AXIALLY SYMMETRIC ELECTROVAC FIELDS FROM GRAVITATIONAL FIELDS

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A method is given by which we can generate asymptotically flat axially symmetric electrovac metrics from solutions representing empty space gravitational fields. The method allows us to generate an infinite chain of solutions of source free Einstein–Maxwell equations. Several new electrovac solutions are obtained from the gravitational fields found by Schwarzschild and Zipoy. A method is also given for generating an analogous stationary solution of source-free Einstein equations.

1. *Introduction*

Various authors have given different classes of axially symmetric static electrovac solutions in different coordinate system (See e.g. [1–5]). Bonnor [2] and Harrison [6] in particular outlined methods of generating electrovac metrics from static fields. In this paper we give a different method of obtaining axially symmetric static solutions of Einstein-Maxwell equations from static vacuum fields. We obtained new electrovac solutions from the gravitational fields of Schwarzschild [7] and Zipoy [8].

All the solutions are asymptotically flat at spatial infinity. When the electrovac field is switched off, the resulting metric is different from the generating vacuum metric. We can generate a more complicated electrovac solution from the resulting metric. Thus we get a series of axisymmetric electrovac solutions starting from one and only one static vacuum metric. We also give a method for generating an analogous stationary solution.

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2. Method of generating electrovac and stationary solutions

We shall need the following two metrics:

$$ds^2 = e^{2\alpha}dt^2 - e^{-2\alpha}[e^{2\lambda}(dx_1^2+dx_2^2)+x_1^2dx_3^2], \tag{1}$$

$$d\bar{s}^2 = e^{2\bar{\alpha}}dt^2 - e^{-2\bar{\alpha}}[e^{2\bar{\lambda}}(dx_1^2+dx_2^2)+x_1^2dx_3^2], \tag{2}$$

where α, $\bar{\alpha}$, λ and $\bar{\lambda}$ are functions of x_1 and x_2 only.

Metrics (1) and (2) correspond to axially symmetric static vacuum (called generating metric) and static electrovac (called generated metric) fields respectively. The equations to be solved for determining the metric (2) are:

$$R_{ik} = -8\pi E_{ik}, \tag{3}$$

$$E^i_{\ k} = -F^{ia}F_{ka}+\tfrac{1}{4}\,\delta^i_{\ k}F^{ab}F_{ab}, \tag{4}$$

$$F_{ij;k}+F_{jk;i}+F_{ki;j} = 0, \tag{5}$$

$$F^{ik}_{\ \ ;k} = 0, \tag{6}$$

the semicolon denoting covariant differentiation. Defining $F_{ij} = K_{i,j}-K_{j,i}$, where K_i is the four potential, equations (3)−(6) reduce to the usual form

$$\nabla^2\bar{\alpha} = e^{-2\bar{\alpha}}(\Phi_1^2+\Phi_2^2), \tag{7}$$

$$\nabla^2\Phi = 2\bar{\alpha}_1\Phi_1+2\bar{\alpha}_2\Phi_2, \tag{8}$$

$$\bar{\lambda}_1 = x_1[(\bar{\alpha}_1^2-\bar{\alpha}_2^2)-e^{-2\bar{\alpha}}(\Phi_1^2-\Phi_2^2)], \tag{9}$$

$$\bar{\lambda}_2 = 2x_1[\bar{\alpha}_1\bar{\alpha}_2-e^{-2\bar{\alpha}}\Phi_1\Phi_2], \tag{10}$$

where

$$K_i = (0, 0, 0, \tfrac{1}{2}\,\pi^{-\frac{1}{2}}\Phi). \tag{11}$$

∇^2 is two dimensional Laplacian operator in the coordinate system concerned. K_i may have the third component also or both the third and fourth components simultaneously. We shall present the solution of (7)−(10) in the general form having both the magnetic and electric scalar potentials.

The field equations derived from $R_{ik} = 0$ for metric (1) take the form

$$\nabla^2\alpha = 0, \tag{12}$$

$$\lambda_1 = x_1(\alpha_1^2-\alpha_2^2), \tag{13}$$

$$\lambda_2 = 2x_1\alpha_1\alpha_2. \tag{14}$$

If we know the solution α, λ of Eqs (12) – (14), then we can obtain $\bar{\alpha}$ and $\bar{\lambda}$ from the following equations. We can verify by direct substitution that the latter satisfy Eqs (7)–(10).

$$e^{2\bar{\alpha}} = \left[\frac{(1-4n^2)e^{\gamma\alpha}}{1-4n^2e^{2\gamma\alpha}}\right]^2, \tag{15}$$

$$\bar{\lambda} = \gamma^2\lambda, \tag{16}$$

$$2\pi^{\frac{1}{2}}K_4 = 2n(1-4n^2)\left[\frac{e^{2\gamma\alpha}}{1-4n^2e^{2\gamma\alpha}} - \frac{1}{1-4n^2}\right], \tag{17}$$

$$F_{ij} = A[K_{i,j}-K_{j,i}] + \frac{B}{2}\,\varepsilon_{ijlm}K^{l;m}, \tag{18}$$

where $A^2-B^2 = 1$, and ε_{ijlm} is the permutation tensor which is antisymmetric with respect to every pair of the indices and $\varepsilon_{1234} = \sqrt{g}$, g denotes the determinant of metric (2) and n is an arbitrary constant $\neq \pm\frac{1}{2}$. Indices have been raised in (18) with respect to metric (2). The constants have been chosen so as to make generated solutions (15)–(16) asymptotically flat at spatial infinity if the generating metric is asymptotically Minkowskian.

If now, the electric and magnetic fields be switched off by making $n = 0$, we get new static vacuum fields with $e^{2\gamma\alpha}$ in place of $e^{2\alpha}$. Repeated use of the new vacuum fields generated by making $n = 0$ each time, shall give rise to a series of electrovac solutions (when $\gamma \neq 0,1$). Thus this method also enables us to generate a series of axially symmetric source-free gravitational fields from a known solutions. Alternatively, the family of electrovac fields generated from the metric (1) can be obtained by simply putting different values of the constant γ in (15) – (18). Singularities associated with the solutions (15) – (18) will be discussed in detail when we give particular applications of the method.

It is also noticed that solutions (15) – (18) may give stationary gravitational fields as follows. We write the axisymmetric stationary metric in the form

$$ds^2 = f(dt-\omega d\varphi)^2 - f^{-1}[e^{2\lambda}(d\varrho^2+dz^2)+\varrho^2 d\varphi^2]. \tag{19}$$

Ernst [9] introduced a complex potential function $\mathcal{E}$ which determines uniquely all the metric coefficients

$$\mathcal{E} = f + i\phi. \tag{20}$$

With the above notations, it is found that f, ϕ, ω etc. can be generated from (15) – (18).

$$f = \frac{(1+4n^2)e^{\gamma\alpha}}{1+4n^2e^{2\gamma\alpha}}, \tag{21}$$

$$\phi = 2n(1+4n^2)\left[\frac{e^{2\gamma\alpha}}{1+4n^2e^{2\gamma\alpha}} + C\right], \tag{22}$$

where C is an arbitrary real constant and

$$\omega_\varrho = 4n\gamma\varrho\alpha_z, \quad \omega_z = -4n\gamma\varrho\alpha_\varrho. \tag{23}$$

Thus axially symmetric stationary gravitational fields can be constructed from a static exterior solution of the Einstein-Maxwell equations given by Eqs (15)-(18).

3. *Application of the method*

(*i*) Schwarzschild solution as the generating metric:

We take the axisymmetric form of the Schwarzschild metric.

$$ds^2 = e^{2\alpha}dt^2 - e^{-2\alpha}[e^{2\mu}(d\varrho^2 + dz^2) + \varrho^2 d\varphi^2], \tag{24}$$

with

$$\alpha = \tfrac{1}{2}\ln\left[\frac{R_1+R_2-2m}{R_1+R_2+2m}\right], \tag{25}$$

$$e^{2\mu} = \frac{(R_1+R_2+2m)(R_1+R_2-2m)}{4R_1R_2},$$

$$R_1^2 = \varrho^2+(z-m)^2, \quad R_2^2 = \varrho^2+(z+m)^2. \tag{26}$$

From (15)-(18) we construct the new electrovac metric in the form

$$ds^2 = \left[\frac{(1-4n^2)(R_1+R_2+2m)^{\frac{\gamma}{2}}(R_1+R_2-2m)^{\frac{\gamma}{2}}}{(R_1+R_2+2m)^\gamma-4n^2(R_1+R_2-2m)^\gamma}\right]^2 dt^2$$

$$-\left[\frac{(R_1+R_2+2m)^\gamma-4n^2(R_1+R_2-2m)^\gamma}{(1-4n^2)(R_1+R_2+2m)^{\frac{\gamma}{2}}(R_1+R_2-2m)^{\frac{\gamma}{2}}}\right]^2\left[\left\{\frac{(R_1+R_2+2m)(R_1+R_2-2m)}{4R_1R_2}\right\}^{\gamma^2}\right.$$

$$\left.(d\varrho^2+dz^2)+\varrho^2 d\varphi^2\right], \tag{27}$$

$$2\pi^{\frac{1}{2}}K_4 = 2n(1-4n^2)\left[\frac{(R_1+R_2-2m)^\gamma}{(R_1+R_2+2m)^\gamma-4n^2(R_1+R_2-2m)^\gamma} - \frac{1}{1-4n^2}\right]. \tag{28}$$

Asymptotic expansions of g_{44} and K_4 in the Schwarzschild coordinates,

$$\varrho^2 = (r^2-2mr)\sin^2\theta, \quad z = (r-m)\cos\theta, \tag{29}$$

are given below

$$g_{44} = 1 - \frac{(1+4n^2)}{(1-4n^2)}\frac{2\gamma m}{r} + \frac{(1+32n^2-208n^4)}{(1-4n^2)^2}\frac{m^2\gamma^2}{r^2} + \ldots, \tag{30}$$

$$2\pi^{\frac{1}{2}}K_4 = -\frac{4nm\gamma}{(1-4n^2)}\frac{1}{r} + \frac{2nm^2\gamma^2(1+16n^2)}{(1-4n^2)}\frac{1}{r^2} + \ldots . \tag{31}$$

Expansions (30) and (31) show that $g_{44} \rightarrow 1$ and $K_4 \rightarrow 0$ when $r \rightarrow \alpha$. Asymptotic behaviour of K_4 shows that the source contains both the monopole and dipole moments. The electric charge, e, is defined by [11]

$$e = \iint F_{41} g_{\theta\theta}^{\frac{1}{2}} g_{\varphi\varphi}^{\frac{1}{2}} d\theta d\varphi. \tag{32}$$

For simplicity, integrating (32) over the surface of a spatial sphere as its radius becomes infinite, we find in our units,

$$e = \frac{8\pi^{\frac{1}{2}} nm\gamma}{(1-4n^2)}. \tag{33}$$

The charge to mass ratio comes out to be,

$$e/m = \frac{8\pi^{\frac{1}{2}} n}{(1+4n^2)}. \tag{34}$$

When $m = 0$, $e = 0$ and we have flat space. We now switch off the electromagnetic field by making $n = 0$, then the metric (27) reduces to the form

$$ds^2 = \left(\frac{R_1+R_2-2m}{R_1+R_2+2m}\right)^{\gamma} dt^2 - \left(\frac{R_1+R_2-2m}{R_1+R_2+2m}\right)^{-\gamma} \left[\left\{\frac{(R_1+R_2+2m)(R_1+R_2-2m)}{4R_1R_2}\right\}^{\gamma^2} \times (d\varrho^2+dz^2)+\varrho^2 d\varphi^2\right]. \tag{35}$$

Transforming to Schwarzschild coordinates by Eqs (26) and (29) we obtain the metric in the form given by Esposito and Witten [10]

$$ds^2 = \left(1-\frac{2m}{r}\right)^{\gamma} dt^2 - \left(1-\frac{2m}{r}\right)^{-\gamma} \left\{\left(\frac{r^2-2mr}{r^2-2mr+m^2\sin^2\theta}\right)^{\gamma^2-1} dr^2 + \frac{(r^2-2mr)^{\gamma^2}}{(r^2-2mr+m^2\sin^2\theta)^{\gamma^2-1}} d\theta^2 + (r^2-2mr)\sin^2\theta d\varphi^2\right\}. \tag{36}$$

$\gamma = 1$ gives the Schwarzschild solution. Esposito and Witten [10] have studied the behaviour of the metric (36). It is spherically symmetric only when $\gamma = 1$. $r = 2m$ is the surface of infinite redshift for all values of γ. But this surface is singular except when $\gamma = 1$. So all the solutions found by Esposito and Witten exhibit singular infinite redshift surfaces except when $\gamma = 1$. Hence Schwarzschild solution is the only one in the family representing a black hole. All others have naked singularities. Further the source with an exterior described by such a solution with $\gamma \neq 1$ could have an area smaller than that of an approximately defined Schwarzschild surface.

Our electrovac solution (27) in Schwarzschild coordinates reduces to the form

$$ds^2 = \left[\frac{(1-4n^2)\left(1-\frac{2m}{r}\right)^{\frac{\gamma}{2}}}{1-4n^2\left(1-\frac{2m}{r}\right)^{\gamma}}\right]^2 dt^2 - \left[\frac{(1-4n^2)\left(1-\frac{2m}{r}\right)^{\frac{\gamma}{2}}}{1-4n^2\left(1-\frac{2m}{r}\right)^{\gamma}}\right]^{-2}$$

$$\times \left\{\left(\frac{r^2-2mr}{r^2-2mr+m^2\sin^2\theta}\right)^{\gamma^2-1} dr^2 + \frac{(r^2-2mr)^{\gamma^2}}{(r^2-2mr+m^2\sin^2\theta)^{\gamma^2-1}}\, d\theta^2\right.$$

$$\left. +(r^2-2mr)\sin^2\theta d\varphi^2\right\}. \tag{37}$$

For $\gamma = 1$, the metric (37) is spherically symmetric and reduces to Reissner–Nordström metric with the following transformations

$$(1-4n^2)r' = (1-4n^2)r+8mn^2.$$

We find here that $r = 2m$ surfaces are still the surfaces of infinite redshift, which are singular except when $\gamma = 1$.

(*ii*) Zipoy solutions as the generating metrics

The equilibrium shape of a rotating star is an oblate spheroid. Zipoy [8] used oblate spheroidal coordinates and showed that solutions for "Newtonian potential" can be written as a linear combination of Legendre polynomials of integral order l. He obtained solutions corresponding to $l = 0$, $l = 1$, and a combination $l = 0, 1$. We use those gravitational solutions to generate electrovac fields by the above method. Metric (1) can be written in oblate spheroidal coordinates as

$$ds^2 = e^{2\sigma}dt^2 - a^2e^{2(\lambda-\sigma)}(\sinh^2 u+\sin^2\theta)(du^2+d\theta^2) - a^2e^{-2\sigma}\cosh^2 u\cos^2\theta d\varphi^2, \tag{38}$$

where

$$x_1 = a\cosh u\cos\theta, \quad x_2 = a\sinh u\sin\theta, \quad x_3 = \varphi, \quad \alpha = \sigma. \tag{39}$$

Case (a) $l = 0$

Zipoy's [8] solution for $l = 0$ is given below:

$$\sigma = -\beta\tan^{-1}(\operatorname{cosech} u), \quad \beta = \frac{m}{a}, \tag{40}$$

$$\lambda = \tfrac{1}{2}\beta^2\ln\left(\frac{\sinh^2 u+\sin^2\theta}{\cosh^2 u}\right), \tag{41}$$

when m is a constant. The electrovac solution can be written immediately from (15)–(18).

$$e^{2\bar{\alpha}} = \left[\frac{(1-4n^2)e^{-\gamma\beta\tan^{-1}(\operatorname{cosech} u)}}{1-4n^2e^{-2\gamma\beta\tan^{-1}(\operatorname{cosech} u)}}\right]^2, \tag{42}$$

$$\bar{\lambda} = \frac{\gamma^2\beta^2}{2}\ln\left(\frac{\sinh^2 u+\sin^2\theta}{\cosh^2 u}\right), \tag{43}$$

$$2\pi^{\frac{1}{2}}K_4 = 2n(1-4n^2)\left[\frac{e^{-2\gamma\beta\tan^{-1}(\operatorname{cosech} u)}}{1-4n^2e^{-2\gamma\beta\tan^{-1}(\operatorname{cosech} u)}} - \frac{1}{1-4n^2}\right]. \tag{44}$$

The asymptotic behaviour of (42) and (44) was studied with the following transformations of coordinates.

$$r = a(\sinh^2 u+\cos^2\theta)^{\frac{1}{2}} \xrightarrow[u\to\alpha]{} a\sinh u, \tag{45}$$

$$\bar{\theta} = \sin^{-1}\frac{a\sinh u\sin\theta}{r} \xrightarrow[u\to\alpha]{} \theta, \tag{46}$$

where $(r, \bar{\theta})$ are spherical polar coordinates but $\bar{\theta}$ is measured here from the equator rather than the pole for comparison, with oblate spheroidal coordinates. We give below the asymptotical expansions in the case $\gamma = 2$ for simplicity

$$e^{2\bar{\alpha}} = 1-\frac{4\beta a(1+4n^2)}{(1-4n^2)}\frac{1}{r}+O\left(\frac{1}{r^2}\right)+\dots, \tag{47}$$

$$2\pi^{\frac{1}{2}}K_4 = -\frac{8n\beta a}{(1-4n^2)}\frac{1}{r}+\frac{16n\beta^2a^2(1+4n^2)}{(1-4n^2)}\frac{1}{r^2}+\dots \tag{48}$$

The charge to mass ratio comes out to be,

$$e/m = \frac{8\pi^{\frac{1}{2}}n}{(1+4n^2)}. \tag{49}$$

The electrovac metric is well behaved and $e = 0$ when $m = 0$. It is seen from the expansion (47) and (48) that the monopole and dipole terms both exist in the electrovac analogue of Zipoy's $l = 0$ solution. Here also we do not get back Zipoy's solution of gravitational field, $l = 0$, when we switch off the electromagnetic field.

We know that $u =$ const. surfaces are oblate spheroids and the above solution (42)–(44) depends on u alone. Hence this has spheroidal symmetry. It should be noted that θ is discontinuous as $\theta =$ const lines cross $u = 0$. We find that $\frac{\partial g_{\mu\nu}}{\partial z}$ is discontinuous over the disc $z = 0$, $\varrho \leqslant a$. Moreover $\frac{\partial K_4}{\partial z}$ is also discontinuous over the disc $z = 0$, $\varrho \leqslant a$.

Case (b): $l = 1$.

Zipoy [8] gives for $l = 1$,

$$\sigma = \delta\{1-\sinh u \tan^{-1}(\operatorname{cosech} u)\}\sin\theta, \quad 0 \leqslant \tan^{-1}(\operatorname{cosech} u) \leqslant \pi, \tag{50}$$

$$\lambda = -\tfrac{1}{2}\delta^2 \ln\left(\frac{\sinh^2 u+\sin^2\theta}{\cosh^2 u}\right) - \tfrac{1}{2}\delta^2 \cos\theta\{[\tan^{-1}(\operatorname{cosech} u)]^2 - [1-\sinh u \tan^{-1}(\operatorname{cosech} u)]^2\}. \tag{51}$$

Proceeding in the same way, we can write down the electrovac solutions easily. Asymptotic expansion of $e^{2\bar{\alpha}}$ and K_4 for $\gamma = 2$ are given by

$$e^{2\alpha} = 1+\tfrac{4}{3}\frac{\delta a^2 \sin\theta}{r^2}\frac{(1+4n^2)}{(1-4n^2)} + \ldots, \tag{52}$$

$$2\pi^{\frac{1}{2}}K_4 = \frac{8n\delta a^2 \sin\theta}{3(1-4n^2)r^2} + \ldots. \tag{53}$$

Asymptotic flatness of $e^{2\bar{\alpha}}$ is retained here but there is no mass term. Secondly K_4 does not contain a monopole but higher multipoles. Since a mass dipole contains equal quantities of positive and negative mass, the total mass vanishes. Here $g_{\mu\nu}$ is discontinuous across the disc $z = 0$, $\varrho \leqslant a$ and so also is $\dfrac{\partial g_{\mu\nu}}{\partial z}$. Dipole moment in (53) vanishes when $n = 0$ and the line element does not reduce back to the gravitational metric of Zipoy for $l = 1$.

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