# LIGHT AND HEAVY CONSTITUENTS IN A LATTICE ALGEBRA

By L. Vékás* and A. Abramovici

University of Timişoara, Timişoara**

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The set $P$ of particles is supposed to be partly ordered by an invariant ordering relation which makes the set $P$ a complemented distributive lattice. As a consequence, hadrons may be regarded simultaneously as bound states of heavy unstable constituents or as particles composed of light stable quasifree constituents, the two basic sets of constituents being in a bootstrap-type connection. Some phenomenological implications (average quark mass, the high energy behaviour of $R(\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-))$ of this way of thinking about constituents are also discussed.

## 1. Introduction

A general property suggested by experiment is that of compositeness: hadrons are built up from more fundamental objects ("constituents"). The great power of constituent models consists in explaining the hadron spectrum, the general features of deep inelastic phenomena together with many other problems of particle physics. The constituents (fractionally or integrally charged quarks) sometimes behave like free particles but are never observed as isolated particles: this is the main puzzle of constituent models. The solutions offered for the "confinement" problem are still in a very speculative state.

In one of the models, suggested by Wilson [1], quark confinement is considered in the framework of field theories with strong coupling defined on a space-time lattice. Quarks are located at lattice sites linked by rigid strings to form string-like hadron states. The masses of these states are obtained by adding the constituent quark and string masses. Quarks are confined: their separation requires an infinite length of string, which has infinite mass in the strong coupling approximation. In this approach the algebraic properties of lattices are not exploited: the space-time lattice is only a mathematical device to obtain meaningful results in the continuum limit.

A lattice with particles and their constituents as elements may be constructed starting from other considerations.

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* Permanent address: Polytechnical Institute of Timişoara, Timişoara, R.S. România.

** Address: University of Timişoara, Timişoara, R.S. România.

Classification of particles involves in fact the existence of two types of relations defined on the set of particles. Building certain multiplets (based on SU(3) or other group) *equivalence relations* are considered, while the fact that hadrons lie on Regge trajectories implies the existence of an *ordering relation*, which partly orders the set of particles. If in this partly ordered set any finite subset has both an infimum and a supremum, then it is a lattice. Such a Regge-lattice bounded below was constructed in [2].

We proceed to examine the consequences which may arise from the assumption that the set of particles, $P$, is partly ordered. We do not try to specify the concrete form of the ordering relation, which depends on the properties of each particle, but we suppose it makes possible to organize the set of particles as a lattice and that it is *independent* of the (inertial or noninertial) reference frame considered.

Using the algebraic properties of the particle lattice it will be suggested that the composite models involving light quasifree constituents and those with strongly bound heavy constituents are in a certain sense dual representations of the structure of hadrons.

## 2. *Particles and their constituents as elements of a lattice*

Hadrons may be built up from more fundamental objects, elements of the lattice $P$, if $P$ is bounded at least at one end. If one requires (as in quark models) a finite total number of constituents, then the lattice $P$ must be bounded below and above too (beside other conditions it must fulfill), that is, it has to satisfy both the minimum and maximum conditions. Consequently, $P$ has a least (denoted by "0") and a greatest element (denoted by "$g$"). It follows that the lattice $P$ is *atomic*, as well as *dually atomic* and that every element of the lattice can be represented as the meet of a finite number of meet-irreducible elements or as the join of a finite number of join-irreducible elements. (The "meet", denoted by $\cap$ and "join", denoted by $\cup$, are the (still unspecified) lattice operations; for the terminology and lattice theoretical results used throughout the paper see e.g. [3].)

If $x \in P$ is a hadron, then it may be represented in two ways:

$$x = p_{i_1} \cup \ldots \cup p_{i_r}, \tag{1}$$

or

$$x = q_{j_1} \cap \ldots \cap q_{j_k}, \tag{2}$$

where $p_i\,(i = 1, \ldots, m)$ are join-irreducible elements (structureless with respect to the operation $\cup$) and $q_j\,(j = 1, \ldots, n)$ meet-irreducible elements (structureless with respect to the operation $\cap$) of the lattice $P$, which in the following will be identified with constituents. We consider only *irredundant* representations, that is $p_{i_1}, \ldots, p_{i_r}$ and $q_{j_1}, \ldots, q_{j_k}$ are the minimum number of constituents ("valence" or "leading" constituents) needed to specify the hadron $x$. (In a redundant representation, beside the minimum number of constituents, there is an arbitrary number of redundant constituents, which may be identified with the "sea" — part of constituents.)

A requirement one has to fulfill is the uniqueness of the representations (1) and (2), which is ensured (by a theorem of Birkhoff) if the lattice $P$ is *distributive*. (A thorough discussion of the join and meet representations and their uniqueness is given by Wille [3].)

### 3. Valuation functions on the particle lattice and the quantum numbers of particles

A *valuation* $V$ defined on the lattice $P$ assigns to each element either a real number or, eventually, one of the symbols $\pm\infty$. If $a$ and $b$ are elements of the lattice, then

$$V(a)+V(b) = V(a \cup b)+V(a \cap b). \tag{3}$$

The valuation $V$ is called order preserving if $a \leqslant b$ implies $V(a) \leqslant V(b)$, while $V$ is positive if $a < b$ implies $V(a) < V(b)$.

We suppose that to each exactly conserved quantum number corresponds a certain valuation defined on the lattice $P$. This is a quite unusual approach, but we supposed only that the particles are elements of a lattice and we did not try to describe them by state vectors in a Hilbert space. This would be a further step related to the problem of representations of the lattice $P$. The relation of the present formalism to quantum physics has yet to be established.

If the join-irreducible elements $p_{i_1}, \ldots, p_{i_r}$ are *atoms* and the meet-irreducible elements $q_{j_1}, \ldots, q_{j_k}$ are *dual atoms* of the lattice $P$ and if they are also linearly independent, it may be shown that for any valuation the following relations hold [3]:

$$V(x) = \sum_{i=i_1}^{i_r} V(p_i)-(r-1)V(0), \tag{4}$$

$$V(x) = \sum_{j=j_1}^{j_k} V(q_j)-(k-1)V(g), \tag{5}$$

corresponding to the representations (1) and (2), respectively. Similar relations may be written for any particle $x \in P$ if $P$ is not only distributive but also a *complemented* lattice: then each join-irreducible element $p_i$ ($i = 1, \ldots, m$) is an atom and each meet-irreducible element $q_j$ ($j = 1, \ldots, n$) is a dual atom of the lattice. Moreover, all atoms, respectively dual atoms are linearly independent, because the lattice $P$ is distributive.

The lattice constructed in this way has a finite number of atoms, equal to that of dual atoms ($m = n$) and the number of elements of the lattice $P$ is also finite.

If $V$ corresponds to an additive quantum number, it follows that $V(0) = V(g) = 0$; in general $V(0) \neq 0$ and $V(g) \neq 0$.

The dual atom $q_i$ may be written as

$$q_i = p_1 \cup \ldots \cup p_{i-1} \cup p_{i+1} \cup \ldots \cup p_m \tag{6}$$

and then

$$p_i \cup q_i = g, \tag{7}$$

$$p_i \cap q_i = 0, \tag{8}$$

$p_i$ and $q_i$ being complemented elements. For a nonadditive quantum number, $N$, which corresponds to a valuation, it follows that

$$N(p_i)+N(q_i) = N(0)+N(g); \quad i = 1, \ldots, m. \tag{9}$$

The angular momentum, $J$, may be considered as a valuation defined on $P$ and then we have

$$J(x) = \sum_{i=i_1}^{i_r} J(p_i)-(r-1)J(0) = \sum_{j=j_1}^{j_k} J(q_j)-(k-1)J(g) \tag{10}$$

which is different from the usual rule of angular momentum addition; in particular, a single value of the total angular momentum is possible. This is resemblant to the Hund rules from atomic physics and by analogy we may argue that the lattice $P$ contains only particles in the *ground* state. The spin quantum numbers of constituents must satisfy the condition (9) which implies each constituent spin to be less than or equal to the sum $J(0)+J(g)$. This means that $J(0)$ and $J(g)$ cannot be both zero, because this would permit only scalar particles to be elements of the set $P$.

Assuming that the energy $E$ corresponds to a valuation of $P$, the mass of particle $x$ is given by

$$M(x) = \sum_{i=i_1}^{i_r} E(p_i)-(r-1)E(0) = \sum_{j=j_1}^{j_k} E(q_j)-(k-1)E(g). \tag{11}$$

## *4. Light and heavy constituents: "preons" or a "bootstrap"-type connection between constituents?*

Let us identify the dual atoms with heavy constituents called "quarks"[1]. Quark confinement is achieved simply if one supposes, as usual, that $M(q_j) > M(x)$ for all quarks $q_j$ $(j = j_1, \ldots, j_k)$, where $x$ denotes any particle from $P$ except $q_j$ $(j = 1, \ldots, m)$ and $g$. As we have seen, $P$ contains only particles in the ground state and then, in the classical limit, for strongly bound heavy quarks it may be written:

$$M(x) = \sum_{j=j_1}^{j_k} M(q_j)-(k-1)E(g). \tag{12}$$

If one would have a quark, $q_l$, for which $M(q_l) \geqslant E(g)$, then for a hadron $x$ composed from $q_l$ and an other quark, $q_s$, results a mass $M(x) = M(q_l)+M(q_s)-E(g) \geqslant M(q_s)$, which contradicts the initial assumption that $M(x) < M(q_s)$. Then we have $M(q_l) < E(g)$, inequality which remains valid in any other reference frame, that is, $E'(q_l) < E'(g)$ because the ordering relation is supposed to be invariant. It follows that *the energy is an order preserving valuation*. As a consequence, in the rest frame of particle $x$ we have $E(0) \leqslant E(p_i) \leqslant M(x)$ for any atom $p_i$ composing $x$, because $0 < p_i < x$, $i = i_1, \ldots, i_r$. Hence the constituents $p_i$ are *light*. The term $(r-1)\,E(0)$ in Eq. (11) plays the role of binding energy, thus the $p_i$'s are also *quasifree* because $E(0)$ is very small.

[1] This is a matter of convenience. We could identify equally well the atoms with quarks, because the ordering relation is unknown.

From the above reasoning it may also be deduced that the masses of particles in the ground state are limited above by $M(g)$. We also may write

$$0 \approx M(0) = \sum_{j=1}^{m} M(q_j)-(m-1)E(g) < m\overline{M}(q)-(m-1)M(g). \tag{13}$$

It follows that the average quark mass, $\overline{M}(q)$, satisfies the inequalities

$$\frac{m-1}{m} M(g) < \overline{M}(q) < M(g). \tag{14}$$

The second inequality is a consequence of the order preserving nature of the energy valuation function.

In conclusion, assuming that on the set of particles, $P$, an invariant ordering relation is defined, which makes $P$ a lattice and that hadrons are built up from constituents of large masses (representation (2)), it follows that the other possible representation, (1), involves light quasifree constituents[2]. Thus a hadron may be regarded simultaneously as a bound state of heavy constituents (say "quarks") or as a particle composed of light quasifree constituents (say "partons").

The electric charge[3] is an additive quantity, which implies that 0 and $g$ are neutral particles, $Q(0) = Q(g) = 0$ and

$$\sum_{i=1}^{m} Q(p_i) = \sum_{i=1}^{m} Q(q_i) = 0, \tag{15}$$

as in some unified models of quarks and leptons. The constituents $p_i$ and $q_i$ must be *integrally* charged in order to avoid fractionally charged hadrons. It was shown above that for any hadron, $x$, holds the inequality: $M(x) > M(p_i)$, $i = i_1, \ldots, i_r$. Thus some of the light constituents may be leptons [4], pointlike objects in the join representation. Dually, among the heavy constituents may be heavy leptons, pointlike objects in the meet representation[4].

All the particles, including heavy and light constituents, 0 and $g$ particles, are *composite* objects, i.e. a kind of global bootstrap emerges. The light constituents $p_i$ and the 0 particle are pointlike with respect to the join composition rule, but they are composed from heavy constituents in the meet representation; the dual statement is valid for the heavy constituents $q_i$ and $g$ particle.

In spite of the composite nature of constituents, a fundamental difference exists between these objects and hadrons: the constituents are composite only in one represen-

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[2] As a consequence of the lattice theoretical duality principle the converse statement is also valid.

[3] We consider the baryon number as a structural characteristic of particles which is not necessary to be strictly conserved [4–5]; $B$ does not correspond to a valuation.

[4] The possibility to build hadrons from leptons was also considered by Faddeev [6] in the framework of a soliton model, while the idea that heavy leptons may serve as hadronic constituents was examined in [7].

tation (meet or join), while hadrons are composite objects both in the meet and join representations. This suggests that the lattice composition rules, "join" and "meet", are related to two kinds of interactions, the structure of constituents being revealed only by a certain type of interaction (see, for example, [8]).

Note that the interaction related to the meet composition rule is a strong one, because the binding energy $(k-1)E(g)$ in the second Eq. (11) has a great value ($E(g) > M(x)$, $x \in P$). This ensures a relative stability to the ground state hadrons.

On the other hand, the heavy constituents are composite only according to the interaction related to the join composition rule, cf. Eq. (6), that is, they are built up from the light $p_i$'s with very small binding energies of order $E(0)$ (see the first of Eq. (11)). It follows that quarks and heavy leptons, as well as the $g$ particle, are *unstable* and therefore hardly observable objects [4, 5] with lifetimes less than that of "stable" ground state hadrons. The light constituents, including leptons, are composite only in the meet representation and, consequently, they are particles of high stability.

It is worth noting that the two dual sets of constituents are a consequence of the very general assumption about the existence of an invariant ordering relation, which makes the set of particles a bounded lattice. The bootstrap-type relation existing between the two basic sets of constituents may be an alternative to the "preon" idea [9, 4] (see also [10]). We suggest that $p_i$ and $q_i$ ($i = 1, \ldots, m$) instead of being truly elementary constituents, are the manifestations of some fundamental "degrees of freedom" [11], having quantum numbers usual to particles, attributed them by the valuation functions defined on $P$. This is a new way of thinking about constituents offered by lattice algebra.

*5. Average quark mass and high energy behaviour of* $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$

Many experimental facts imply that a strikings imilarity exists among the final states in various collisions, including $e^+e^-$ annihilations [12–14]. A good example is the universal curve of mean charged multiplicity versus available energy, which indicates that the production of particles tends to be independent of the incident particles as the available energy exceeds several GeV's.

Indeed, due to the bootstrap-type connection between constituents, any particle may be considered as a potential carrier of all degrees of freedom, because (see Eq. (6))

$$x = q_{j_1} \cap \ldots \cap q_{j_k} = (p_1 \cup \ldots \cup p_{j_1-1} \cup p_{j_1+1} \cup \ldots \cup p_m) \cap$$

$$\ldots \cap (p_1 \cup \ldots \cup p_{j_k-1} \cup p_{j_k+1} \cup \ldots \cup p_m), \quad (j_1 \neq j_k).$$

It follows that each constituent $p_i$ ($i = 1, \ldots, m$) may be produced (or, equivalently each degree of freedom may be excited), when the available energy has the order of magnitude of $M(g)$. Then the strong binding of $q$'s provided by the interaction related to the "$\cap$"-composition rule is broken and the structure of $q$'s relative to the "$\cup$"-type interaction becomes apparent. This leads to the formation of an unstable (the binding energy being of order $E(0)$) intermediate state, consisting of all $p$'s, which is equivalent to an excited state of $g$ particle[5], $g = p_l \cup \ldots \cup p_m$, regardless to the initial particles. This qualitatively explains the fact that as the energy increases, the charged particle multiplicities for different collisions tend to a unique curve, independent of the types of colliding particles.

From Eq. (14) and the multiplicity data one may then conclude that the magnitude of average quark mass is at a few GeV's[6]. If heavy leptons do exist, their masses would be of the same order of magnitude.

In the framework of the parton model the ratio $R = \dfrac{\sigma(e^+e^- \to \text{hadrons})}{(e^+e^- \to \mu^+\mu^-)}$ depends on the charged constituents. Experimental results indicate $R \approx 2.5$ in the lower energy region, followed by an increase of $R$ between 3.5 and 4.5 GeV to $R \approx 5$, which is usually attributed to the opening of new (higher mass) quark-antiquark production channels. In the higher energy region presently available $R$ remains roughly constant at this value. From the possible connection between the multiplicity data and average quark mass shown above, it seems that in this higher energy region the value $R \approx 5$ already includes the contributions of all constituents. Therefore we expect that there will be no further increase of $R$ even at higher values of available energy.

The high degree of generality of the experimental results discussed above, requires the use of very general theoretical arguments, perhaps like these based on lattice algebra.

## 6. *Conclusions*

Assuming that on the set of particles, $P$, an invariant ordering relation is defined, the set $P$ may be organized as a complemented distributive lattice. As a consequence, hadrons may be regarded simultaneously as bound states of heavy unstable constituents or as particles composed of light stable quasifree constituents. This gives a possible solution to the confinement problem.

All the particles, including heavy and light constituents, are composite objects. The hadrons are composite particles in each representation. The light constituents are pointlike in a representation, but are composed from heavy constituents in the other representation; the dual statement is valid for the heavy constituents. The bootstrap-type connection existing between the two basic sets of constituents may be an alternative to the "preon" idea.

The way of thinking about constituents outlined in the paper shows that each particle is a potential carrier of all degrees of freedom and qualitatively explains the fact that at high energies the charged multiplicity curves tend to a unique curve regardless the initial particles.

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[5] We have taken into account the idempotency of lattice operations and the properties of complemented distributive lattices [3].

[6] A more precise statement is difficult to give, because the multiplicity data are still affected by large errors.

From the possible connection between average heavy constituent mass and charged multiplicity data, it may be concluded that heavy constituents have a mass of the order of several GeV's. This also suggests that there will be no further increase of the ratio $R(\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-))$ at higher energies.

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