HIGH-ENERGY BREMSSTRAHLUNG AND PAIR PRODUCTION IN THE COULOMB FIELD: BETHE AND MAXIMON VERSUS CHENG AND WU APPROACHES

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High-energy bremsstrahlung and pair production in the Coulomb field are investigated. The Cheng and Wu impact formula for the amplitude (obtained for a screened potential) is evaluated in the limit when the screening is removed; it is compared then with the high-energy limit of the amplitude calculated by Bethe and Maximon for the unscreened potential. The two limits are shown to be identical provided we calculate correctly the no-screening limit of the Coulomb scattering amplitude. In Cheng and Wu paper this calculation was wrong what led to double counting of the Bethe-Heitler amplitude for pair creation.

1. *Introduction*

The processes of high energy electromagnetic bremsstrahlung and pair production in the Coulomb field of a nucleus are of considerable practical interest and have been studied theoretically for quite a long time.

Bethe and Heitler [1] were the first to obtain relativistic amplitudes for these reactions. The Born approximation which they used is sufficiently accurate for light nuclei, i.e. when $Z\alpha \ll 1$ ($\alpha = 1/137$). For heavy elements one encounters deviations from the Bethe-Heitler result by up to 10%, caused by multiple photon exchange. For instance, for lead $Z\alpha$ is as large as 0.6 and the Born approximation is no longer valid.

The next step was done in 1954 by Bethe and Maximon [2] (hereafter referred to as BM). They investigated high-energy bremsstrahlung and pair production in the unscreened field of a pointlike charge and were able to derive closed expressions for the amplitudes and cross-sections, valid to the lowest order in α and to all orders in $Z\alpha$. This was achieved

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by using the exact (in the high-energy limit) solutions of the Dirac equation in the external potential. The result of this calculation is valid for any $Z\alpha$ and for energies of all electrons and/or positrons involved much higher than the electron mass, m.

More recently, Cheng and Wu (CW) carrying out their extensive program of studying QED at high-energies obtained simple expressions for the bremsstrahlung and pair creation amplitudes [3] in the so-called impact picture [4] (see also [5, 6]). Their formulae include the Coulomb interaction to all orders in $Z\alpha$ and can be used for screened potentials as well, but otherwise have a more limited range of applicability than the BM result. They are only valid in the limit where the incident energy ω tends to infinity (in practice $\omega \gg m$), all longitudinal momenta being fixed fractions of ω and all transverse momenta being fixed and much smaller than ω, the transverse momentum transfer also satisfying $|\Delta_\perp| \gg m^2/\omega$[1]. The last restriction on the momentum transfer is a consequence of the singular nature of the Coulomb interaction. Therefore the impact picture formulas cannot be used for calculations of the total cross-sections because integrated cross-sections acquire important contributions from the almost forward direction, $|\Delta_\perp| \sim m^2/\omega$ [12].

There exist then two completely different calculations of the bremsstrahlung and pair production amplitudes. Still, no comparison of the CW results with those of BM appeared in the literature so far, this being apparently hindered by the mathematical complexity of the CW pair production amplitude, given in the form of a nontrivial integral[2] rather than an explicit algebraic expression. Apart from obvious practical reasons, such a comparison might have more general implications in establishing the validity of the impact picture for the unscreened Coulomb potential. Let us now discuss the last point in some more detail.

Historically, the impact picture was derived by CW in the case of Delbrück scattering in the *unscreened* Coulomb potential [7]. This calculation, however, was done in the lowest order in $Z\alpha$, i.e. taking into account two-photon exchange only. On the other hand, in derivations of the impact formula to all orders in the external potential [4]–[6] it has always been assumed that the potential is of *finite range*. The reason for this restriction becomes evident when one considers the physical intuition underlying the impact picture: in the case of bremsstrahlung, for instance, one imagines the incident high-energy electron fluctuating into a system of an electron and a photon. This virtual system can then materialize by receiving a necessary momentum transfer from the external potential. Alternatively, the incident electron can go off-mass-shell by absorbing some momentum from the external potential and then decay into an electron and a photon.

To the lowest order in $Z\alpha$ and the lowest order in α this is, indeed, all that can happen. To higher orders in $Z\alpha$, however, the incident electron could scatter off the external potential, emit a photon, and then scatter again; such processes are not taken into account in the impact picture. In fact, if the potential is of finite range, they do not contribute in the high-energy limit. This can be most easily seen by using the Lorentz dilation argument: since the life-time of the virtual state grows with the energy, there will be eventually only a negligible probability of the photon emission just inside the *finite* region of interaction with the potential.

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[1] For simplicity we will refer to this limit as the "high-energy, fixed momentum transfer limit".

[2] Strictly speaking this integral, as it stands, is ill-defined and requires a regularization.

For an unscreened infinite range potential this argument does not hold. One may then consider a limit of an impact formula for a screened potential when the screening parameter, say, the "photon mass" κ, tends to zero. This is how the results of Ref. [3] were obtained. However, the limit $\kappa \to 0$ is a nontrivial one and, indeed, as we will see, was not evaluated correctly in Ref. [3]. Furthermore, it is a priori not known whether the limits $\omega \to \infty$ and $\kappa \to 0$ are interchangeable (the interchangeability has only been explicitly proved in the two-photon exchange Delbrück scattering).

In view of the above considerations it should be clear that the bremsstrahlung and pair production processes offer a unique possibility of checking the validity of the above-mentioned limiting procedure $\kappa \to 0$. This can be done by comparing the impact picture results with those of the BM approach, in which the infinite range of the potential is taken into account right from the begining. Such a comparison is the objective of the present paper[3].

In the case of both bremsstrahlung (discussed in Section 2) and pair production (Section 3) our strategy is the following:

(A) We start with discussing the appropriate impact formula obtained for $\kappa > 0$, and take the limit $\kappa \to 0$.

(B) Then we turn to the BM no-screening amplitude and take its high-energy, fixed momentum transfer limit.

What we find is that, indeed, both limits give the same answer. Nevertheless, we find that only the bremsstrahlung amplitude obtained in (B) agrees with that arrived at by CW [3]; their pair production amplitude, on the other hand, differs from ours. The difference lies, as we show, in a misinterpretation of the factor

$$S^{\pm}(q_{\perp}) = \mp 4\pi \nu i (q_{\perp}^2)^{-1\mp i\nu},$$

(with $\nu = Z\alpha$) appearing in the impact formula. Contrary to the commonly made conjecture [5, 3] this expression, when appropriately regularized, should be interpreted as the S-matrix and *not* T-matrix element for positron or electron Coulomb scattering in the eikonal approximation[4]. This misinterpretation led Cheng and Wu to double counting of the Bethe-Heitler amplitude for pair creation.

2. *Bremsstrahlung*

2A. No-screening limit of the impact formula

The amplitude of the bremsstrahlung process is given (when radiative corrections are neglected) by a set of Feynman diagrams of Fig. 1, where the crosses denote interactions with the external potential. The sum runs over all possible numbers of interactions before and after the emission of the photon. In the case of the screened Coulomb potential

$$V_{\kappa}(r) = \frac{Ze}{4\pi r} e^{-\kappa r}, \tag{2.1}$$

[3] A preliminary version of this work appeared as a preprint TPJU-5/1974.

[4] This result was anticipated in the special case of production of an e^+e^- pair with zero relative momentum [8].

with each cross there is associated a factor $-ie^2Z\gamma_0(\boldsymbol{q}^2+\kappa^2)^{-1}$, where κ is the screening parameter (the "photon mass") and $\boldsymbol{q}$ is the three-momentum transfer in a static potential (the zeroth component of the four-momentum is conserved).

Let us sketch now how the high-energy impact formula can be obtained in our particular case. Consider the high-energy, fixed momentum transfer limit (see the first footnote

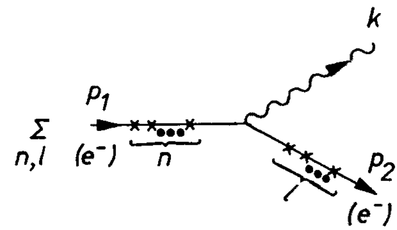

Fig. 1. The Feynman diagrams for the bremsstrahlung amplitude

in the Introduction). It is easy to check that in this limit, taken with κ = const $\neq 0$, only those diagrams survive in which all the interactions with the potential take place either *before* or *after* the emission of the photon (this follows from considering singularities in the appropriate integrals over q_z, the longitudinal momentum transfer). In this way the sum of Fig. 1 reduces to that of Fig. 2, where the heavy dots denote the multiple interaction with the potential. When in the diagrams of this figure the eikonal approximation

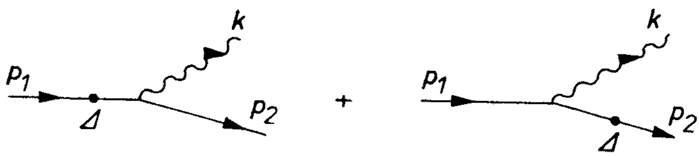

Fig. 2. The impact diagrams for Bremsstrahlung. The momenta are parametrized: $\boldsymbol{p}_1 = (\boldsymbol{0}_\perp, \omega)$, $\boldsymbol{p}_2 = (-\boldsymbol{k}_\perp+\boldsymbol{\Delta}_\perp, (1-\beta)(\omega+\Delta_z))$, $\boldsymbol{k} = (\boldsymbol{k}_\perp, \beta(\omega+\Delta_z))$

is made in the electron propagation between successive scatterings, one arrives at the "impact picture" formula [3][5]

$$\mathcal{M}^{e,e\gamma}_{\mathrm{IP}}(\kappa,\omega) = -2i\omega\frac{1}{e^2}\mathcal{I}^{e,e\gamma}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp)S^-_\kappa(\boldsymbol{\Delta}_\perp), \tag{2.2}$$

where $\boldsymbol{r}_\perp = \frac{1}{2}\boldsymbol{\Delta}_\perp$, $\boldsymbol{\Delta} = \boldsymbol{p}_2+\boldsymbol{k}-\boldsymbol{p}_1$, $\boldsymbol{p}=\frac{1}{2}(\boldsymbol{k}-\boldsymbol{p}_2)$ and where the normalization conventions of Ref. [3] have been adopted. The remaining parameters are defined in Fig. 2. In the "impact factor", given according to [3], by

$$\mathcal{I}^{e,e\gamma}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp) = \lim_{\omega\to\infty}\left(-\frac{1}{2\omega}\right)e^3\beta(1-\beta)$$

$$\times\bar{u}(p_2)\left[\frac{\gamma_j(\hat{p}_1+\hat{\Delta}+m)\gamma_0}{(\boldsymbol{k}_\perp-\beta\boldsymbol{\Delta})^2+m^2\beta^2} - \frac{\gamma_0(\hat{p}_2-\hat{\Delta}+m)\gamma_j}{(1-\beta)\,(\boldsymbol{k}_\perp^2+m^2\beta^2)}\right]u(p_1), \tag{2.3}$$

[5] For simplicity of notation in writing the amplitudes we omit the fixed arguments $\boldsymbol{p}_{1\perp}, \boldsymbol{p}_{2\perp}, \boldsymbol{k}_\perp$ and β.

the two terms in the square bracket correspond to the two terms of Fig. 2. The index j in γ_j denotes the emitted photon polarization and $\hat{a} \equiv \boldsymbol{a} \cdot \boldsymbol{\gamma}$. Finally, the quantity

$$S_\kappa^\pm(\boldsymbol{q}_\perp) = \int d^2x_\perp e^{-i\boldsymbol{q}_\perp \cdot \boldsymbol{x}_\perp} e^{\mp 2i\nu K_0(\kappa|\boldsymbol{x}_\perp|)} \tag{2.4}$$

is the non-spin-flip S-matrix element for the $e^\pm$-Coulomb scattering in the eikonal approximation (here K_0 is the modified Bessel function [9]).

Following the original derivation by Cheng and Wu we take now the limit $\kappa \to 0$. According to Refs [4, 5] for any $\boldsymbol{q}_\perp \neq 0$ there exists the limit

$$\lim_{\substack{\kappa\to 0+ \\ \boldsymbol{q}_\perp \neq 0}} e^{\mp i\Phi(\kappa)} S_\kappa^\pm(\boldsymbol{q}_\perp) = S^\pm(\boldsymbol{q}_\perp), \tag{2.5}$$

where[6]

$$S^\pm(\boldsymbol{q}_\perp) = \mp i \frac{4\pi\nu}{(\boldsymbol{q}_\perp^2)^{1\pm i\nu}}, \tag{2.6}$$

and

$$\Phi(\kappa) = 2\nu \ln (C\kappa) + 2 \operatorname{Arg} \Gamma(1+i\nu), \tag{2.7}$$

with $C = e^\gamma$, $\gamma \simeq 0.5772$ being the Euler constant. It is seen thus that for $\kappa \to 0$ the only κ-dependence is contained in a phase factor. Cheng and Wu have omitted this "infinite phase" and arrived at the formula (in our notation)

$$\mathcal{M}_{\rm CW}^{e,e\gamma}(\omega) = -2i\omega \frac{1}{e^2} \mathcal{I}^{e,e\gamma}(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp) S^-(\boldsymbol{\Delta}_\perp). \tag{2.8}$$

The result obtained by keeping all the phase factors is then related to theirs by

$$\mathcal{M}_{\rm CW}^{e,e\gamma}(\omega) = \lim_{\substack{\kappa\to 0+ \\ \boldsymbol{\Delta}_\perp \neq 0}} e^{i\Phi(\kappa)} \mathcal{M}_{\rm IP}^{e,e\gamma}(\kappa, \omega). \tag{2.9}$$

2B. High-energy limit of the unscreened case

Our next task will be to show that (2.8) is also the high-energy, fixed momentum transfer limit of the amplitude for the unscreened potential. In this case the Feynman diagrams of Fig. 1 can effectively be summed up by solving the Dirac equation in the external potential $V_0(r)$ (i.e. with $\kappa = 0$). The solutions of this equation are well known and their approximate forms (for large angular momentum) are relatively simple [10, 2]. For instance, the wave function of the incoming electron of momentum $\boldsymbol{p}_1$ is in this approximation given by[7]

$$\psi_1(\boldsymbol{x}) = N_1 e^{i\boldsymbol{p}_1\cdot\boldsymbol{x}} \left(1 - \frac{i}{2E_1} \boldsymbol{\alpha} \cdot \nabla_x\right) F_1(\boldsymbol{x}) u(\boldsymbol{p}_1), \tag{2.10}$$

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[6] Our $S^\pm(\boldsymbol{q}_\perp)$ is related to the amplitudes in Ref. [3] by $S^\pm(\boldsymbol{q}_\perp) = \pm ieV_\pm(\boldsymbol{q}_\perp)$.

[7] To simplify the notation we approximate the velocities of the electrons by 1.

and that of the outgoing electron by

$$\psi_2^\dagger(\boldsymbol{x}) = N_2^* e^{-ip_2\cdot x}\left(1-\frac{i}{2E_2}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_x\right)F_2^*(\boldsymbol{x})u^\dagger(\boldsymbol{p}_2), \tag{2.11}$$

where F_1 and F_2 are confluent hypergeometric functions,

$$F_1(\boldsymbol{x}) = {}_1F_1(i\nu, 1; ip_1|\boldsymbol{x}|-i\boldsymbol{p}_1\cdot\boldsymbol{x}), \tag{2.12}$$

$$F_2^*(\boldsymbol{x}) = {}_1F_1(i\nu, 1; ip_2|\boldsymbol{x}|+i\boldsymbol{p}_2\cdot\boldsymbol{x}), \tag{2.13}$$

and

$$N_1 = N_2 = \Gamma(1-i\nu)e^{\pi\nu/2},$$

are the normalization constants. The amplitude

$$\mathscr{M}_{\mathrm{NS}}^{e,e\gamma}(\omega) = \int d^3x\,\psi_2^\dagger(\boldsymbol{x})\gamma_j e^{-ik\cdot x}\psi_1(\boldsymbol{x}), \tag{2.14}$$

(with "NS' standing for "no screening") now can be cast in the form [2]

$$\mathscr{M}_{\mathrm{NS}}^{e,e\gamma}(\omega) = e\frac{\pi\nu}{\sinh\pi\nu}e^{\pi\nu}\bar{u}(\boldsymbol{p}_2)\left[\gamma_j I_1+\gamma_j\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{I}_2+\boldsymbol{\gamma}\cdot\boldsymbol{I}_3\gamma_0\gamma_j\right]u(\boldsymbol{p}_1), \tag{2.15}$$

where

$$I_1 = \lim_{\lambda\to 0+}\int d^3x e^{-i\Delta\cdot x}e^{-\lambda|x|}F_2^*F_1, \tag{2.16}$$

$$\boldsymbol{I}_2 = \lim_{\lambda\to 0+}\frac{-i}{2E_1}\int d^3x e^{-i\Delta\cdot x}e^{-\lambda|x|}F_2^*(\boldsymbol{\nabla}_x F_1), \tag{2.17}$$

$$\boldsymbol{I}_3 = \lim_{\lambda\to 0+}\frac{i}{2E_2}\int d^3x e^{-i\Delta\cdot x}e^{-\lambda|x|}(\boldsymbol{\nabla}_x F_2^*)F_1 \tag{2.18}$$

(the factor $e^{-\lambda|x|}$ is introduced here in order to regularize the integrals). Since the integrals (2.16) and (2.17) are also explicitly evaluated in Ref. [2] it is only a question of algebra to find a relation between the amplitudes (2.9) and (2.15). For instance, in our high-energy, fixed momentum transfer limit the integral I_1 takes the form

$$I_1 \propto -2\cdot 4\pi\nu\frac{\sinh\pi\nu}{\pi\nu}e^{-\pi\nu}(4p_1p_2)^{-i\nu}(\Delta_\perp^2)^{-1+i\nu}\left(\frac{p_1}{D_1}+\frac{p_2}{D_2}\right),$$

where

$$D_1 = -\Delta^2-2\boldsymbol{p}_1\cdot\Delta \simeq \frac{(\boldsymbol{k}_\perp-\beta\Delta_\perp)^2+m^2\beta^2}{\beta(1-\beta)},$$

and

$$D_2 = -\Delta^2+2\boldsymbol{p}_2\cdot\Delta \simeq -\frac{k_\perp^2+m^2\beta^2}{\beta}.$$

Using a high-energy approximation $E_i \simeq p_i$ and spinor identities

$$2E_1 u(p_1) = (\hat{p}_1 + m)\gamma_0 u(p_1), \quad 2E_2 \bar{u}(p_2) = \bar{u}(p_2)\gamma_0(\hat{p}_2 + m),$$

we can write the term in (2.15) proportional to I_1 as

$$-Ze^2\beta(1-\beta)(4p_1p_2)^{-i\nu}(\Delta_\perp^2)^{-1+i\nu}\bar{u}(p_2)\left[\frac{\gamma_j(\hat{p}_1+m)\gamma_0}{(k_\perp - \beta\Delta_\perp)^2 + m^2\beta^2} - \frac{\gamma_0(\hat{p}_2+m)\gamma_j}{(1-\beta)(k_\perp^2 + m^2\beta^2)}\right]u(\vec{p}_1)_i$$

This is, up to a phase factor, a part of the formula (2.9). The remaining piece comes from the I_2 and I_3 terms, as can be readily checked knowing that [2]

$$I_{2,3} \propto 4\pi\nu e^{-\pi\nu}\frac{\sinh \pi\nu}{\pi\nu}(4p_1p_2)^{-i\nu}(\Delta_\perp^2)^{-1+i\nu}\left(\frac{\mp\Delta_\perp}{D_{1,2}}\right),$$

where the minus sign refers to I_2 and plus to I_3. Finally we obtain the relation

$$\frac{1}{\omega}\mathcal{M}_{\mathrm{CW}}^{e,e\gamma}(\omega) = \lim_{\substack{\omega\to\infty \\ \Delta_\perp \neq 0}} \frac{1}{\omega}[4\omega^2(1-\beta)]^{i\nu}\mathcal{M}_{\mathrm{NS}}^{e,e\gamma}(\omega). \tag{2.19}$$

The unobservable overall phase factor here results from an arbitrariness of the phase of the Coulomb wave function.

The numerical identity of the limits of Eq. (2.2) and (2.15) might have been expected on the grounds that, as noted by BM as well as CW, their amplitudes including multi--photon exchange differ from the first order Born amplitude by an irrelevant phase factor only; the Born approximation itself is obviously the same in both limits considered.

This agreement is, however, not at all obvious when we realize that two *different* sets of Feynman diagrams are summed up in $\mathcal{M}_{\mathrm{IP}}$ and $\mathcal{M}_{\mathrm{NS}}$. As we have mentioned above, to $\mathcal{M}_{\mathrm{NS}}$ contribute all the Feynman diagrams of Fig. 1, whereas to $\mathcal{M}_{\mathrm{IP}}$ only those of Fig. 2. This means that in the impact picture the electron scattering occurs always either *before* or *after* emitting the photon. It may, therefore, be interesting to note that in the amplitude $\mathcal{M}_{\mathrm{NS}}$ one can also find a specific manifestation of this phenomenon. Namely, it turns out that in the fixed Δ high-energy limit $\mathcal{M}_{\mathrm{NS}}$ splits into a sum of two terms: in the first term the electron scatters with a finite momentum transfer ($\simeq \Delta_\perp$) before emiting the photon, and after the emission it undergoes only an almost forward scattering with an infinitesimally small momentum transfer; and in the second term vice versa. Thus these two terms correspond, in some sense, to the two terms in the impact picture (Fig. 2).

To formulate this statement more precisely, let us rewrite the amplitude $\mathcal{M}_{\mathrm{IP}}$ in the momentum space. In order to simplify the algebra, we shall only consider the term proportional to the integral I_1; the remaining terms can be treated along exactly the same lines. Thus we have

$$I_1 = \lim_{\lambda\to 0+} (2\pi)^{-3}\int d^3q_1 d^3q_2 \delta^{(3)}(\Delta - q_1 - q_2)\tilde{F}_2^*(q_2)\tilde{F}_1(q_1), \tag{2.20}$$

where

$$\tilde{F}_1(\boldsymbol{q}) = \int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} e^{-\lambda|\boldsymbol{x}|} F_1(\boldsymbol{x}), \tag{2.21}$$

and similarly for $F_2^*(\boldsymbol{q})$. In Eq. (2.20) $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are the net momentum transfers received (in multiple interaction with the potential) by the electron before and after emitting the photon. Let us then choose, for transverse momenta fixed and $\omega \to \infty$, a small number ξ, such that $m^2/\omega \ll \xi \ll |\boldsymbol{\Delta}_\perp|$, and write the integral I_1 as a sum of three terms,

$$I_1 = I_1^{(0)} + I_1^{(1)} + I_1^{(2)}, \tag{2.22}$$

where the $\boldsymbol{q}_\perp$-integration in $I_1^{(i)}$ is restricted to $|\boldsymbol{q}_{i\perp}| \leqslant \xi$. It can be shown now (see Appendix A) that in the limit $\omega \to \infty$, $\boldsymbol{\Delta}_\perp$ fixed, the term $I_1^{(0)}$ is negligible in comparison with the two next terms (its modulus tends to a constant, whereas the whole sum I_1 grows linearly with ω). This means that, indeed, the main contributions to bremsstrahlung in the unscreened field come from the processes with a very small momentum transfer $\boldsymbol{q}_2$ after the photon emission (term $I_1^{(2)}$) or with a very small momentum transfer $\boldsymbol{q}_1$ before the photon emission (term $I_1^{(1)}$).

3. Pair production

3A. No-screening limit of the impact formula

The Feynman diagrams contributing to the pair production in the static external potential are given in Fig. 3, where notation is also explained. In the high-energy, fixed momentum transfer limit, using e.g. the rules of Ref. [4], one arrives at the impact formula [3] (see Fig. 4)

$$\mathscr{M}_{\mathrm{IP}}^{\gamma,ee}(\kappa,\omega) = i\omega(2\pi)^{-2}\int d^2q_\perp \frac{1}{e^2}\mathscr{I}^{\gamma,ee}(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp, \boldsymbol{q}_\perp)$$

$$\times\{S_\kappa^+(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)S_\kappa^-(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp) - (2\pi)^4\delta^2(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)\delta^2(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)\} \tag{3.1}$$

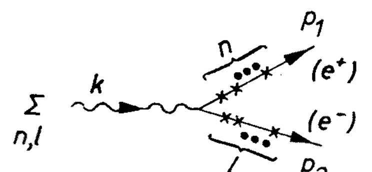

Fig. 3

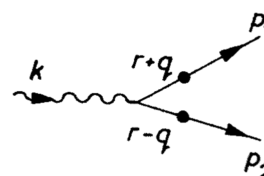

Fig. 4

Fig. 3. The Feynman diagrams for the pair production amplitude

Fig. 4. The impact diagram for pair production. The momenta are parametrized: $\boldsymbol{k} = (\boldsymbol{0}_\perp, \omega)$, $\boldsymbol{p}_i = (\boldsymbol{p}_{i\perp}, \beta_i(\omega+\Delta_z))$ $(i = 1, 2)$, with $\beta_1+\beta_2 = 1$

where, according to [3],

$$\mathscr{I}^{\gamma,ee}(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp, \boldsymbol{q}_\perp) = \lim_{\omega\to\infty}(-e^3)\bar{u}(p_2)\Bigg[2\beta_1\beta_2\gamma_j + \frac{1}{\omega}\beta_2\gamma_j\gamma_0(\hat{\boldsymbol{r}}_\perp+\hat{\boldsymbol{q}}_\perp)$$

$$+\frac{1}{\omega}\beta_1(\hat{\boldsymbol{r}}_\perp-\hat{\boldsymbol{q}}_\perp)\gamma_0\gamma_j\Bigg]v(p_1)\,[(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)^2+m^2]^{-1}, \tag{3.2}$$

where $2\boldsymbol{r}_\perp = \boldsymbol{\Delta}_\perp \equiv \boldsymbol{p}_{1\perp}+\boldsymbol{p}_{2\perp}-\boldsymbol{k}_\perp, \boldsymbol{p}_\perp = \frac{1}{2}(\boldsymbol{p}_{1\perp}-\boldsymbol{p}_{2\perp})$, and $S_\kappa^\pm$ are the eikonal S-matrix elements (2.4) for $e^\pm$ scattering in the screened potential.

Consider now the limit $\kappa \to 0$. As mentioned in Section 2 the $\kappa \to 0$ limit of $S_\kappa^\pm(\boldsymbol{q}_\perp)$ for any $\boldsymbol{q}_\perp \neq 0$ is given, after extracting a phase factor, by $S^\pm(\boldsymbol{q}_\perp)$, Eq. (2.6). However, in the formula (3.1) above, there is an integration over all values of $\boldsymbol{q}_\perp$ and therefore we must know what is the behaviour of the $\kappa \to 0$ limit when $\boldsymbol{q}_\perp \to 0$ (note that the convergence of $S_\kappa^\pm(\boldsymbol{q}_\perp)$ to $S^\pm(\boldsymbol{q}_\perp)$ is *not* uniform in $\boldsymbol{q}_\perp$).

In particular Cheng and Wu [4, 3] interpreted $S^\pm(\boldsymbol{q}_\perp)$ for all $\boldsymbol{q}_\perp$'s as the *amplitude* and conjectured implicitly that the limit of the S-matrix element $S_\kappa^\pm$ is given by

$$\lim_{\kappa\to 0+} e^{\mp i\Phi(\kappa)} S_\kappa^\pm(\boldsymbol{q}_\perp) \stackrel{?}{=} (2\pi)^2\delta^2(\boldsymbol{q}_\perp)+S^\pm(\boldsymbol{q}_\perp),$$

(indeed, in the *lowest order* in v, $S^\pm(\vec{q}_\perp)$ reduces to the Born amplitude). In this way they have been led to the following expression (in our notation) for the pair production amplitude [3]

$$\mathcal{M}_{\mathrm{CW}}^{\gamma,ee}(\omega) = i\omega(2\pi)^{-2}\int d^2q_\perp \frac{1}{e^2}\mathcal{I}^{\gamma,ee}(\boldsymbol{r}_\perp.\boldsymbol{p}_\perp,\boldsymbol{q}_\perp)$$
$$\times\{[(2\pi)^2\delta^2(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)+S^+(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)][(2\pi)^2\delta^2(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)+S^-(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)]$$
$$-(2\pi)^4\delta^2(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)\delta^2(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)\}. \tag{3.3}$$

The integral in this formula is, in fact, divergent unless somehow regularized. In similar cases Cheng and Wu used implicitly a regularization equivalent to the following[8]:

$$\mathcal{M}_{\mathrm{CW}}^{\gamma,ee}(\omega) \stackrel{?}{=} \lim_{\varrho\to 0+} i\omega(2\pi)^{-2}\int d^2q_\perp \frac{1}{e^2}\mathcal{I}^{\gamma,ee}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp,\boldsymbol{q}_\perp)$$
$$\times\{[(2\pi)^2\delta^2(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)+\bar{S}_\varrho^+(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)][(2\pi)^2\delta^2(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)+\bar{S}_\varrho^-(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)]$$
$$-(2\pi)^4\delta^2(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)\delta^2(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)\} \tag{3.4}$$

where

$$\bar{S}_\varrho^\pm(\boldsymbol{q}_\perp) = \mp i\frac{4\pi v}{(q_\perp^2)^{1\pm iv-\varrho}}. \tag{3.5}$$

Below we will show, however, that the $\kappa \to 0$ limit of the amplitude (3.1) is not given by the above expression. We obtain instead (for $\Delta_\perp \neq 0$)

$$\lim_{\substack{\kappa\to 0+\\ \Delta_\perp\neq 0}} \mathcal{M}_{\mathrm{IP}}^{\gamma,ee}(\kappa,\omega) = \lim_{\varrho\to 0+} i\omega(2\pi)^{-2}\int d^2q_\perp \frac{1}{e^2}\mathcal{I}^{\gamma,ee}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp,\boldsymbol{q}_\perp)$$
$$\times\{\bar{S}_\varrho^+(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)\bar{S}_\varrho^-(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)-(2\pi)^4\delta^2(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)\delta^2(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)\}. \tag{3.6}$$

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[8] See, e.g., Eq. (A10) of Ref. [11]. In this formula an analytic continuation in v is understood, equivalent to our regularization.

The difference between the last two formulae i. e. (3.4) and (3.6), is quite important, as it is given by

$$\frac{i\omega}{e^2}\{\mathscr{I}^{\gamma,ee}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp,\boldsymbol{r}_\perp)S^+(\boldsymbol{\Delta}_\perp)+\mathscr{I}^{\gamma,ee}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp,-\boldsymbol{r}_\perp)S^-(\boldsymbol{\Delta}_\perp)\},$$

which is (in the lowest order in ν) just the one photon exchange (Bethe–Heitler) amplitude.

The proof of Eq. (3.6) is quite straightforward. First we note that for any positive number ξ the convergence of $S_\kappa^\pm(\boldsymbol{q}_\perp)$ to $S^\pm(\boldsymbol{q}_\perp)$, as given by Eq. (2.5), is uniform in the region $|\boldsymbol{q}_\perp|>\xi$. Obviously, the convergence of $\bar{S}_\varrho^\pm(\boldsymbol{q}_\perp)$ to $S^\pm(\boldsymbol{q}_\perp)$ when $\varrho\to 0$ is also uniform in this region. It follows that

$$\lim_{\kappa\to 0+}\int_{\Xi} d^2q_\perp \mathscr{I}^{\gamma,ee}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp,\boldsymbol{q}_\perp)S_\kappa^+(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)S_\kappa^-(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)$$
$$=\lim_{\varrho\to 0+}\int_{\Xi} d^2q_\perp \mathscr{I}^{\gamma,ee}(\boldsymbol{r}_\perp,\boldsymbol{p}_\perp,\boldsymbol{q}_\perp)\bar{S}_\varrho^+(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)\bar{S}_\varrho^-(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp), \tag{3.7}$$

where Ξ denotes the region where $|\boldsymbol{r}_\perp+\boldsymbol{q}_\perp|>\xi$ *and* $|\boldsymbol{r}_\perp-\boldsymbol{q}_\perp|>\xi$. Now, since ξ may be arbitrarily small (in particular $\xi\ll|\boldsymbol{r}_\perp|$) and since $\mathscr{I}^{\gamma,ee}$ is regular for $\boldsymbol{q}_\perp\simeq\pm\boldsymbol{r}_\perp$, it is sufficient to prove that

$$\lim_{\kappa\to 0+} e^{\mp i\Phi(\kappa)}(2\pi)^{-2}\int_{|\boldsymbol{q}_\perp|\leqslant\xi} d^2q_\perp S_\kappa^\pm(\boldsymbol{q}_\perp) = \lim_{\varrho\to 0+}(2\pi)^{-2}\int_{|\boldsymbol{q}_\perp|\leqslant\xi} d^2q_\perp \bar{S}_\varrho^\pm(\boldsymbol{q}_\perp). \tag{3.8}$$

In appendix B we show that this equality holds and, incidentally, its LHS = RHS = $(\xi^2)^{\pm i\nu}$.

This completes the proof of our statement (3.6) and, quite generally, provides a precise definition of the $\kappa\to 0$ limit of the S-matrix. Some of its properties are discussed in more detail in Appendix B.

3B. High-energy limit of the unscreened case

Let us consider now the high-energy, fixed momentum transfer limit of the amplitude calculated by BM for the unscreened potential. We shall show that this limit is equal to the limit previously considered,

$$\frac{1}{\omega}\lim_{\kappa\to 0+}\mathscr{M}_{\mathrm{IP}}^{\gamma,ee}(\kappa,\omega) = \lim_{\omega\to\infty}\frac{1}{\omega}\left(\frac{\beta_1}{\beta^2}\right)^{-i\nu}\mathscr{M}_{\mathrm{NS}}^{\gamma,ee}(\omega). \tag{3.9}$$

The no-screening amplitude in this case is given by an overlap of the wave functions analogous to (2.9) and (2.10) but with the replacement of ν by $-\nu$ and $\boldsymbol{p}_1$ by $-\boldsymbol{p}_1$ in the wave function ψ_1 (this wave function describes then the outgoing positron). As before, the amplitude can be expressed in the form

$$\mathscr{M}_{\mathrm{NS}}^{\gamma,ee}(\omega) = e\frac{\pi\nu}{\sinh\pi\nu}\bar{u}(p_2)\left[\gamma_j I_1+\gamma_j\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{I}_2+\boldsymbol{\gamma}\cdot\boldsymbol{I}_3\gamma_0\gamma_j\right]v(p_1) \tag{3.10}$$

where the integrals I_1, $\boldsymbol{I}_2$, $\boldsymbol{I}_3$ are defined by (2.16)–(2.18) with (2.13) and with Eq. (2.12) replaced by

$$F_1(\boldsymbol{x}) = {}_1F_1(-i\nu,1;ip_1|\boldsymbol{x}|+i\boldsymbol{p}_1\cdot\boldsymbol{x}). \tag{3.11}$$

Similarly to what we have done in the preceding section (see Eq. (2.20)) we rewrite now the integrals I_1, I_2, I_3 in momentum space, using the Fourier transforms of the functions F_1 and F_2^*. We are then able to show that, for instance, I_1 reduces in the high--energy limit ($\omega \to \infty$) to

$$I_1 \propto -2i(4\pi\nu)^2 \frac{\sinh \pi\nu}{\pi\nu} \omega\beta_1\beta_2 \left(\frac{\beta_1}{\beta_2}\right)^{i\nu}$$

$$\times (2\pi)^{-2} \lim_{\varrho\to 0+} \int d^2 q_\perp [(\boldsymbol{r}_\perp + \boldsymbol{q}_\perp)^{2(1+i\nu-\varrho)}(\boldsymbol{r}_\perp - \boldsymbol{q}_\perp)^{2(1-i\nu-\varrho)}]^{-1}[(\boldsymbol{p}_\perp - \boldsymbol{q}_\perp)^2 + m^2]^{-1}. \quad (3.12)$$

The regularization with $\varrho \to 0+$ arises here from considering the integral I_1 as a function of a complex variable ν. More precisely, I_1 can be written as

$$I_1 = \lim_{\substack{\lambda\to 0+\\ \varrho\to 0+}} \int d^3x e^{-i\boldsymbol{\Delta}\cdot\boldsymbol{x}} e^{-\lambda|\boldsymbol{x}|} {}_1F_1(i\nu-\varrho; 1; ip_2|\boldsymbol{x}| + i\boldsymbol{p}_2\cdot\boldsymbol{x})$$

$$\times {}_1F_1(-i\nu-\varrho; |; ip_1|\boldsymbol{x}| + i\boldsymbol{p}_1\cdot\boldsymbol{x}). \quad (3.13)$$

The last integral is convergent for any $\varrho \geqslant 0$ and the result of integration is an analytic function of ϱ. If we now take the limit $\omega \to \infty$, it appears that some limiting transitions under the integral sign are permissible as long as $\varrho > 0$. This allows us to perform the q_z integration, resulting in Eq. (3.12). The details of this calculation are given in Appendix C.

Inserting Eq. (3.12) into (3.10) we obtain the contribution proportional to the first term in the curly brackets in Eq. (3.2). The remaining integrals I_2 and I_3 can be calculated in the entirely analogous way. It turns out that when $\omega \to \infty$ only $(I_{2,3})_\perp$ are different from zero, because in $(I_{2,3})_z$ the integrand is an antisymmetric function of the integration variable q_z. The same result was obtained in Ref. [2]. The I_2 and I_3 terms in (3.10) are proportional to $\boldsymbol{r}_\perp + \boldsymbol{q}_\perp$ and $\boldsymbol{r}_\perp - \boldsymbol{q}_\perp$ respectively because of the configuration space differentiation involved in (2.17) and (2.18). Adding up the terms with I_1, I_2 and I_3 gives us finally Eq. (3.9).

Note also that proving Eq. (3.9) provides us also with explicit analytic expressions for the $q_\perp$-intgrals appearing in the impact formula (3.6). The necessary formulae are listed in Appendix C.

4. *Summary and conclusions*

The most important findings of this paper can be summarized in the two equations below:

$$\frac{1}{\omega} \lim_{\kappa\to 0+} e^{i\Phi(\kappa)} \mathcal{M}_{\mathrm{IP}}^{e,e\gamma}(\kappa, \omega) = \lim_{\omega\to\infty} \frac{1}{\omega} [4\omega^2(1-\beta)]^{i\nu} \mathcal{M}_{\mathrm{NS}}^{e,e\gamma}(\omega)$$

$$= -2i\omega \frac{1}{e^2} \mathcal{J}^{e,e\gamma}(\boldsymbol{r}_\perp, \boldsymbol{p}_\perp) S^-(\Delta_\perp), \quad (4.1)$$

and

$$\frac{1}{\omega}\lim_{\kappa\to 0+}\mathscr{M}_{\text{IP}}^{\gamma,ee}(\kappa,\omega) = \lim_{\omega\to\infty}\frac{1}{\omega}\left(\frac{\beta_1}{\beta_2}\right)^{-i\nu}\mathscr{M}_{\text{NS}}^{\gamma,ee}(\omega)$$

$$= \lim_{\varrho\to 0+} i\omega(2\pi)^{-2}\int d^2q_\perp \frac{1}{e^2}\mathscr{I}^{\gamma,ee}(\mathbf{r}_\perp,\mathbf{p}_\perp,\mathbf{q}_\perp)$$

$$\times\{\bar{S}_S^+(\mathbf{r}_\perp+\mathbf{q}_\perp)\bar{S}_\varrho^-(\mathbf{r}_\perp-\mathbf{q}_\perp)-(2\pi)^4\delta^2(\mathbf{r}_\perp+\mathbf{q}_\perp)\delta^2(\mathbf{r}_\perp-\mathbf{q}_\perp)\}. \tag{4.2}$$

Here $\mathscr{M}_{\text{IP}}$ is the high-energy amplitude calculated in the impact picture for a screened Coulomb potential ($\kappa > 0$) and $\mathscr{M}_{\text{NS}}$ is the amplitude calculated using the Coulomb wave functions (with no screening) at finite energy. The impact factors $\mathscr{I}$ and the Coulomb "S-matrix elements" $S^\pm$ and $\bar{S}_\varrho^\pm$ are given by Eqs (2.3), (3.2), (2.6) and (3.5).

The two equations above state that, indeed, in the cases considered the limits $\omega\to\infty$ and $\kappa\to 0$ are interchangeable. One can thus say that the impact picture (originally obtained for $\kappa > 0$) holds also for the infinite range potential ($\kappa = 0$) in the sense that the limit $\lim_{\kappa\to 0}$ [(phase factor). $\mathscr{M}_{\text{IP}}(\kappa,\omega)$] is *numerically* the same as the high-energy amplitude calculated with $\kappa = 0$.

This result is not surprising for pair production where the incident photon does not interact with the potential and, in the impact picture language, the scattering occurs only *after* the "fluctuation" independently of whether the potential is of a finite or an infinite range. The validity of the impact picture is not so obvious in bremsstrahlung where, for finite $\Delta_\perp$, two different sets of Feynman diagrams contribute in the cases $\kappa > 0$ and $\kappa = 0$ when $\omega\to\infty$. In spite of this difference there is still a noticeable physical similarity of these two cases. In the screened potential the amplitude becomes at high-energy a sum of two terms: in the first term there is no electron scattering after the emission of the photon and in the second term no scattering before the photon emission. In the nonscreened field instead of no-scattering there occurs (multiple) scattering with a very small ($\sim m^2/\omega$) momentum transfer.

Another result contained in Eq. (4.2), which seems to be rather important, is that the $\kappa\to 0$ limit of the (eikonal) S-matrix element $S_\kappa^\pm(\mathbf{q}_\perp)$ is given (after extracting an appropriate phase factor) by the quantity $S^\pm(\mathbf{q}_\perp)$ or, more precisely[9], by the regularized expression $\lim_{\varrho\to 0+}\bar{S}_\varrho^\pm(\mathbf{q}_\perp)$, and not by $(2\pi)^2\delta^2(\mathbf{q}_\perp)+S^\pm(\mathbf{q}_\perp)$. The latter assertion, commonly used in the literature [3, 5], leads to incorrect results in the case of pair production.

We should note here, however, that in some other processes both assertions give the same result. This is, for instance, the case for Delbrück scattering where, due to the symmetry properties of the amplitude, the δ-function parts of the S-matrix elements do not contribute.

To conclude, our results show that in the cases considered the procedure of taking the $\kappa\to 0$ limit of the impact formula (obtained for $\kappa > 0$) reproduces correctly the high-

[9] The mathematical properties of the $\varkappa\to 0$ limit of the Coulomb amplitudes are discussed in more detail in Appendix B.

-energy amplitude for $\kappa = 0$ (no screening). This should enable us to apply the very useful impact picture techniques to the processes involving unscreened Coulomb fields except for the almost forward region, $|\Delta_\perp| \sim m^2/\omega$.

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APPENDIX A

We estimate here the integral $I_1^{(0)}$ appearing in Eq. (2.22). Using Eqs (6.7) and (6.12) of Ref. [2] we can easily calculate the Fourier transform $\tilde{F}_1(\boldsymbol{q})$ as defined by Eq. (2.21),

$$\tilde{F}_1(\boldsymbol{q}) = 4\pi\nu e^{-2\pi\nu} 2p_1(\boldsymbol{q}^2+\lambda^2)^{-1+i\nu}(\boldsymbol{q}^2+\lambda^2+2\boldsymbol{p}_1\cdot\boldsymbol{q}-2i\lambda p_1)^{-1-i\nu},$$

and similarly for $\tilde{F}_2^*(\boldsymbol{q})$. Substituting this into Eq. (2.20) and setting $q_{1z} = r_z+q_z$, $q_{2z} = r_z-q_z$ with $r_z = \frac{1}{2}\Delta_z$ we get in the high-energy limit

$$I_1^{(0)} \propto (4\pi\nu)^2[4\omega^2(1-\beta)]^{-i\nu}(2\pi)^{-3} \int\limits_{\Xi_0} d^2q_{1\perp}d^2q_{2\perp}\delta^2(\Delta_\perp-\boldsymbol{q}_{1\perp}-\boldsymbol{q}_{2\perp})$$

$$\times \int dq_z[\boldsymbol{q}_{1\perp}^2+(q_z+r_z)^2]^{-1+i\nu}[\boldsymbol{q}_{2\perp}^2+(q_z-r_z)^2]^{-1+i\nu}$$

$$\times \left[\frac{\boldsymbol{q}_{1\perp}^2+(q_z+r_z)^2}{2\omega}+q_z+r_z-i\varepsilon\right]^{-1-i\nu}\left[\frac{\boldsymbol{q}_{2\perp}^2+(q_z-r_z)^2}{2(1-\beta)\omega}+q_z-r_z-i\varepsilon\right]^{-1-i\nu}.$$

Here Ξ_0 denotes the region of integration where both $|q_{1\perp}| > \xi$ and $|q_{2\perp}| > \xi$.

In the q_z-integral above the singularities contributed by the last two factors (their location being dependent on ω) fall both above the real axis. The other singularities, originating from the first two factors, are placed at $q_z = r_z \pm i|\boldsymbol{q}_{1\perp}|$ and $q_z = r_z \pm i|\boldsymbol{q}_{2\perp}|$, and their distance from the real axis is never smaller than ξ. Therefore, even if $\omega \to \infty$, the integration contour is never pinched between the singularities in the upper and lower half-planes. It follows that the $\omega \to \infty$ limit of the integral exists and $I_1^{(0)}$ is, except for the trivial phase factor, ω-independent.

APPENDIX B

In this Appendix we discuss some properties of the Coulomb scattering amplitudes in the limit $\kappa \to 0$.

Let us first prove Eq. (3.8). We write the integral on the LHS as

$$(2\pi)^{-2} \int\limits_{|q_\perp|\leqslant\xi} d^2q_\perp S_\kappa^\pm(\boldsymbol{q}_\perp) = 1+(2\pi)^{-2} \int\limits_{|\vec{q}_\perp|\leqslant\xi} d^2q_\perp \int d^2x_\perp e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{x}_\perp}[e^{\mp i\chi(|x_\perp|)}-1]$$

$$= 1+\int\limits_0^\xi qdq \int\limits_0^\infty xdxJ_0(qx)\,[e^{\mp i\chi(x)}-1],$$

where, according to (2.4),

$$\chi(x) = 2vK_0(\kappa x).$$

Since tha last x-integral is convergent uniformly in q, the q-integration can be performed first yielding

$$1+\int_0^\infty xdx\,\frac{\xi}{x}\,J_1(\xi x)\,[e^{\mp i\chi(x)}-1] = \xi\int_0^\infty dxJ_1(\xi x)e^{\mp i\chi(x)}.$$

This integral is convergent uniformly with respect to κ, which allows us to take the $\kappa \to 0$ limit inside the integral. If we use then the expansion for small z [9],

$$K_0(z) = -\ln(\tfrac{1}{2}Cz)+O(z^2\ln z),$$

and recall Eq. (2.7), we obtain for the LHS of Eq. (3.8) the value $(\xi^2)^{\pm iv}$ which is equal to the RHS.

Let us consider now some mathematical properties of the $\kappa \to 0$ limit of the Coulomb S-matrix. This limit, say $\varphi^{\pm}$,

$$\lim_{\kappa\to 0+} e^{\mp i\Phi(\kappa)}S_\kappa^{\pm}(\boldsymbol{q}_\perp) = \varphi^{\pm}(\boldsymbol{q}_\perp), \tag{B.1}$$

is a distribution defined by specifying its action on a (sufficiently regular) test function $f(\boldsymbol{q}_\perp)$. In the standard notation we have

$$(\varphi^{\pm}, f) = \lim_{\varrho\to 0+}\int d^2q_\perp \bar{S}_\varrho^{\pm}(\boldsymbol{q}_\perp)f(\boldsymbol{q}_\perp),$$

with $\bar{S}_\varrho^{\pm}$ given by Eq. (3.5).

It is interesting to observe that the distribution (B.1) possesses an expansion in powers of v. It can readily be checked that

$$\varphi^{\pm}(\boldsymbol{q}_\perp) = (2\pi)^2\delta^2(\boldsymbol{q}_\perp)+\sum_{n=1}^{\infty} v^n\varphi_n^{\pm}(\boldsymbol{q}_\perp),$$

where the expansion coefficients are distributions defined by

$$(\varphi_n^{\pm}, f) = \lim_{\xi\to 0+}\left\{(2\pi)^2\frac{(\mp i\ln\xi^2)^n}{n!}f(\boldsymbol{0}_\perp)+\int\limits_{|\boldsymbol{q}_\perp|\geq\xi} d^2q_\perp\,\frac{4\pi(\mp i)^n}{(n-1)!}\,\frac{(\ln\boldsymbol{q}_\perp^2)^{n-1}}{\boldsymbol{q}^2}f(\boldsymbol{q}_\perp)\right\}.$$

The distribution (B.1) contains then the δ-function term analogous to that in the expansion of the S-matrix for $\kappa > 0$

$$S_\kappa^{\pm}(\boldsymbol{q}_\perp) = (2\pi)^2\delta^2(q_\perp)\mp i\,\frac{4\pi v}{\vec{q}_\perp^{\,2}+\kappa^2}+\dots.$$

It is seen that, for instance, the $\kappa \to 0$ limit of the lowest order ($\sim v$) Born amplitude does not exist, neither as an ordinary function nor a distribution (in a space of test functions non-vanishing at $\boldsymbol{q}_\perp = 0$). However, there exists, as a distribution, the limit

$$\lim_{\kappa\to 0+}\left\{\mp i\,\frac{4\pi v}{\vec{q}_\perp^{\,2}+\kappa^2}\mp 2iv\ln\kappa\right\} = v\varphi_1^{\pm}(\boldsymbol{q}_\perp).$$

APPENDIX C

In this Appendix we will prove Eq. (3.12) for the integral I_1 appearing in Eq. (3.10). More precisely, we shall evaluate the limit

$$L \equiv \lim_{\omega\to\infty} \frac{1}{\omega} I_1. \tag{C.1}$$

To start with, we analytically continue the wave functions F_1 and F_2^* in ν in order to regularize all the subsequent integrals. Thus, instead of F_1 and F_2^* as given by Eqs (3.11) and (2.13) we will use

$$F_1(\boldsymbol{x}) = \lim_{\varrho\to 0+} {}_1F_1(-i\nu+\varrho, 1; ip_1|\boldsymbol{x}|+i\boldsymbol{p}_1\cdot\boldsymbol{x}) \equiv \lim_{\varrho\to 0+} F_1(\varrho, \boldsymbol{x}), \tag{C.2}$$

$$F_2^*(\boldsymbol{x}) = \lim_{\varrho\to 0+} {}_1F_1(i\nu+\varrho, 1; ip_2|\boldsymbol{x}|+i\boldsymbol{p}_2\cdot\boldsymbol{x}) \equiv \lim_{\varrho\to 0+} F_2^*(\varrho, \boldsymbol{x}). \tag{C.3}$$

Now the space integration in (2.16) is uniformly convergent in $\boldsymbol{x}$ for any ϱ, provided $\lambda > 0$. Therefore we can take $\lim\limits_{\varrho\to 0+}$ out of the integral obtaining

$$\frac{1}{\omega} I_1 = \lim_{\lambda\to 0+} \lim_{\varrho\to 0+} \frac{1}{\omega} \int d^3x e^{-\lambda|\boldsymbol{x}|} F_1(\varrho, \boldsymbol{x}) F_2^*(\varrho, \boldsymbol{x}) \equiv \lim_{\lambda\to 0+} \lim_{\varrho\to 0+} A(\omega, \lambda, \varrho). \tag{C.4}$$

An analytic expression for $I_1(\lambda, \varrho)$ can be achieved by simply substituting $a_1 = \nu+i\varrho$ and $a_2 = \nu-i\varrho$ in the general formulae given by BM (Eqs (6.7) and (6.12) of Ref. [2]). An examination of the formula obtained shows that $A(\omega, \lambda, \varrho)$ is regular at the point $\lambda = \varrho = 0$ and $\omega \to \infty$. Therefore, the limits can be interchanged with the result

$$L = \lim_{\varrho\to 0+} \lim_{\omega\to\infty} \lim_{\lambda\to 0+} A(\omega, \lambda, \varrho). \tag{C.5}$$

Next let us go over to the momentum space. Using again the formulae given by BM we can easily calculate the Fourier transforms of the wave functions $F_1(\varrho, \boldsymbol{x})$ and $F_2^*(\varrho, \boldsymbol{x})$ and obtain

$$\begin{aligned} A(\omega, \lambda, \varrho) &= \frac{1}{\omega}(2\pi)^{-3} \int d^3q_1 d^3q_2 \delta^3(\boldsymbol{\Delta}-\boldsymbol{q}_1-\boldsymbol{q}_2)\tilde{F}_1(\varrho, \boldsymbol{q}_1)\tilde{F}_2^*(\varrho, \boldsymbol{q}_2) \\ &= -(4\pi\nu)^2 4\beta_1\beta_2(2\pi)^{-3} \int d^2q_\perp dQ[(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)^2+(r_z+Q/\omega)^2+\lambda^2]^{-1-i\nu+\varrho} \\ &\quad \times[(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)^2+(r_z-Q/\omega)^2+\lambda^2]^{-1+i\nu+\varrho} \\ &\quad \times[(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)^2+(r_z+Q/\omega)^2+\lambda^2-2\boldsymbol{p}_{1\perp}\cdot(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)-2\beta_1\omega(r_z+Q/\omega)-2i\lambda p_1]^{-1+i\nu-\varrho} \\ &\quad \times[(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)^2+(r_z-Q/\omega)^2+\lambda^2-2\boldsymbol{p}_{2\perp}\cdot(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)-2\beta_2\omega(r_z-Q/\omega)-2i\lambda p_2]^{-1-i\nu-\varrho}, \end{aligned} \tag{C.6}$$

where we have introduced a new variable $\boldsymbol{q}$, such that $\boldsymbol{q}_1 = \boldsymbol{r}+\boldsymbol{q}$, $\boldsymbol{q}_2 = \boldsymbol{r}-\boldsymbol{q}$, and rescaled q_z, $Q = \omega q_z$.

Our aim now is to perform (in the limit considered) the Q-integration in the formula (C.6). This will leave us eventually with the $\boldsymbol{q}_\perp$ integral (3.12).

To this end let us divide the region of the $q_\perp$-integration into three regions: Ξ_1, Ξ_2, and the rest, say Ξ_0. The regions $\Xi_i (i = 1, 2)$ are defined by the condition $|q_{i\perp}| \leq \xi$, the constant ξ being chosen such that $\lambda \ll \xi \ll |r_\perp|$. Eq. (C.5) can then be rewritten identically as

$$L = \lim_{\varrho\to 0+} \lim_{\xi\to 0+} \lim_{\omega\to\infty} \lim_{\lambda\to 0+} (A_0+A_1+A_2), \tag{C.7}$$

where the subscripts 0, 1, 2 refer to the regions Ξ_0, Ξ_1, Ξ_2 in the $q_\perp$-integration. Below we will see that in A_0 the Q-integral can be evaluated explicitly whereas the terms A_1 and A_2 give a vanishing contribution in the limit of Eq. (C.7).

Let us then consider the Q-integral with $q_\perp$ in the region Ξ_0. The location of the singularities of the integrand in the complex Q plane is shown in Fig. 5. The branch points $c_k (k = 1, \ldots, 4)$ come from the Coulomb amplitudes, and the branch points a_k from

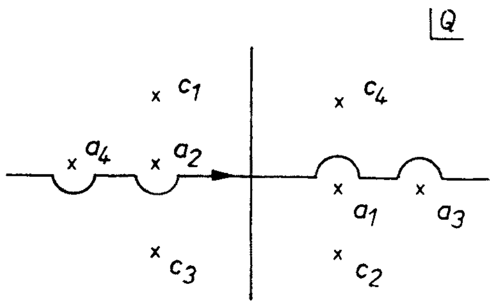

Fig. 5. Singularities of the integrand of Eq. (C.6)

the rest of the integrand in Eq. (C.6). A crucial point now is that, when $\lambda \to 0$ and $\omega \to \infty$, the integration contour never gets pinched between the singularities. Indeed, as long as $q_\perp$ in confined to Ξ_0, i.e. $|q_{1\perp}| > \xi$ and $|q_{2\perp}| > \xi$, we have $\operatorname{Im} c_1 = \omega\sqrt{\lambda^2+q_{1\perp}^2} > \omega\xi$ and $\operatorname{Im} c_2 = -\omega\sqrt{\lambda^2+q_{2\perp}^2} < -\omega\xi$. Consequently, the integration contour can be distorted to such a path Γ (Fig. 5), that the integral along Γ is uniformly convergent in the limit $\lim_{\omega\to\infty} \lim_{\lambda\to 0+}$. Taking this limit inside the integral yields

$$-(4\pi v)^2(\beta_1/\beta_2)^{iv} \lim_{\varrho\to 0+} \lim_{\xi\to 0+} (2\pi)^{-3} \int_{\Xi_0} d^2q_\perp[(r_\perp+k_\perp)^2]^{-1-iv+\varrho}$$
$$\times[(r_\perp-q_\perp)^2]^{-1+iv+\varrho} \int_\Gamma dQ[-Q+b_1]^{-1+iv-\varrho}[Q+b_2]^{-1-iv-\varrho},$$

where

$$b_1 = \frac{1}{2\beta_1}[(r_\perp+q_\perp)^2-2p_{1\perp}\cdot(r_\perp+q_\perp)]-\omega r_z, \tag{C.11}$$

$$b_2 = \frac{1}{2\beta_2}[(r_\perp-q_\perp)^2-2p_{2\perp}\cdot(r_\perp-q_\perp)]-\omega r_z. \tag{C.12}$$

The Q-integration can then be easily done giving

$$-i(4\pi\nu)^2(\beta_1/\beta_2)^{i\nu}\frac{\sinh\pi\nu}{\pi\nu}\lim_{\varrho\to 0+}\lim_{\xi\to 0+}(2\beta_1\beta_2)^{1+2\varrho}(2\pi)^{-2}$$
$$\times\int_{\Xi_0} d^2q_\perp[(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)^2]^{-1-i\nu+\varrho}[(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)^2]^{-1+i\nu+\varrho}[(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)^2+m^2]^{-1-2\varrho}.$$

It is easy to see now that, as long as $\varrho > 0$, the limit $\xi \to 0$ exists and is equal to the integral over the whole space. In the last factor in the integrand we can also put $\varrho = 0$, thus recovering Eq. (3.12).

The last step is to show that the integrals over Ξ_1 and Ξ_2 (i.e. terms A_1 and A_2) do not contribute to the limit (C.5). For definiteness let us consider the region Ξ_2, i.e. $|\boldsymbol{q}_{2\perp}| \equiv |\boldsymbol{r}_\perp-\boldsymbol{q}_\perp| < \xi$ (the other region can be treated analogously). The contribution from Ξ_2 to the limit (C.6) can be written (remember $\xi \ll |\boldsymbol{r}_\perp|$) as

$$-(4\pi\nu)^2(\beta_1/\beta_2)^{i\nu}\lim_{\varrho\to 0+}\lim_{\xi\to 0+}\lim_{\omega\to\infty}\lim_{\lambda\to 0+}\omega^{2-2i\nu-2\varrho}(\Delta_\perp^2)^{-1-i\nu+\varrho}(2\pi)^{-3}\int_{|q_{2\perp}|\leqslant\xi} d^2\boldsymbol{q}_{2\perp}G(\boldsymbol{q}_{2\perp}) \tag{C.13}$$

where

$$G(\boldsymbol{q}_{2\perp}) = \int dQ(-Q+b_1-i\varepsilon_1)^{-1+i\nu-\varrho}(Q+b_2-i\varepsilon_2)^{-1-i\nu-\varrho}$$
$$\times(Q-c_1)^{-1+i\nu+\varrho}(Q-c_3)^{-1+i\nu+\varrho}. \tag{C.14}$$

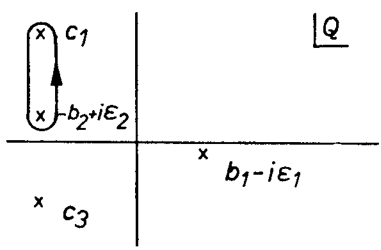

Fig. 6. Singularities of the integrand of (C.14)

Here b_1 and b_2 are given by (C.11) and (C.12), $\varepsilon_i = 2\beta_i\lambda\omega$, and $c_1 = r_z+i\omega\sqrt{\lambda^2+q_{2\perp}^2}$, $c_3 = \omega r_z-i\omega\sqrt{\lambda^2+q_{2\perp}^2}$. The positions of the singularities in the Q-plane are shown in Fig. 6.

To estimate the integral in (C.13) it turns out to be convenient to split again the $\boldsymbol{q}_{2\perp}$-integration region into two regions, $B/\omega \leqslant |\boldsymbol{q}_{2\perp}| < \xi$ and $|q_{2\perp}| \leqslant B/\omega$, with $B \gg |b_1+b_2|$ and $B \gg |\omega r_z|$. It is easy to see that in the first region $G(\boldsymbol{q}_{2\perp}) \sim (\omega^2 q_{2\perp}^2)^{-1+i\nu+\varrho}$, and the contribution to Eq. (C. 13) is $\sim \xi^{i\nu+\varrho}-(B/\omega)^{i\nu+\varrho}$, which vanishes in the limit considered. In the second region the Q-integration path can be pinched between the branch points $-b_2+i\varepsilon_2$ and c_3. The pinch-point contribution can be estimated by closing the integration contour around the points $-b_2+i\varepsilon_2$ and c_1 (see Fig. 6) and noting that in the limit $|\boldsymbol{q}_{2\perp}| \to 0$ and $\lambda \to 0$ these two points coincide producing a second-order pole. Its residue is $\sim (\omega^2 q_{2\perp}^2)^{-1+i\nu/2+\varrho/2}$, and the contribution to Eq. (C.13), $\sim \omega^{-i\nu-\varrho}(B/\omega)^{i\nu+\varrho} = \omega^{-2i\nu-2\varrho}B^{i\nu+\varrho}$, again vanishes in the limit considered.

This completes the derivation od Eq. (3.12). We should also note that this equation and similar equations for the integrals I_2 and I_3 provide us with explicit analytic expressions for the $\boldsymbol{q}_\perp$-integrals occuring in Eq. (3.6)[10]. They can be obtained by taking an appropriate limit in the analytic expressions derived by BM (Eq. (6.3) of Ref. [2]). Denoting

$$f_\varrho(\boldsymbol{q}_\perp) = [(\boldsymbol{r}_\perp+\boldsymbol{q}_\perp)^{2(1+i\nu-\varrho)}(\boldsymbol{r}_\perp-\boldsymbol{q}_\perp)^{2(1-i\nu-\varrho)}]^{-1}[(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)^2+m^2]^{-1},$$

we have

$$\lim_{\varrho\to 0+}(2\pi)^{-2}\int d^2q_\perp f_\varrho(\boldsymbol{q}_\perp) = \frac{-i}{4\pi\nu}\,\frac{\pi\nu}{\sinh\pi\nu}\left(\frac{R_1}{R_2}\right)^{i\nu} \times\left\{\frac{V(\xi)}{D}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)-\frac{i\nu W(\xi)}{R_1R_2}\left(\frac{m^2}{R_1}+\frac{m^2}{R_2}-1\right)\right\}, \tag{C.15}$$

$$\lim_{\varrho\to 0+}(2\pi)^{-2}\int d^2q_\perp(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)f_\varrho(\boldsymbol{q}_\perp) = \frac{-i}{4\pi\nu}\,\frac{\pi\nu}{\sinh\pi\nu}\left(\frac{R_1}{R_2}\right)^{i\nu} \times\left\{\frac{V(\xi)}{D}\left(\frac{\boldsymbol{p}_\perp-\boldsymbol{r}_\perp}{R_1}-\frac{\boldsymbol{p}_\perp+\boldsymbol{r}_\perp}{R_2}\right)-\frac{i\nu W(\xi)}{R_1R_2}m^2\left(\frac{\boldsymbol{p}_\perp-\boldsymbol{r}_\perp}{R_1}+\frac{\boldsymbol{p}_\perp+\boldsymbol{r}_\perp}{R_2}\right)\right\}, \tag{C.16}$$

where

$$D = 4r_\perp^2,\quad R_1 = (\boldsymbol{p}_\perp-\boldsymbol{r}_\perp)^2+m^2,\quad R_2 = (\boldsymbol{p}_\perp+\boldsymbol{r}_\perp)^2+m^2,\quad V(\xi) = {}_2F_1(-i\nu, i\nu; 1; \xi),$$

$$W(\xi) = {}_2F_1(1-i\nu, 1+i\nu; 2; \xi),\qquad \xi = 1-m^2D/R_1R_2.$$

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[10] A direct evaluation of these integrals by introducing Feynman parameters was attempted in Ref. [8] in the special case $\boldsymbol{p}_\perp = 0$. The result was expressed in terms of generalized hypergeometric functions and we were not able to prove that it coincides with the above Eq. (C.14) and Eq. (C.15) below. However, the agreement was found in some special cases, and in particular the first few terms in the expansion in ν turned out to be identical.