Vol. B8 (1977) ACTA PHYSICA POLONICA No 11

1/*N* EXPANSION FOR NONLINEAR SCALAR INTERACTIONS

By P. Rembiesa

Institute of Physics, Jagellonian University, Cracow*

(*Received May 18, 1977*)

Nonlinear scalar interactions of the form $N\sum_{i=1}^{\infty}\frac{c_i}{i!}(\Phi^2/N)^i$ are considered in the large N limit. The renormalization of the generating functional for Green's functions in three dimensions is performed up to the next to leading order. The induced derivative couplings are absent to this order. The generality of the approach allows one to study all nonlinear $O(N)$-symmetric scalar interactions, provided they are expandible in a Taylor series in Φ^2/N.

1. *Introduction*

This note is intended primarily as an effort to further development of the formalism of the $1/N$ expansion [1, 2] in application to the nonlinear scalar interactions of the form:

$$V_0\left(\frac{\Phi^2}{N}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\Phi^2}{N}\right)^k V_0^k(0), \tag{1.1}$$

where

$$\Phi^2 = \sum_{a=1}^{N}\Phi_a\Phi_a. \tag{1.2}$$

Considerations concerning similar but one-component models often appear in the literature especially as of the last ten years. We do not even attempt to include a list of references here. Most of the effort in this direction has been based on the conventional perturbation technique using the expansion in powers of the coupling constant. None of the invented methods has been fully successful. Except renormalizable theories in two dimensions (Sine-Gordon!) all horrors of the infinite number of the induced derivative couplings are present even in the second order of the expansion in the coupling constant.

* Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

The above difficulties have halted the progress in the theory of bounded interactions [3] where one believes in the good large momentum behavior of Green's functions but a skillful method avoiding the expansion (1.1) has not yet been invented.

Recently considerable effort has been made to understand the meaning of nonrenormalizability [4, 5] and it is apparent that because of the nonanalyticity in the coupling constant the conventional expansion techniques are not powerful enough to face non-renormalizable interactions. Therefore attention has been called to the methods which avoid the Taylor expansion in the coupling constant. The large N expansion seems to be suitable for this task [6]. It was shown for the somewhat academical example of the quartic interaction in more than four dimensions that in the large N limit the resulting theory is finite, free of the ambiguities and possessing the ground state [6, 7].

Schnitzer [8] gave a careful analysis of the leading order $1/N$ approximation of theories described by Lagrangians with interaction parts of the form (1.1) in two and three dimensional space-time. The main result of his work was that the leading order Green's functions can be renormalized without the usual proliferation of the parameters and that the large Φ^2 behavior of the theory is just as for the renormalizable ones.

In the present paper we show that the situation does not deteriorate if the next to leading corrections are included, provided we deal with an essentially infinite sum in (1.1). We give the explicit formulae for all counterterms that allows to study all interactions of scalar particles, provided the interaction part of the Lagrangian is a Taylor expandible function of Φ^2/N.

2. *Effective action*

Let us consider the $O(N)$ symmetric Lagrangian with the interaction part expandible in powers of Φ^2/N

$$\mathscr{L}[\Phi] = \tfrac{1}{2}(\partial_\mu \Phi)^2 - NV_0\left(\frac{\Phi^2}{N}\right), \tag{2.1}$$

where

$$V_0\left(\frac{\Phi^2}{N}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\Phi^2}{N}\right)^k V_0^{(k)}(0), \tag{2.2}$$

and

$$\Phi^2 = \sum_{i=1}^{N} \Phi_a \Phi_a. \tag{2.3}$$

We shall calculate the generating functional of the 1-particle irreducible Green's functions (effective action) $\Gamma(\varphi, G)$ using the method of Cornwall, Jackiw and Tomboulis [9]

$$\Gamma(\varphi, G) = W(\varphi) + \tfrac{1}{2} i\hbar \operatorname{tr} \log G^{-1} + \tfrac{1}{2} i\hbar \operatorname{tr} \mathscr{D}^{-1}(\varphi) G + \Gamma_2(\varphi, G), \tag{2.4}$$

where $W(\varphi)$ denotes the classical action in terms of the classical field φ, $\mathscr{D}$ is the free propagator, G the exact propagator to be determined from the requirement:

$$\frac{\delta\Gamma(\varphi, G)}{\delta G_{ab}(x, y)} = 0. \tag{2.5}$$

Finally Γ_2 is the sum of the 2-particle irreducible vacuum graphs constructed from the exact propagators $G_{ab}(x, y)$ and the vertices generated by the shifted Lagrangian $\mathscr{L}(\Phi+\varphi)$ whose interaction part equals

$$\mathscr{L}_{\text{int}}(\Phi, \varphi) = -NV_0\left(\frac{(\Phi+\varphi)^2}{N}\right) + NV_0\left(\frac{\varphi^2}{N}\right) + 2\varphi_a\Phi_a V_0^{(1)}\left(\frac{\varphi^2}{N}\right)$$

$$+\Phi_a\Phi_b\left[\delta_{ab}V_0^{(1)}\left(\frac{\varphi^2}{N}\right) + 2\frac{\varphi_a\varphi_b}{N}V_0^{(2)}\left(\frac{\varphi^2}{N}\right)\right]. \tag{2.6}$$

Here and in the following $V_0^{(n)}\left(\frac{\varphi^2}{N}\right)$ denotes the n-th derivative of $V_0\left(\frac{\varphi^2}{N}\right)$ with respect to φ^2/N

$$V_0^{(n)}\left(\frac{\varphi^2}{N}\right) = \sum_{k=0}^{\infty}\frac{V_0^{(k+n)}(0)}{k!}\left(\frac{\varphi^2}{N}\right)^k. \tag{2.7}$$

The inverse free propagator equals

$$i\mathscr{D}_{ab}^{-1}(\varphi; x, y) = \frac{\delta^2 I(\varphi)}{\delta\varphi_a(x)\delta\varphi_b(y)}$$

$$= -\left\{\left[\Box + 2V_0^{(1)}\left(\frac{\varphi^2}{N}\right)\right]\delta_{ab} + 4\frac{\varphi_a(x)\varphi_b(x)}{N}V_0^{(2)}\left(\frac{\varphi^2}{N}\right)\right\}\delta(x-y). \tag{2.8}$$

We find rewarding to decompose (2.8) onto the transversal and longitudinal modes with respect to the versor $\hat{\varphi} = \varphi/|\varphi|$ in the isospin space

$$\mathscr{D}_{ab}^{-1} = i\left[\Box + 2V_0^{(1)}\left(\frac{\varphi^2}{N}\right)\right](\delta_{ab} - \hat{\varphi}_a\hat{\varphi}_b)$$

$$+i\left[\Box + 2V_0^{(1)}\left(\frac{\varphi^2}{N}\right) + \frac{4V_0^{(2)}\left(\frac{\varphi^2}{N}\right)}{N}\right]\hat{\varphi}_a\hat{\varphi}_b. \tag{2.9}$$

A similar decomposition was introduced by Townsend [10] in his study of the $\frac{1}{N}\Phi_3^6$ model.

The same decomposition of the exact propagator yields

$$G_{ab} = g(\delta_{ab} - \hat{\varphi}_a\hat{\varphi}_b) + \tilde{g}\hat{\varphi}_a\hat{\varphi}_b. \tag{2.10}$$

The form of g and $\tilde{g}$ is implicitly given by analogy to (2.5)

$$\left.\frac{\delta\Gamma(\varphi, G)}{\delta g}\right|_{\tilde{g}} = \left.\frac{\delta\Gamma(\varphi, G)}{\delta \tilde{g}}\right|_{g} = 0. \tag{2.11}$$

Substituting (2.9) and (2.10) into (2.4) we obtain

$$\Gamma(\varphi, G) = W(\varphi) + \frac{\hbar}{2}\int d^n x\left[i(N-1)\log g^{-1} - (N-1)\left(\Box + 2V_0^{(1)}\left(\frac{\varphi^2}{N}\right)\right) + i\log \tilde{g}^{-1}\right]$$

$$-\frac{\hbar}{2}\int d^n x\left[\Box + 2V_0^{(1)}\left(\frac{\varphi^2}{N}\right) + 4\frac{\varphi^2}{N}V_0^{(2)}\left(\frac{\varphi^2}{N}\right)\right]\tilde{g} + \Gamma_2, \tag{2.12}$$

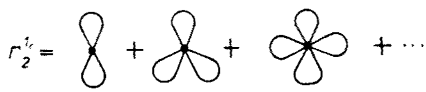

Fig. 1. Vacuum diagrams contributing to Γ_2 in the leading, next to leading and higher orders

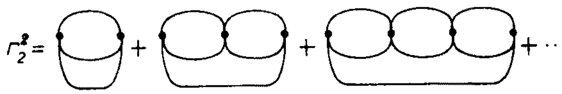

Fig. 2. Another kinds of graphs contributing to Γ_2. These graphs do not contribute in the leading order. Dots stand for the "effective vertices" as defined on Figs 4 and 5

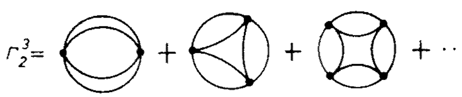

Fig. 3. Same as Fig. 2. Also these contributions are absent in the leading order

where Γ_2 collects contributions steming from the graphs on Figs 1–3. Using the $1/N$ power counting arguments we find that the relevant Feynman rules generated by (2.6) are

$$i\left.\frac{\partial^{2k}\mathscr{L}_{\text{int}}(\Phi, \varphi)}{\partial\Phi_{a_1}(x)\ldots\partial\Phi_{a_{2k}}(x)}\right|_{\Phi=0} = -iN\left(\frac{2}{N}\right)^k V_0^{(k)}\left(\frac{\varphi^2}{N}\right)[\delta_{a_1a_2}\ldots\delta_{a_{2k-1}a_{2k}}$$

$$+\text{distinct permutations}], \tag{2.13}$$

$$i\left.\frac{\partial^{2k+1}\mathscr{L}_{\text{int}}(\Phi, \varphi)}{\partial\Phi_{a_1}(x)\ldots\partial\Phi_{a_{2k}}(x)\partial\Phi_b(x)}\right|_{\Phi=0} = -iN\left(\frac{2}{N}\right)^{k+1}\varphi_b V_0^{(k+1)}\left(\frac{\varphi^2}{N}\right)[\delta_{a_1a_2}\ldots\delta_{a_{2k-1}a_{2k}} + \ldots], \tag{2.14}$$

and

$$i\left.\frac{\partial^{2k}\mathscr{L}_{\text{int}}(\Phi, \varphi)}{\partial\Phi_{a_1}(x)\ldots\partial\Phi_{a_{2k-2}}(x)\partial\Phi_b(x)\partial\Phi_c(x)}\right|_{\Phi=0}$$

$$= -iN\left(\frac{2}{N}\right)^{k+1}\varphi_b\varphi_c V_0^{(k+1)}\left(\frac{\varphi^2}{N}\right)[\delta_{a_1a_2}\ldots\delta_{a_{2k-3}a_{2k-2}} + \ldots]. \tag{2.15}$$

We are now in a position to evaluate the contributions Γ_2^1, Γ_2^2 and Γ_2^3 from the diagrams of Figs 1, 2 and 3, respectively. After some algebra we find that these of Fig.1 yield

$$\Gamma_2^1 = -N\int d^n x\left[V_0\left(\frac{\varphi^2}{N}+\hbar g(x,x)\right)-V_0\left(\frac{\varphi^2}{N}\right)-\hbar g(x,x)V_0^{(1)}\left(\frac{\varphi^2}{N}\right)\right]$$
$$-\hbar\int d^n x\left\{\hbar g^2(x,x)V_0^{(2)}\left(\frac{\varphi^2}{N}+\hbar g(x,x)\right)-(g-\tilde{g})V_0^{(1)}\left(\frac{\varphi^2}{N}-\hbar g\right)\right.$$
$$\left.+(g-\tilde{g})V_0^{(1)}\left(\frac{\varphi^2}{N}\right)+2\frac{\varphi^2}{N}\tilde{g}V_0^{(2)}\left(\frac{\varphi^2}{N}-\hbar g\right)-2\frac{\varphi^2}{N}\tilde{g}V_0^{(2)}\left(\frac{\varphi^2}{N}\right)\right\}. \tag{2.16}$$

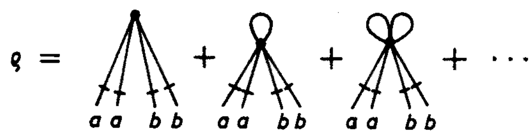

Fig. 4. The definition of the "effective" Φ^4 vertex

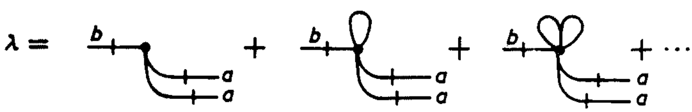

Fig. 5. The definition of the "effective" Φ^3 coupling

Before we proceed the evaluation of the remaining contributions to Γ_2 let us introduce the subsidiary Φ^4 and Φ^3 vertices ϱ and λ as defined on Fig. 4 and Fig. 5. The calculations require only a little of combinatorics and give

$$\varrho = V_0^{(2)}\left(\frac{\varphi^2}{N}+\hbar g(x,x)\right) \tag{2.17}$$

and

$$\lambda = 2\varphi_b V_0^{(2)}\left(\frac{\varphi^2}{N}+\hbar g(x,x)\right) = 2\varphi_b\varrho. \tag{2.18}$$

The above analysis exhibits the topological similarity of the general $\frac{1}{N}\Phi^{2n}$ theory to the $\frac{1}{N}\Phi^4$ one. This fact was stated by Schnitzer for the leading order and now we see that this property persists also in the next to leading order. We have a convenient tool available for a relatively simplified method of evaluating the complete diagrammatical expansion for (2.4). We need not to calculate graphs containing all the vertices defined by (2.13)–(2.15). Instead we may confine ourselves to diagrams involving only the *Vierbein* coupling and then, in the result replace the coupling constant by the full series standing on the right hand side of (2.17).

The similar technique was used by Townsend [10] in $\frac{1}{N}\Phi_3^6$ theory. Now it is apparent that the situation in the Φ^6 model is a reflection of much more general topological property of the $1/N$ expansion. It is apparent that this pattern of calculation can be easily generalized for higher orders in $1/N$ but obviously the number of the "effective couplings" will increase order by order.

The next observation which will simplify the calculations and justifies the need for the decomposition (2.10) is that the next to leading approximation of the transversal mode g of G_{ab} is completely determined by the leading order results for Γ. Indeed, decomposing g into the $O(1)$ and $O(N^{-1})$ parts

$$g = g_1 + g_{1/N} + O(N^{-2}) \tag{2.19}$$

we find

$$V(\varphi; g, \tilde{g}) = V(\varphi; g_1, \tilde{g}) + g_{1/N} \left. \frac{\partial V(\varphi; g, \tilde{g})}{\partial g} \right|_{g=g_1} \tag{2.20}$$

Therefore

$$V(\varphi; g, \tilde{g}) = V(\varphi; g_1; \tilde{g}) + O(N^{-1})$$

because

$$\partial V(\varphi; g_1, \tilde{g})/\partial g_1 = 0,$$

by definition of g_1 (compare (2.11)).

Differentiating (2.12) with (2.16) substituted for Γ_2 we recover Schnitzer's result for g^{-1} [8]

$$g^{-1} = i\left[\Box + 2V_0^{(1)}\left(\frac{\varphi^2}{N} + \hbar g(x, x)\right)\right], \tag{2.21}$$

or in momentum representation

$$g^{-1} = -i(k^2 - M^2), \tag{2.22}$$

M^2 being the solution of the "gap equation"

$$M^2 = 2V_0^{(1)}\left(\frac{\varphi^2}{N} + B_1\right), \tag{2.23}$$

where we have introduced

$$B_1 = i\hbar \int \frac{d^n p}{(2\pi)^n} \frac{1}{k^2 - M^2}. \tag{2.24}$$

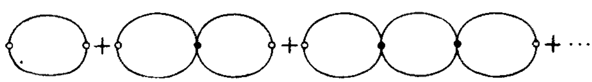

Fig. 6. The 'bubble chain" diagrams contributing to I. Heavy dots mean the "effective" coupling ϱ. Empty circles recall that the factors at the ends of the chain are omitted

We are now ready to evaluate the remaining contributions to Γ_2. It is convenient first to sum the "bubble chain" graphs of Fig. 6. The procedure is standard and gives

$$I(k) = -i \frac{B_2(k)}{1 + 4\varrho B_2(k)}, \tag{2.25}$$

where

$$B_2(k) = -i\frac{\hbar}{2}\int\frac{d^np}{(2\pi)^n}\frac{1}{p^2-M^2}\frac{1}{(k+p)^2-M^2}. \tag{2.26}$$

Using (2.25) we obtain for the contributions Γ_2^2 of diagrams of Fig. 2

$$\Gamma_2^2 = -8\hbar\int\frac{d^nk}{(2\pi)^n}\left(\frac{\varphi^2}{N}\right)\frac{\varrho^2B_2(k)}{1+4\varrho B_2(k)}\tilde{g}(k), \tag{2.27}$$

and for Γ_2^3 of Fig. 3

$$\Gamma_2^3 = -\frac{\hbar}{2}\int\frac{d^nk}{(2\pi)^n}\log[1+4\varrho B_2(k)]+2\hbar\varrho\int\frac{d^nk}{(2\pi)^n}B_2(k). \tag{2.28}$$

Collecting all terms together we obtain

$$\Gamma = \Gamma_{(N)}+\Gamma_{(1)}+O(N^{-1}) \tag{2.29}$$

where the $O(N)$ part $\Gamma_{(N)}$ reproduces the result of Schnitzer [8]

$$\Gamma_{(N)}(\varphi) = \int d^nx\left\{\tfrac{1}{2}(\partial_\mu\varphi)^2-N\left[V_0\left(\frac{\varphi^2}{N}+\hbar g\right)-\hbar gV_0^{(1)}\left(\frac{\varphi^2}{N}+\hbar g\right)\right]\right\}$$
$$+\tfrac{1}{2}Ni\hbar\ \mathrm{tr}\log\left[\Box+2V_0^{(1)}\left(\frac{\varphi^2}{N}+\hbar g\right)\right] \tag{2.30}$$

and the $O(N^0)$ part $\Gamma_{(1)}$ equals

$$\Gamma_{(1)}(\varphi) = B_1^2V_0^{(2)}\left(\frac{\varphi^2}{N}+B_1\right)+B_1V_0^{(1)}\left(\frac{\varphi^2}{N}+B_1\right)-\hbar B_1V_0^{(1)}\left(\frac{\varphi^2}{N}\right)$$
$$+\hbar\int\frac{d^np}{(2\pi)^n}\left\{\frac{i}{2}\log\frac{\tilde{g}^{-1}}{p^2-M^2}-\frac{i}{2}\frac{p^2-2V_0^{(1)}\left(\frac{\varphi^2}{N}\right)}{p^2-M^2}+\tfrac{1}{2}\left[p^2-2V_0^{(1)}\left(\frac{\varphi^2}{N}\right)\right.\right.$$
$$\left.-\frac{4\varphi^2}{N}V_0^{(2)}\left(\frac{\varphi^2}{N}\right)\right]\tilde{g}-\tilde{g}V_0^{(1)}\left(\frac{\varphi^2}{N}+B_1\right)+\tilde{g}V_0^{(1)}\left(\frac{\varphi^2}{N}\right)-\frac{2\varphi^2}{N}\tilde{g}V_0^{(2)}\left(\frac{\varphi^2}{N}+B_1\right)$$
$$\left.+\frac{2\varphi^2}{N}\tilde{g}V_0^{(2)}\left(\frac{\varphi^2}{N}\right)+\tfrac{1}{18}\frac{\varphi^2}{N}\tilde{g}\frac{\varrho^2B_2(p)}{1+4\varrho B_2(p)}+\tfrac{1}{2}\log(1+4\varrho B_2(p))+2\varrho B_2(p)\right\}. \tag{2.31}$$

Solving the equation $\partial\Gamma/\partial\tilde{g} = 0$ for $\tilde{g}$ we find

$$\tilde{g} = i\left[p^2-M^2-\frac{4\varphi^2}{N}\varrho+16\hbar\frac{\varphi^2}{N}\frac{\varrho^2B_2(p)}{1+4\varrho B_2(p)}\right]^{-1}. \tag{2.32}$$

Inserting (2.32) back into (2.31) and changing to Euclidean momenta we find after unpleasant algebraic manipulations

$$\Gamma = \Gamma_{(N)} - \frac{\hbar}{2} \int\limits_E \frac{d^n p}{(2\pi)^n} \log \frac{(p^2+M^2)(1+4\varrho B_2)+4q \dfrac{\varphi^2}{N}}{p^2+M^2} . \tag{2.33}$$

It is easy to check by direct substitution that our general result reduces to the specific results obtained by Root [11] for the $\frac{1}{N}$ Φ^4 theory and Townsend [10] for the $\frac{1}{N}$ Φ^6 model. This seems to confirm the reliability of our calculations.

3. *Renormalization*

The expression (2.33) for the effective action contains the unpleasant logarithm so it is rewarding to work with the derivative of the effective potential rather than directly with Γ. In the following $V_{(N)}$ will denote the $O(N)$ and $V_{(1)}$ the $O(1)$ part of the effective potential. We have

$$\frac{\partial V_{(N)}}{\partial \varphi^2/N} = V_0^{(1)}\left(\frac{\varphi^2}{N} + B_1\right), \tag{3.1}$$

and

$$\frac{\partial V_{(1)}}{\partial \varphi^2/N} = \frac{\hbar}{2} \int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2+M^2)(1+4\varrho B_2)+4\varrho\varphi^2/N}$$

$$\times \left\{ \frac{4(p^2+M^2)\varrho\partial B_2/\partial M^2 - 4\varrho\varphi^2/N}{p^2+M^2} \frac{\partial M^2}{\partial \varphi^2/N} + \left[4(p^2+M^2)B_2 + 4\frac{\varphi^2}{N}\right] \frac{\partial \varrho}{\partial \varphi^2/N} + 4\varrho \right\}. \tag{3.2}$$

From now on we shall carry out all calculations in three dimensional space time. All divergences which occur in the first $1/N$ order of any scalar theory in three dimensions can be soaked up by normal ordering. This normal ordering is equivalent to shifting the arguments of the expansion coefficients on the right hand side of (2.2) [8].

More precisely, the renormalized parameters $V^{(k)}$ are related to the bare ones $V_0^{(k)}$ by the equation

$$V^{(k)}(0) = V_0^{(k)}(c), \tag{3.3}$$

where c is the linearly divergent constant proportional to the cutoff Λ

$$c = -\frac{\hbar\Lambda}{2\pi^2} . \tag{3.4}$$

The renormalized version of (3.1) is

$$\frac{\partial V_{(N)}}{\partial \varphi^2/N} = V^{(1)}\left(\frac{\varphi^2}{N} - \frac{\hbar M}{4\pi}\right), \tag{3.5}$$

where $-\hbar M/4\pi$ is the finite part of B_1.

All the counterterms required to cancel the divergences of the expression (3.2) are of order $1/N$. Therefore their relevant contribution to $\dfrac{\partial V}{\partial \varphi^2/N}$ while calculated up to the next to leading order stems only from the diagrams which are topologically of order N. The renormalization procedure will replace the coefficients $V^{(j)}$ of the expansion of V by $\overline{V}^{(j)}$ of the form

$$\overline{V}^{(j)}(0) = V^{(j)}(0) + \frac{1}{N}\,\delta^{(j)}. \tag{3.6}$$

All the $\delta^{(j)}$'s are of order 1 and are determined by the divergent part of (3.2). Expanding $V_0^{(1)}$ around its renormalized value $\overline{V}^{(1)}$ we obtain

$$V_0^{(1)}\left(\frac{\varphi^2}{N} + B_1\right) = \overline{V}^{(1)}\left(\frac{\varphi^2}{N} - \frac{\hbar M}{4\pi}\right) + \frac{1}{N}\sum_j \delta^{(j)}\, \frac{\partial \overline{V}^{(1)}\left(\dfrac{\varphi^2}{N} - \dfrac{\hbar M}{4\pi}\right)}{\partial \overline{V}^{(j)}(0)}, \tag{3.7}$$

where

$$\frac{\partial \overline{V}^{(1)}\left(\dfrac{\varphi^2}{N} - \dfrac{\hbar M}{4\pi}\right)}{\partial \overline{V}^{(j)}(0)} = \frac{1}{(j-1)!}\left(\frac{\varphi^2}{N} - \frac{\hbar M}{4\pi}\right)^{j-1}\left(1 + \frac{\varrho \hbar}{4\pi M}\right)^{-1}, \tag{3.8}$$

or

$$\frac{\partial \overline{V}^{(1)}\left(\dfrac{\varphi^2}{N} - \dfrac{\hbar M}{4\pi}\right)}{\partial \overline{V}^{(j)}(0)} = \frac{1}{2\varrho}\,\frac{1}{(j-1)!}\left(\frac{\varphi^2}{N} - \frac{\hbar M}{4\pi}\right)^{j-1}\frac{\partial M^2}{\partial \varphi^2/N}. \tag{3.9}$$

Our present task is to separate the divergent part of (3.2) and to express it in the form of the sum of terms having the same shape as (3.9), the most suitable for determining the $\delta^{(j)}$'s. In three dimensions B_2 is finite and equals

$$B_2(p) = \frac{\hbar}{8\pi p}\arcsin\sqrt{p^2/p^2+4M^2} = \frac{\hbar}{16}(p^2+M^2)^{-1/2}\left\{1 - \frac{4}{\pi}\,M(p^2+M^2)^{-1/2} + \ldots\right\},$$

thus

$$\frac{\partial B_2}{\partial M^2} = -\frac{\hbar}{8\pi M}\,\frac{1}{p^2+M^2}. \tag{3.11}$$

We also find

$$\frac{\partial \varrho}{\partial \varphi^2/N} = V^{(3)}\left(\frac{\varphi^2}{N} - \frac{\hbar M}{4\pi}\right)\frac{1}{2\varrho}\,\frac{\partial M^2}{\partial \varphi^2/N}. \tag{3.12}$$

Using (3.11), (3.12) and the fact that $\partial B_1/\partial M^2$ is cutoff-independent and equal to $-\hbar/8\pi M$ we get

$$2\frac{\partial V_{(1)}}{\partial\varphi^2/N}=\hbar\int\frac{d^3p}{(2\pi)^3}\frac{1}{(p^2+M^2)(1+4\varrho B_2)+4\varrho\varphi^2/N}\left\{4\varrho+4(p^2+M^2)B_2V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)\right.$$
$$\left.+\frac{4\varphi^2}{N}V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)-2\varrho\left(\frac{4\varrho\varphi^2/N}{p^2+M^2}\right)\right\}\frac{1}{2\varrho}\frac{\partial M^2}{\partial\varphi^2/N}. \tag{3.13}$$

Expanding the integrand in inverse powers of $\sqrt{p^2+M^2}$, neglecting the finite terms and performing the cutoff in the divergent ones we obtain

$$\frac{2\partial V_{(1)}}{\partial\varphi^2/N}=\left\{2V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)c^2-4\varrho c-4V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)\left[\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right]c\right.$$
$$+\frac{\hbar^2}{16}V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)c-\hbar\log\frac{\Lambda^2/\mu^2}{4\pi}\left[\hbar\varrho^2+2\hbar\varrho V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)\left[\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right]\right.$$
$$\left.\left.-4\hbar\varrho^2\left(\frac{\hbar}{16}\right)^2V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)\right\}\frac{1}{2\varrho}\frac{\partial M^2}{\partial\varphi^2/N}, \tag{3.14}$$

where c is the normal-ordering constant (3.4). The expression (3.14) contains both power and logarithmical divergences which can now be subtracted because all members of (3.14) have the shape of (3.12). To make this feature more apparent we shall expand ϱ, ϱ^2, $\varrho V^{(3)}$ and $\varrho^2 V^{(3)}$ in powers of $\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)$. Let us recall

$$\varrho=\sum_{k=0}^{\infty}\frac{1}{k!}V^{(k+2)}(0)\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)^k, \tag{3.15}$$

therefore

$$\varrho^2=\sum_{k=0}^{\infty}\frac{A_k}{k!}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)^k, \tag{3.16}$$

$$\varrho V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)=\sum_{k=0}^{\infty}\frac{B_k}{k!}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)^k, \tag{3.17}$$

and

$$\varrho^2 V^{(3)}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)=\sum_{k=0}^{\infty}\frac{c_k}{k!}\left(\frac{\varphi^2}{N}-\frac{\hbar M}{4\pi}\right)^k, \tag{3.18}$$

where

$$A_k = \sum_{l=0}^{k} \binom{k}{l} V^{(l+2)}(0) V^{(k-l+2)}(0), \tag{3.19}$$

$$B_k = \sum_{l=0}^{k} \binom{k}{l} V^{(l+2)}(0) V^{(k-l+3)}(0), \tag{3.20}$$

and

$$C_k = \sum_{l=0}^{k} \sum_{m=0}^{l} \binom{k}{l} \binom{l}{m} V^{(k-l+2)}(0) V^{(l-m+2)}(0) V^{(m+3)}(0). \tag{3.21}$$

Substituting (3.16)–(3.18) into (3.14) and arranging in powers of $\left(\frac{\varphi^2}{N} - \frac{\hbar M}{4\pi}\right)$ we obtain

$$\frac{2\partial V_{(1)}}{\partial \varphi^2/N} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\varphi^2}{N} - \frac{\hbar M}{4\pi}\right)^k \Big\{ 2c^2 V^{(k+3)}(0) - 4(k+1) c V^{(k+2)}(0)$$

$$+ \frac{\hbar^2}{16} c V^{(k+3)}(0) - \hbar^2 \log \frac{\Lambda^2/\mu^2}{4\pi} \left[A_k + 2kB_{k-1} - 4\left(\frac{\hbar}{16}\right)^2 C_k \right] \Big\} \frac{1}{2\varrho} \frac{\partial M^2}{\partial \varphi^2/N}. \tag{3.22}$$

Comparing (3.22) with (3.6)–(3.9) we find the explicit formula for the next to leading counterterms $\delta^{(j)}$

$$\delta^{(k)} = -\Big\{ 2c^2 V^{(k+2)}(0) - 4kc V^{(k+1)}(0) + \frac{\hbar^2}{16} c V^{(k+2)}(0)$$

$$- \hbar^2 \log \frac{\Lambda^2/\mu^2}{4\pi} \left[A_{k-1} + 2(k-1) B_{k-2} - 4\left(\frac{\hbar}{16}\right)^2 C_{k-1} \right] \Big\}. \tag{3.33}$$

4. Conclusions

We have demonstrated that the next to leading radiative corrections do not destroy the general structure of the three dimensional theory whose interaction part of the Lagrangian is a Taylor expandible function of Φ^2/N. In particular the induced derivative couplings are absent to this order. We have derived the formula allowing to calculate the explicit expressions for all counterterms.

The details will depend on the particular form of the interaction term (2.2) but it is obvious that consistency requires (2.2) to be a polynomial of infinite degree[1]. Nothing can be said about higher orders unless by no means simple calculations to the order $1/N$

[1] Or of the trivial, renormalizable $\frac{\lambda}{N}\Phi^4 + \frac{g}{N^2}\Phi^6$ form.

will be done. Probably in next orders the recourse to more sophisticated methods should be made, like improving the Green's functions in the spirit of [6] or using nonperturbative Symanzik's constants [4].

Our present knowledge does not allow us to do more than raise this possibility, however our analysis demonstrates that the possible nonrenormalizable features of nonlinear theories are suppressed by a factor of at least $1/N^2$. This result is independent of the particular choice of the coupling constants $V^{(k)}$ and confirms the possibility that the inability of the conventional perturbative methods to deal with interactions of the form (2.2) is only a symptom not a cause of the difficulties one usually encounters on this ground.

REFERENCES

[1] K. G. Wilson, *Phys. Rev.* **D7**, 2911 (1973).
[2] H. J. Schnitzer, *Phys. Rev.* **D10**, 1800, 2042 (1974).
[3] J. Rayski, *Acta Phys. Pol.* **35**, 403 (1969); **36**, 1003 (1969); TPJU 4/76 preprint.
[4] K. Symanzik, *Comm. Math. Phys.* **45**, 79 (1975); DESY 75/24 preprint.
[5] J. R. Klauder, *Phys. Lett.* **27B**, 523 (1973); *Acta Phys. Austriaca* Suppl. **11**, 341 (1973); J. R. Klauder, H. Narnhofer, *Phys. Rev.* **D13**, 257 (1976).
[6] G. Parisi, *Nucl. Phys.* **B100**, 368 (1975).
[7] P. Rembiesa, *Phys. Rev.* **D**, in press.
[8] H. J. Schnitzer, *Nucl. Phys.* **B109**, 297 (1976).
[9] J. M. Cornwall, R. Jackiw, E. Tomboulis, *Phys. Rev.* **D10**, 2428 (1974).
[10] P. K. Townsend, *Phys. Rev.* **D12**, 2269 (1975); **D14**, 1715 (1976); *Nucl. Phys.* **B118**, 199 (1977).
[11] R. G. Root, *Phys. Rev.* **D10**, 3235 (1974).