BACK REACTION OF A QUANTIZED FIELD IN THE GAUGE TREATMENT OF GRAVITY

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A quantized scalar field in an external time-dependent gravitational field is considered. The complete Lagrangian contains terms quadratic in the curvature. These terms are connected with the removing of infinities appearing in the vacuum expectation value of the energy-momentum tensor and characterize a back reaction effect. The methods of the gauge fields theory are used. It is shown that in the isotropic time-dependent gravitational field the back reaction effect leads to the absence of solutions with physical singularities. It is noted that the particle creation is small in the considered case.

The problem of back reaction of a quantized field on a classical gravitational background is one of the most actual problems. Its complete solution is possible only in the framework of the quantum theory of gravity. However, the back reaction effect is also essential for a number of semiclassical problems. The purpose of this article is to demonstrate this fact for a particular case of a quantized scalar field.

The semiclassical approach consists in investigation of a quantized field (or fields) in an external classical field [1]. This method is discussed, for example, in connection with the problem of particle creation in a time-dependent gravitational field [2–7].

In an external time-dependent gravitational field we consider a quantized noncharged scalar field φ with the Lagrangian $\mathscr{L}_s$ conformally invariant in the limit of vanishing mass [8]

$$\mathscr{L}_S = \tfrac{1}{2}\sqrt{-g}\left[\nabla^a\varphi\nabla_a\varphi - \left(m^2 - \frac{R}{6}\right)\varphi^2\right]. \tag{1}$$

In the discussion of the Fock representation of canonical commutation relations (CCR) it becomes clear that the scalar field Hamiltonian is in general non-diagonal. Thus, it is not a self-adjoined operator. This difficulty can be formally removed if we replace the creation operators $C_J^{(+)}$ and the annihilation operators $C_J^{(-)}$ of the Fock representation

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by Bogolubov transformations to some new $\tilde{C}_J^{(\pm)}$. Then the vacuum state, defined by the relations

$$C_J^{(-)}(\lambda_J)\,|0, \lambda_J\rangle = 0,$$

where λ_J are the parameters of the Bogolubov transformations ($\lambda_J = \lambda_J(t)$; $|\lambda_J| < 1$), is a time-dependent state. The classical density is in general nonzero[1].

Massless scalar particles are not created in a conformally flat model due to conformal invariance of the field equations. The creation of massive scalar particles is small for the modern stage of the expanding Universe [4, 5], because in the isotropic world, where $\alpha \simeq 1+Ht$ (the Hubble constant $H \simeq 2.5 \cdot 10^{-18}\ \mathrm{sec}^{-1}$), the density

$$n \simeq \frac{mH^2}{256\pi}.$$

By analogy with the phenomena of electron-positron pair creation in an external electromagnetic field it is natural to suppose [4, 5] that the particle creation in an external gravitational field is essential in strong and rapidly variable gravitational fields. The corresponding realistic models must have physical singularities. This condition, for example, is not satisfied in the Robertson-Walker model with

$$R = \mathrm{const.}$$

In the considered framework the geometry of the model is not arbitrary but may be defined as a solution of the field equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = \kappa \langle 0|T_{ik}|0\rangle, \tag{2}$$

where the right side is a vacuum expectation value of the energy-momentum tensor. It has been proved in [1] that the left side of Eq. (2) must also contain the counterterms of a completely new type proportional to the squares of spacetime curvature which remove logarithmic divergences in the right side of Eq. (2)[2] (see also [6, 7]).

Thus, the complete Lagrangian would contain the term

$$\mathscr{L}_g = \tfrac{1}{4}\sqrt{-g}\,(\lambda_1 R_{ijkl} R^{ijkl} + \lambda_2 R_{ilk}{}^{l} R^{ilk}{}_{l} + \lambda_3 R_{ij}{}^{ij} R_{kl}{}^{kl}), \tag{3}$$

where $\lambda_1, \lambda_2, \lambda_3$ are new coupling constants, which may be interpreted as a back reaction of a quantized scalar field on a spacetime geometry.

In terms of our interpretation the conditions $\lambda_1 = \lambda_2 = \lambda_3 = 0$ mean the absence of a back reaction effect. In the framework of the gauge treatment of gravity, as we shall see, the back reaction term (3) may lead to the absence of isotropic time-dependent solutions with physical singularities.

In the present article, as before [12], we use the methods of the gauge fields theory (GFT) and the gauge approach to the gravitational field. The Lagrangian is quadratic in

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[1] There exists another approach to the particle creation problem, based on the Feynman propagator properties, by which the creation of particles does not occur in some time-dependent gravitational fields [9].

[2] The recent development of quantum gravity demonstrated that the removing of ultraviolet divergences in one-loop diagrams with internal graviton lines leads to similar nonminimal counterterms [10].

the gauge field tensor in the GFT, because it contains the electromagnetic field theory as a particular case. Therefore it is natural to consider a quadratic Lagrangian for the gravitational field, which corresponds to the Poincaré group in the GFT. At the same time, the field equations of a purely quadratic theory cannot be the equations of a macroscopic theory of gravitation [12]. This fact leads to the necessity of considering the linear Lagrangian $\mathscr{L}_E = \frac{1}{2}\sqrt{-g}R$ where $R = g^{km}R^i R^i_{kim}$, together with the quadratic Lagrangians.

According to the GFT, one must vary the complete Lagrangian with respect to the connection coefficients and the metric tensor components, independently. Supposing that the effects of the spacetime torsion are insignificant in comparison with the other effects of gravitational fields, one examines, as before [12], a special form of the general field equations corresponding to the Riemannian connection. To get this form it is necessary to set the torsion tensor in the general field equations equal to zero. Our restriction of the theory is not essential and may be justified by mathematical difficulties which occur due to the absence of the adequate method of investigation for the general field equations.

As shown in the work [12], after this procedure the vacuum class of solutions of the field equations coincides with the class of the Einstein spaces ($R_{ik} = 0$) for most physical problems of interest. In general the set of field equations is overdetermined because the number of independent equations is larger than that of independent functions. To remove this defect one supposes that the Lagrangian of the scalar field contains the self-interaction term characterizing the interaction of the scalar field with the torsion tensor field.

Thus, the complete Lagrangian has the form

$$\mathscr{L} = \mathscr{L}_E + \mathscr{L}_g + \kappa\mathscr{L}_S + \mathscr{L}_i,$$

where $\kappa = 8\pi G/c^4$ and $\mathscr{L}_S$, $\mathscr{L}_g$ are defined by the formulae (1), (3). The self-interaction term

$$\mathscr{L}_i = \sqrt{-g}\sum_{r=1}^{m-k} q_r F_r(\varphi), \tag{4}$$

where q_r are coupling constants, $F_r(\varphi)$ are arbitrary functions of a scalar field φ [3], m is the number of independent field equations, k is the number of independent metric components[4].

The field equations corresponding to $\mathscr{L}$ have the form[5]

$$fR_{ik} - \frac{1}{2}\mu g_{ik} + \lambda(2R_{il}R^l_k + 2R_{ijkl}R^{jl} - g_{ik}R_{jl}R^{jl})$$
$$= -\frac{1}{6}\kappa(4\nabla_i\varphi\nabla_k\varphi - 2\varphi\nabla_i\nabla_k\varphi - g_{ik}\nabla^a\varphi\nabla_a\varphi + 2g_{ik}\varphi\nabla^a\nabla_a\varphi) + g_{ik}(q_r F_r), \tag{5}$$

$$\lambda\nabla^a R_{aijk} + (\lambda + \bar{\lambda})g_{i[j}\nabla_{k]}(\kappa m^2\varphi^2 + q_r\varphi(F_r)'_\varphi - 4q_r F_r) = 0, \tag{6}$$

$$\nabla^a\nabla_a\varphi + (m^2 - R/6)\varphi - \kappa^{-1}q_r(F_r)'_\varphi = 0, \tag{7}$$

[3] A scalar field with the self-interaction term (4) allows one to reduce the Poincaré group invariance to the Lorentz group invariance. This effect is analogous to the Higgs phenomena [11].

[4] Since the torsion tensor $Z^i_{jk} = 0$, in general $m \neq k$.

[5] The general field equations, from which Eqs (5, 6) follow, are given e.g. in [12]. Due to the Lanczos identities [13] all the constants λ_i (see the Lagrangian $\mathscr{L}_g$) vanish with the exception of the two constants λ, $\bar{\lambda}$. The equations of motion follow from Eq. (5). It is easy to see that Eq. (7) is a corollary of Eq. (5).

where the symbol $\sum_{r=1}^{m-k}$ is omitted and

$$\lambda = \lambda_1 + \tfrac{1}{4}\lambda_2, \quad \bar{\lambda} = \lambda_3 - \lambda_1,$$

$$f = f(\varphi) = 1 - \tfrac{1}{6}\kappa\varphi^2 - \tfrac{1}{2}\bar{\lambda}(\kappa m^2\varphi^2 + q_r\varphi(F_r)'_\varphi - 4q_rF_r),$$

$$\mu = \mu(\varphi) = -(\kappa m^2\varphi^2 + q_r\varphi(F_r)'_\varphi - 4q_rF_r)(1 + \tfrac{1}{6}\kappa\varphi^2) + \tfrac{1}{4}\bar{\lambda}(\kappa^2 m^2\varphi^2 + q_r\varphi(F_r)'_\varphi$$
$$- 4q_rF_r)^2 - \kappa m^2\varphi^2 + \tfrac{1}{2}\bar{\lambda}\kappa(\kappa m^2\varphi^2 - 4q_rF_r).$$

In particular, from Eq. (5) it follows that

$$R = -\kappa m^2\varphi^2 + 4q_rF_r - q_r\varphi(F_r)'_\varphi. \tag{8}$$

If the coupling constants $\lambda = \bar{\lambda} = 0$, then the set of equations (5, 6) coincides with the traditional Einstein equations.

Consider a Riemannian space-time M^4 with the metric

$$ds^2 = \alpha^2(dx^1)^2 + \beta^2(dx^2)^2 + \gamma^2(dx^3)^2 - (dx^4)^2, \tag{9}$$

where $\alpha = \alpha(x^4)$, $\beta = \beta(x^4)$, $\gamma = \gamma(x^4)$ are arbitrary functions.

For the reper

$$x = \theta^1 = \alpha dx^1, \quad y = \theta^2 = \beta dx^2,$$

$$z = \theta^3 = \gamma dx^2, \quad t = \theta^4 = dx^4,$$

the curvature form $\Omega(= d\omega + \omega \wedge \omega$, where ω is a connection 1-form) is

$$\Omega = a_1(P \vee P) + a_2(\tilde{X} \vee \tilde{X}) + a_3(X' \vee X') \tag{10}$$
$$+ a_4(\tilde{Y} \vee \tilde{Y}) + a_5(Y' \vee Y') + a_6(\tilde{Q} \vee \tilde{Q}),$$

where

$$\begin{aligned} a_1 &= \tfrac{1}{2}\alpha'\beta'\alpha^{-1}\beta^{-1}, & a_2 &= \tfrac{1}{2}\alpha'\gamma'\alpha^{-1}\gamma^{-1}, \\ a_3 &= -\tfrac{1}{2}(\alpha'\alpha^{-1})', & a_4 &= \tfrac{1}{2}\beta'\gamma'\beta^{-1}\gamma^{-1}, \\ a_5 &= -\tfrac{1}{2}(\beta'\beta^{-1})', & a_6 &= -\tfrac{1}{2}(\gamma'\gamma^{-1})'; \end{aligned} \tag{11}$$

a prime denotes differentiation with respect to the time variable,

$$P = x \wedge y, \; \tilde{X} = z \wedge x, \; X' = t \wedge x, \; \tilde{Y} = z \wedge y, \; Y' = t \wedge y, \; \tilde{Q} = z \wedge t,$$

the symbols $\vee$, $\wedge$ denote the symmetrized product and the exterior product, respectively[6].

The field equations (5) for the metric (9) are written in the Appendix.

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[6] Here we apply the method of calculation, developed in Ref. [12].

Eq. (6) is equivalent to the following set of equations

$$R'_\varphi \cdot \nabla_j \varphi = 0, \quad j = 1, 2, 3, \tag{12}$$

$$a'_\delta + a_\delta(\beta'\beta^{-1} + \gamma'\gamma^{-1}) - \tfrac{1}{2}(\lambda + \bar{\lambda})R' = 0; \quad \delta = 3, 5, 6, \tag{13}$$

where a_δ are defined by the formulae (10) and λ, $\bar{\lambda}$ are arbitrary constants.

Eq. (12) gives the following two cases: 1) $R'_\varphi = 0$; 2) $\nabla_j \varphi = 0 (j = 1, 2, 3)$, $\varphi = \varphi(t)$. In the first case using Eq. (8), one obtains the relation

$$\sum_{r=1}^{m-k} \tfrac{3}{4} q_r (F_r)'_\varphi - \tfrac{1}{4} q_r \varphi (F_r)''_\varphi = \frac{\kappa m^2}{2} \varphi. \tag{14}$$

In the second case Eq. (7) may be transformed to[7]

$$Z'' + W^2(t) Z = -\kappa^{-1} \Big(\sum_{r=1}^{m-k} q_r (F_r)'_\varphi \exp\big[\tfrac{1}{2} \int_{t_0}^{t} (\alpha'\alpha^{-1} + \beta'\beta^{-1} + \gamma'\gamma^{-1}) d\xi\big]\Big), \tag{15}$$

where

$$W^2(t) = -\tfrac{1}{4}(\alpha'\alpha^{-1} + \beta'\beta^{-1} + \gamma'\gamma^{-1})^2 - \tfrac{1}{2}(\alpha''\alpha^{-1} +$$
$$+ \beta''\beta^{-1} + \gamma''\gamma^{-1} - \alpha'^2\alpha^{-2} - \beta'^2\beta^{-2} - \gamma'^2\gamma^{-2}) - (m^2 - R/6)$$

and

$$R = 2(a_1 + a_2 + a_4 - a_3 - a_5 - a_6). \tag{16}$$

Let $\beta = \gamma = \alpha(t)$ (M^4 is an isotropic space-time).
Then

$$a_2 = a_4 = a_1 = \tfrac{1}{2}(\ln \alpha)'^2, \tag{17}$$
$$a_5 = a_6 = a_3 = -\tfrac{1}{2}(\ln \alpha)''.$$

From (17) it follows that

$$(2a_1)^{1/2} - (2a_1)_0^{1/2} = -2 \int_{t_0}^{t} a_3 d\xi, \tag{18}$$

where the index "0" denotes an initial value.

Substituting (17) and (18) into Eqs (13), one obtains the equation

$$a'_3 + a_3\Big[-2 \int_{t_0}^{t} a_3 d\xi + (2a_1)_0^{1/2}\Big] - \tfrac{1}{2}(\lambda + \lambda)R' = 0. \tag{19}$$

Integrating (19), one finds the relation

$$(4\lambda + 3\bar{\lambda})(R - R_0) = 0,$$

[7] In the limit of vanishing coupling constants q_r the quantization problem for the oscillator (15) with an arbitrary frequency $W(t)$ has been solved in [14].

where, as before, λ, $\bar{\lambda}$ are arbitrary. Thus $R = R_0 \equiv 3c$ and using the relations (17), (18), one gets the equation

$$v' + v^2 - c = 0, \qquad v \equiv (\ln \alpha)'. \tag{20}$$

Integrating (20), one finds the following two solutions

$$\alpha = \alpha_0 e^{\pm t\sqrt{c}}, \tag{21}$$

$$\alpha = \alpha_0 \frac{\pm 1 + Ae^{\pm 2t\sqrt{c}}}{\pm(1+A)e^{\pm t\sqrt{c}}}, \tag{22}$$

where

$$A = [\sqrt{c} - (\ln \alpha)'_0]^2 / c - (\ln \alpha)'^2_0.$$

In particular, these solutions give $\alpha/\alpha_0 \simeq 1 + Ht$ if c is a small parameter.[8]

Due to (8)

$$F'_\varphi = \frac{\kappa m^2}{q} \varphi + \mu\varphi^3,$$

thus, Eq. (7) has the form

$$\nabla^a \nabla_a \varphi + m^2_{\text{eff}} \varphi - \frac{\mu q}{\kappa} \varphi^3 = 0, \tag{23}$$

where the "effective" mass

$$m^2_{\text{eff}} = -\frac{c}{2}. \tag{24}$$

It is evident that the other equations of the complete set of field equations are identically fulfilled.

Summarizing the results for an isotropic space-time, we get the following facts.

a) The models given by (21) and (22) have no physical singularities.

b) The restriction of spacetime geometry by the metrics (21, 22) is removed if the quadratic terms are absent ($\lambda = \bar{\lambda} = 0$). In terms of our interpretation this result connects the particle creation with the back reaction effect.

c) According to [4, 5] the particle creation effect in the metrics (21, 22) is small.[9]

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[8] In realistic models this condition is fulfilled due to the equality $c = \text{const} = R/3$.

[9] It is interesting to note this fact in connection with its possible interpretation in the framework of the generalized Einstein–Cartan theory (see Ref. [15, 16]).

APPENDIX

Computing the coefficients of the form $\nabla\nabla\varphi = N_1(xx)+N_2(xy)+N_3(zx)+N_4(tx)+N_5(yy)+N_6(zy)+N_7(ty)+N_8(zz)+N_9(tz)+N_{10}(tt)$ one obtains the following formulae:

$$N_1 = \alpha^{-2}\varphi_{11}-\varphi'\alpha'\alpha^{-1}, \quad N_2 = 2\alpha^{-1}\beta^{-1}\varphi_{12},$$

$$N_3 = 2\alpha^{-1}\gamma^{-1}\varphi_{13}, \quad N_4 = 2(\varphi_1\alpha^{-1})',$$

$$N_5 = \beta^{-2}\varphi_{22}-\varphi\,\beta'\beta^{-1}, \quad N_6 = 2\beta^{-1}\gamma^{-1}\varphi_{23},$$

$$N_7 = 2(\varphi_2\beta^{-1})' \quad N_8 = \gamma^{-2}\varphi_{33}-\varphi'\gamma'\gamma^{-1},$$

$$N_9 = 2(\varphi_3\gamma^{-1})', \quad N_{10} = \varphi'',$$

where the lower indices denote differentiation in the variables x^1, x^2, x^3, respectively.

Eq. (5) is equivalent to the following ones:

$$f(a_1+a_2-a_3)+8\lambda(a_1+a_2-a_3)^2-8\lambda(-a_4+a_5+a_6)^2$$
$$-8\lambda(a_1a_2-a_1a_3-a_2a_3)-8\lambda(a_4a_5+a_4a_6-a_5a_6)-\tfrac{1}{2}\mu$$
$$=\tfrac{1}{6}\kappa(3b_1^2-b_2^2-b_3^2+b_4^2+2\varphi N_5+2\varphi N_8-2\varphi N_{10})-q_rF_r, \tag{5xx}$$

$$f(a_1+a_4-a_5)+8\lambda(a_1+a_4-a_5)^2-8\lambda(-a_3+a_3+a_6)^2$$
$$-8\lambda(a_1a_4-a_1a_5-a_4a_5)-8\lambda(a_2a_3+a_1a_6-a_3a_6)-\tfrac{1}{2}\mu$$
$$=\tfrac{1}{6}\kappa(3b_2^2-b_1^2-b_3^2+b_4^2+2\varphi N_1+2\varphi N_8-2\varphi N_{10})-q_rF_r, \tag{5yy}$$

$$f(a_2+a_4-a_6)+8\lambda(a_2+a_4-a_6)^2-8\lambda(-a_1+a_3+a_5)^2$$
$$-8\lambda(a_2a_4-a_2a_6-a_4a_6)-8\lambda(a_1a_3+a_1a_5-a_3a_5)-\tfrac{1}{2}\mu$$
$$=\tfrac{1}{6}\kappa(3b_3^2-b_1^2-b_2^2+b_4^2+2\varphi N_1+2\varphi N_5-2\varphi N_{10})-q_rF_r, \tag{5zz}$$

$$f(a_3+a_5+a_6)-8\lambda(a_3+a_5+a_6)^2+8\lambda(a_3a_5+a_3a_6+a_5a_6)$$
$$+8\lambda(a_1+a_2+a_4)^2-8\lambda(a_1a_2+a_1a_4+a_2a_4)+\tfrac{1}{2}\mu$$
$$=\tfrac{1}{6}\kappa(3b_4^2+b_1^2+b_2^2+b_3^2-2\varphi N_1-2\varphi N_5-2\varphi N_8)+q_rF_r, \tag{5tt}$$

$$4b_1b_2-\varphi N_2 = 0, \quad \text{(5xy)} \qquad 4b_2b_3-\varphi N_6 = 0, \tag{5yz}$$

$$4b_1b_3-\varphi N_3 = 0, \quad \text{(5xz)} \qquad 4b_2b_4-\varphi N_7 = 0, \tag{5yt}$$

$$4b_1b_4-\varphi N_4 = 0, \quad \text{(5xt)} \qquad 4b_3b_4-\varphi N_9 = 0, \tag{5zt}$$

where $b_1 = \varphi_1\alpha^{-1}$, $b_2 = \varphi_2\beta^{-1}$, $b_3 = \varphi_3\gamma^{-1}$, $b_4 = \varphi'$ are coefficients of the form $\nabla\varphi = b_1x+b_2y+b_3z+b_4t$.

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