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REDUCED MASS FOR RELATIVISTIC QUANTUM-MECHANICAL PROBLEMS

By A. Staruszkiewicz

Institute of Physics, Jagellonian University, Cracow*

and K. Zalewski

Institute of Nuclear Physics, Cracow**

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A relativistic generalization of the reduced mass is proposed. Using it, it is possible to reduce approximately some relativistic two-body problems to the problem of a single non-relativistic particle interacting with a fixed potential.

In the non-relativistic theory of two-body processes the concept of reduced mass plays a very important role. Using it, it is possible to reduce two-body problems to the simpler problems of one particle interacting with a fixed potential. It would be very helpful (e.g. in nuclear physics) to have a similar device also for relativistic two-body problems. Strictly speaking this cannot be done. We point out, however, that at least for electromagnetic interactions it is possible to use the single particle Schrödinger equation with a suitably defined reduced mass and the ordinary non-relativistic potential to obtain very good approximations to the exact results.

It has been proved by one of us [1] that the relative motion of two classical particles interacting via a particular Lienard-Wiechert potential[1] can be described exactly by a Hamiltonian, which to a very good approximation is simply the classical, single particle Hamiltonian, but with changed parameters. The corresponding Schrödinger equation is

$$\left\{\frac{1}{2m}p^2+V(x)\right\}\psi(x) = \frac{k^2}{2m}\psi(x),\ V(x) = \frac{\alpha Z_1 Z_2}{|x|}. \tag{1}$$

* Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

** Address: Instytut Fizyki Jądrowej, Kawiory 26a, 30-055 Kraków, Poland.

[1] One particle acts only as emitter and the other only as absorber; thus the first particle moves in the advanced (retarded) Lienard-Wiechert potential of the second particle, and the second particle in the retarded (advanced) potential of the first.

Here $x = x_1 - x_2$ is the relative coordinate, the two one-particle coordinates x_1 and x_2, however, are evaluated at different times: $|t_1 - t_2| = |x_1 - x_2|$ and either always $t_1 \geq t_2$ or always $t_2 \geq t_1$. The process is described in the centre of mass frame and p is the momentum conjugated to x. For large intraparticle distances, where $V(x)$ is negligible, p reduces to the momentum of particle 1, because the "centre of mass coordinate" has been chosen as

$$X = \frac{1}{\sqrt{s}} (\varepsilon_1 x_1 + \varepsilon_2 x_2), \tag{2}$$

where ε_i denotes the c.m.s. energy of particle i and $\sqrt{s} = \varepsilon_1 + \varepsilon_2$. Consequently, k^2 is the square of the c.m.s. momentum of particle 1 calculated assuming no interaction

$$k^2 = \frac{1}{4s} [s - (m_1 + m_2)^2] [s - (m_1 - m_2)^2]. \tag{3}$$

The reduced mass

$$m = \frac{1}{2\sqrt{s}} (s - m_1^2 - m_2^2). \tag{4}$$

Note that for slow particles both (3) and (4) reduce correctly to their nonrelativistic limits.

We will show now that equation (1) yields good values for the small angle scattering cross-section for a pair of spinless particles interacting through the standard Coulomb potential. The solution of Eq. (1) is known (cf. e.g. [2]) and yields the cross-section

$$\frac{d\sigma}{d\Omega} = \left(\frac{m\alpha Z_1 Z_2}{2k^2}\right)^2 \sin^{-4} \frac{\Theta}{2}, \tag{5}$$

where α is the fine structure constant. This formula is valid here in spite of $t_1 \neq t_2$, because the particle momentum p has the same meaning in Eq. (1) and in its non-relativistic analogue, and x is in both cases conjugated to p. The exact result for the differential cross--section of two scalar particles interacting through a Coulomb potential may be well approximated by the one photon exchange contribution. A standard calculation (cf. e.g. [3]) yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left(1 + \frac{1}{v_1 v_2} - \sin^2 \frac{\Theta}{2}\right)^2 \sin^{-4} \frac{\Theta}{2}, \tag{6}$$

where v_i is the c.m.s. velocity of particle i. This is to be compared with the formula obtained by substituting the reduced mass (4) into expression (5)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left(1 + \frac{1}{v_1 v_2}\right)^2 \sin^{-4} \frac{\Theta}{2}. \tag{7}$$

It is seen that expression (7) is not exact, but that it is a very good approximation if either at least one of the particles is slow ($v_1 v_2 \ll 1$), or the scattering angle is small ($\sin^2 \theta/2 \ll 1$). In the general case the relative error introduced by using (7) instead of (6) is

$$\frac{1}{4} \frac{q^2}{p^2 + \varepsilon_1 \varepsilon_2 - q^2}, \tag{8}$$

where q^2 is the square of the momentum transfer. In most practical cases this is very small and formula (7) may be safely used. We checked that formula (7) holds with an accuracy to terms of order θ^2 also for the non-flip amplitudes in the scattering of spin 1/2 particles. Equation (1) gives moreover a reasonable spectrum of bound states, provided the squared energy of the ground state

$$s_0 > m_1^2 + m_2^2. \tag{9}$$

Our conclusions can be summarized as follows:

1. Equation (1) with the reduced mass (4) yields the correct cross-section for forward Coulomb scattering of spinless particles. Conversely formula (4) is the only one, which gives the correct forward scattering cross-section. In this sense it is the best possible formula for the reduced mass.
2. For scattering angles different from zero the approximation deteriorates, though numerically it remains satisfactory for most cases of practical interest. Since the reduced mass should not depend on the scattering angle, this cannot be remedied by a redefinition of the reduced mass. This discrepancy is not due to the terms omitted, when deriving Eq. (1), because these terms are of order α^2 and do not contribute to the α^2 term in the differential cross-section.
3. It is plausible that also processes other than electromagnetic could be treated along similar lines i.e. by using a Schrödinger equation with the potential known from low energy (non-relativistic) situations and substituting the reduced mass (4). As immediately seen from the formulae of the eikonal approximation [2], in this approach energy independent potentials would correspond to energy independent, non-zero cross-sections at very high energies.

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