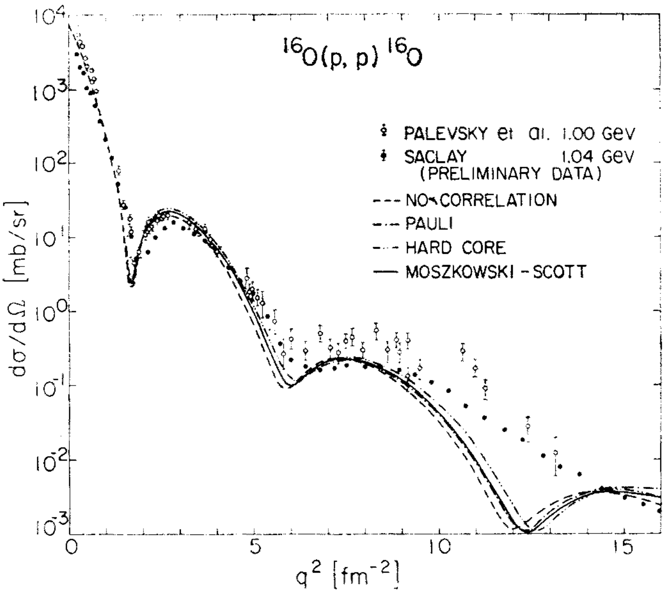

Fig. 15. Same as Fig. 14 for elastic $p+{}^{16}O$ scattering

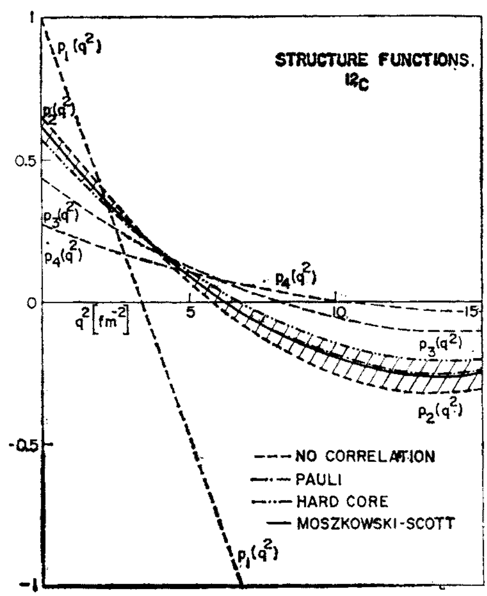

Fig. 16. The structure functions $p_n(q^2)$ (Eq. (5.39), (5.40)) and $\bar{p}_2(q^2)$ (cf. Eq. (5.47)) for different correlations in elastic $p+{}^{12}C$ scattering

correlations, there might be a better chance to detect them in this way. The major problem in determining correlations in high-energy proton-nucleus scattering lies in the fact that correlations are only corrections to the double and higher-order multiple scattering terms. To see this let us look at Fig. 16 where we have plotted the structure functions $p_n(q^2)$

(Eq. (5.40)) as a function of q^2 in the no-correlation approximation ($n = 1, 2, 3, 4$) and $\bar{p}_2(q^2)$ for Pauli, hard-core, and Moszkowski–Scott correlations. The differences are very small and it is very doubtful if it will ever be possible to determine short-range dynamical correlations in this way. Furthermore, when we start discussing very small effects we must go back and reexamine our basic understanding of the reaction mechanism to this level and include spin effects, inelastic shadowing, target dynamics effects, etc.[8] In contrast to inelastic sum rules, elastic scattering probes only integral properties of the correlation function, and therefore the effects are just too small to identify unambiguously. These conclusions that short-range two-body correlations give small effects here are similar to those arrived at by Feshbach and coworkers [33], as well as others[9].

6. *Other possibilities*

We would like to very briefly mention three selected topics that directly involve two-body densities and two-body correlations in nuclei.

A. Local field corrections

There are corrections to the multiple scattering expansions, or the optical potential, for the scattering of hadrons from nuclear targets coming from processes where the incident projectile is multiply reflected between target particles. This is not contained in the Glauber analysis of the optical potential presented previously. An example of this is the Lorentz–Lorenz effect of Ericson in low-energy pion-nucleus scattering which has been examined recently by Eisenberg et al. [35]. Keister [36] has shown that at high energy these local field corrections to the optical potential, or corrections coming from multiple-struck target particles, go away very rapidly with increasing energy because the particles are anti-correlated, or held apart. It is essential that the particles be held apart to have this local field correction disappear. Here again we have an example of the anti-correlations in the nucleus making nuclear physics simple and allowing us to describe high-energy scattering in terms of the elementary projectile-target-particle amplitude and the one-body density. Down in the resonance region the effects may be large [37, 38]. However, the uncertainty in the actual scattering mechanism is probably sufficient to rule out the possibility of learning anything about two-particle correlations in this fashion[10].

B. Lifetime for $\Lambda+N \rightarrow N+N$

Another class of processes where correlations play an important role is non-leptonic weak interactions. Since the Pauli principle forbids the ordinary decay $\Lambda \rightarrow N+\pi$ for a Λ at rest in nuclear matter, the predominant decay mechanism for a Λ is expected to be the

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[8] See, e.g., Ref. [32].

[9] Compare Ref. [34].

[10] Similar uncertainty about the reaction mechanism makes it unlikely that we get definitive information on correlations in the near future from pion capture on nucleon pairs.

collision decay $\Lambda + N \rightarrow N + N$. This is a weak interaction It may take place through $1-\pi$ exchange, and since one knows the couplings involved, the $1-\pi$ exchange contribution can be computed Because of the anti-correlations between the baryons, one might expect $1-\pi$ exchange to be the dominant mechanism and, in fact, one gets substantially different reaction rates depending on the detailed nature of the correlation between the baryons at small distances [39]. The lifetimes for these heavy Λ-nuclei have not been measured yet, however, similar considerations, i.e., how are the weak interactions transmitted between nucleons at small distances and what are the strong correlations at these distances, are essential in estimating the degree of parity violation in nuclear physics introduced by the non-leptonic weak interactions [40].

C. Coincidence experiments (e, e′x)

Finally, we will say just a few words about electron scattering coincidence experiments with nuclear targets and nuclear correlations. deForest has looked at the process (e, e′N) through the Coulomb interaction assuming a Fermi gas for the target with hard-core correlations between the nucleons [41]. He works consistently to second order in the dimensionless parameter c. His idea, an extension of work by Czyż and Gottfried [42], is that if one looks far enough away from the quasielastic peak by varying both the energy transfer and momentum transfer, one can get to kinematic regions where the major contribution to the double-differential cross section can only come from processes where nucleons are in collision during the time of the electromagnetic interaction with the electron (See also Ref. [43]). One interesting feature of his results is a large response for backward--scattered neutrons. It is difficult to see how one can get backward scattered neutrons in competition with this. While amusing, it is not immediately possible to relate these processes to the two-body density in nuclei as we have been able to with the previous ones discussed[11].

Finally, in our opinion, one of the most promising experiments to directly look at two-nucleon correlations is the (e, e′2N) reaction. This involves a triple coincidence, but by varying the energy and momentum transfer to the pair one should in principle be able to map out the two-body wavefunction for the two target particles[12]. It is difficult, and it is complicated by the final state interaction of the nucleons, but it should provide a direct handle on the two-body wavefunction Further analysis of this process, both theoretical and experimental, is clearly called for.

7. *Summary*

In conclusion, we would like to reiterate that it can be argued that the short-range anti-correlations between nucleons in nuclei and the short-range healing of the wavefunction to the unperturbed value are responsible for most of nuclear physics as we know it. We can understand the saturation properties of nuclear matter in the independent-

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[11] The effects of correlations in photonuclear processes, including pion photo-production, has been recently analyzed in Ref. [44, 45].

[12] This experiment was suggested by David Yu [46].

-pair approximation; we can justify the independent-pair approximation and understand the suppression of higher cluster contributions to the energy; we can understand the validity and success of the single-particle shell model, even though the nucleon-nucleon force is strong and short-range; we can understand why exchange currents are not important in nuclei; we can understand why many-body forces are not important. If there could be strong overlap of nucleons in nuclei, the situation could be entirely different. The presence of another nucleon could profoundly modify the meson fields surrounding other nucleons in the nucleus, and nuclei would not look like the relatively low-density collection of non-interacting free particles that they do; we can also understand why the local-field corrections and multiply struck target particles are not important in high-energy scattering.

Because these effects take place at small relative distances and the nucleus is essentially a low-density system, it is not easy to directly detect the effects of these short-range correlations. Although the effects on the inelastic Coulomb sum rules are small, they should be measured because this is the cleanest determination of the correlations[13]. The effects are larger in the inelastic sum rule for the scattering of high-energy nucleons or other hadrons from nucleons. In particular, it should be possible to rule out large classes of not--unreasonable correlations between nucleons on the basis of such experiments. The most promising experiment to look directly at correlations is probably the (e, e′2N) triple coincidence experiment. Detailed analysis of this process, both theoretical and experimental lies in the future. Finally, the relatively low mass of the pion and consequent low-energy meson production may put a fundamental limitation on our ability to describe nuclei in terms of elementary nucleons.

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[13] The effects of correlations on these sum rules has recently also been reexamined by Bertsch and Borysowicz [47].

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