A DYNAMICAL MODEL FOR MULTIPLICITY DISTRIBUTIONS IN πp^- AND pp-COLLISIONS

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The independent emission model for the production of particles is generalized so as to include the pion-nucleon interaction in a simplified form. In this new model excited states of hadronic matter play an important role. Assuming that a thermal equilibrium between these states is established it is then possible to calculate the multiplicity distributions of charged and neutral pions. For πp- and pp-reactions a comparison with experimental results is given.

1. *Introduction*

In this paper we would like to formulate a model for the production of particles on the assumption that in the high energy collisions highly excited states of hadronic matter are created. In order to describe these states we suppose that they can be considered as the eigen-states of the Hamiltonian which gives the interaction between pions and nucleons. In general it will be impossible to construct these states. However, by restricting ourselves to static nucleons and mesons, the problem can be reduced to finding the eigen-vectors of a real symmetric matrix, which can be solved numerically. As a consequence of this static approximation it will not be possible to calculate momentum distributions and we will not be able to determine other than multiplicity distributions and multiplicity correlations. As long as we restrict our considerations to π mesons this static approximation excludes the possibility to form anti-symmetric pion clusters. In this case we therefore do not expect to get good agreement with experiment for the neutral-charge correlations, which are presumably due to cluster formation.

However, by extending our model to include ϱ-production, we find a much better agreement with experiment.

For each excited state the distribution of the number of charged particles can be calculated. It is found that the width of this distribution is proportional to the square

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root of the average number of particles, like in a Poisson distribution. This width changes, however, when for the final state of the reaction we assume a distribution over the excited states. In fact, by taking a thermal distribution and letting the temperature increase linearly with the average number of particles, we obtain a linear dependence of the dispersion on this average number of particles. This well-established "Wróblewski-relation" [1] can be made to coincide with the experimental values by fitting two energy-independent parameters of our model. It then turns out that also the higher central moments of the charged particle distribution are linear functions of the average number of charged particles. Hence we also find KNO-scaling [2].

In our model the isospin is conserved exactly. As a result we find not only strong correlations between the number of neutral and charged particles, but also a distribution of neutral pions, which is completely different from what one obtains by assuming Czyżewski-Rybicki [3]-like distribution for the total number of pions and then using the Cerulus weight factors to project out the distributions of charged and neutral pions [4].

These distributions and also the distribution of the total number of pions can be decomposed into components carrying isospin zero, one or two. It turns out that for proton-proton scattering the component with $I_\pi = 2$ can be neglected. From a calculation of the distribution of neutral pions it is furthermore seen that these pions are preferably produced in pairs.

A more detailed discussion of our results will, however, be postponed until Section 3 and 4. There we will also discuss how the production of ϱ and other mesons can be incorporated into our model. A few simple calculations are performed in order to show that by these extensions the form of the π^0-distribution and the charge-neutral correlation can be improved considerably.

In the next section we first explain our ideas by using a simple model for particles without isospin. The extension to the cases of $\pi^\pm$p- and pp-interactions will be sketched, but details of the calculations are omitted.

2. *Description of the model*

In order to explain our ideas we will first consider a simplified version of our model, in which the mesons and nucleons have no isospin. A Hamiltonian describing such a case with static particles is

$$H = a^*a + g(a + a^*), \tag{1}$$

where a annihilates a meson. The constant kinetic energy of the nucleon has been omitted, while the meson mass is taken as the unit of energy. This Hamiltonian can be written in diagonal form by the substitution $c = a + g$. This gives $H = c^*c - g^2$, which has the eigen-values $\varepsilon_k = k - g^2$ $(k = 0, 1, 2, \ldots)$ and the eigen-states

$$|k) = \frac{c^{*k}}{\sqrt{k!}}|0), \tag{2}$$

where the groundstate is given by

$$|0) = e^{-\frac{1}{2}g^2}e^{-ga^*}|0). \tag{3}$$

For this groundstate, also called coherent state, the distribution of the total number of particles can be calculated and one obtains a Poisson-distribution, as is well known. Since this coherent state is the state which is always chosen to describe independent or uncorrelated particle emission, a generalization of this independent production present itself immediately.

We will assume that the final state in a high energy collision is not described only by the groundstate of the Hamiltonian (1), but rather by an incoherent superposition of all excited states $|k)$. Each state will occur with a certain probability, and assuming that immediately after the collision the system of hadronic matter is in a thermal equilibrium at temperature T, this probability is proportional to the Boltzmann factor $e^{-\varepsilon_k/T}$. Later the temperature will be chosen as a special function of the energy or rather of the average number of particles. For any operator A acting in the space spanned by the many particle states $|n\rangle = \frac{a^{*n}}{\sqrt{n!}}|0\rangle$, we can now calculate the average in the final state as follows

$$\overline{A(T)} = Z^{-1}\sum_{k=0}^{\infty} e^{-\beta\varepsilon_k}\langle k|A|k\rangle, \tag{4}$$

where $\beta = 1/T$ and the partition function Z is

$$Z = \sum_{k=0}^{\infty} e^{-\beta\varepsilon_k} = \frac{e^{\beta g^2}}{1-e^{-\beta}}. \tag{5}$$

In particular we find for the average number of particles

$$\bar{n} = g^2 + \frac{1}{e^{\beta}-1} \tag{6}$$

and for the square of the dispersion D:

$$D^2 = \overline{(n-\bar{n})^2} = \bar{n}^2+\bar{n}-g^4. \tag{7}$$

Keeping the coupling constant g fixed and letting the temperature approach zero we find the result of the independent emission model, i.e.

$$\bar{n} \to g^2 \quad \text{and} \quad D^2 \to \bar{n}. \tag{8}$$

For T going to infinity, however, we obtain

$$\bar{n} = T+g^2-\tfrac{1}{2}+O\left(\frac{1}{T}\right) \tag{9}$$

and

$$D = \bar{n}+\tfrac{1}{2}+O\left(\frac{1}{\bar{n}}\right). \tag{10}$$

It is seen that the width of the multiplicity distribution no longer increases like $\sqrt{\bar{n}}$, as for Poisson-distribution, but like $\bar{n}$, as is observed experimentally and as is expressed in the "Wróblewski-relation" $D = a\bar{n} - b$. The coefficient a, however, is less than one and b is positive, so that the values of a and b as given by Eq. (10) are certainly not correct. This can be remedied by letting also g^2 increase linearly with $\bar{n}$ or with T, which has the same effect. Then, as can be seen from Eq. (7), the coefficient a can be made less than one. Indeed, by taking the proper values for A and B in

$$g^2 = AT + B \tag{11}$$

the numbers a and b can be recovered. In this way the Wróblewski-relation can be satisfied for all energies by adjusting the two energy independent parameters A and B.

When we compare our formulae (6) and (7), for the case $g = 0$, with the corresponding formulae for black body radiation,

$$\bar{n} = \sum_{\vec{p}} \frac{1}{e^{\beta E(p)} - 1} \tag{12}$$

and

$$D^2 = \bar{n} + \sum_{\vec{p}} \frac{1}{(e^{\beta E(p)} - 1)^2}, \tag{13}$$

with $E(p) = |\vec{p}|$, we observe that the only difference is the occurrence of only one mode in our treatment and an infinity of momentum modes in the black body case. For the black body case, however, one finds easily by converting the summations into integrations that $D = c\sqrt{\bar{n}}$, where c is a numerical constant. From this follows that our result $D \sim \bar{n}$ is a consequence of the assumption that in the final state only one mode, at least not a continuum of modes, can be excited. In other words: it is essential to assume that the final fireball is an incoherent superposition of excited states of one or of a few particles and not of a field in a box.

Another feature of our model is that the multiplicity distribution satisfies KNO-scaling. This can be proved in general by showing that all central moments $[\overline{(n-\bar{n})^k}]^{1/k}$ increase linearly with $\bar{n}$. For the simple case $g = 0$, however, it can be shown directly that the distribution

$$P_n(\beta) = (1 - e^{-\beta}) e^{-\beta n} \quad (n = 0, 1, 2, \ldots) \tag{14}$$

scales in the sense that

$$\psi(z) = \lim_{n \to \infty} \bar{n} P_n(\beta) \tag{15}$$

depends only on the variable $z = \dfrac{n}{\bar{n}}$. We find

$$\psi(z) = e^{-z/t}, \tag{16}$$

provided also $\frac{T}{n} = t$ is kept constant. This is the same condition as required for the Wróblewski-relation to be satisfied.

We notice that in our model the distribution of observed particles is represented as a sum over a number of components. For each of these components the particle distribution is Poisson-like. Multi-component models of this type have been considered before [10]. If also KNO-scaling is satisfied, which is the case in our theory, these models can be used to explain the observed leading proton spectrum [11].

So far we have only considered scalar mesons. It is possible, however, to incorporate the isospin of the pions and nucleons and its conservation, but keeping the restriction of static particles. This is done in a way which is completely analogous to the case of scalar mesons. For pion-nucleon scattering we consider the Hamiltonian [5]

$$H_1 = \vec{a}^* \cdot \vec{a} + g(\vec{a} + \vec{a}^*) \cdot \vec{\tau}, \tag{17}$$

where τ_1, τ_2, τ_3 are the Pauli spin operators acting on the isospin of the nucleon. For pp-interactions we take the Hamiltonian

$$H_2 = \vec{a}^* \cdot \vec{a} + g(\vec{a} + \vec{a}^*) \cdot (\vec{\tau}_{(1)} + \vec{\tau}_{(2)}). \tag{18}$$

The idea is now the same as in the scalar case: calculate the eigen-values and eigen-states of the relevant Hamiltonian, assume a thermal equilibrium between these states, and calculate any particle distribution. Again a relation of the form of Eq. (11) is assumed and the constants A and B are determined such that the Wróblewski-relation is satisfied.

In order to show the isospin structure we write down the form of the eigen-states. For H_1 they are:

$$|\pi^+ \mathrm{p}\rangle \Rightarrow |k\rangle = \alpha_k A_1^1(k)\,|\mathrm{p}\rangle + \frac{\beta_k}{\sqrt{5}} [2A_2^2(k)\,|\mathrm{n}\rangle - A_2^1(k)\,|\mathrm{p}\rangle] \tag{19}$$

and

$$|\pi^- \mathrm{p}\rangle \Rightarrow w_1 |k, \tfrac{1}{2}\rangle + w_2 |k, \tfrac{3}{2}\rangle, \tag{20}$$

with

$$|k, \tfrac{1}{2}\rangle = \alpha_k^1 A_0^0(k)\,|\mathrm{n}\rangle + \beta_k^1 [\sqrt{\tfrac{1}{3}}\, A_1^0(k)\,|\mathrm{n}\rangle - \sqrt{\tfrac{2}{3}}\, A_1^{-1}(k)\,|\mathrm{p}\rangle] \tag{21}$$

and

$$|k, \tfrac{3}{2}\rangle = \alpha_2'' [\sqrt{\tfrac{1}{3}}\, A_1^{-1}(k)\,|\mathrm{p}\rangle + \sqrt{\tfrac{2}{3}}\, A_1^0(k)\,|\mathrm{n}\rangle] + \beta_k'' [\sqrt{\tfrac{2}{5}}\, A_2^0(k)\,|\mathrm{n}\rangle - \sqrt{\tfrac{3}{5}}\, A_2^{-1}(k)\,|\mathrm{p}\rangle]. \tag{22}$$

For H_2 we consider only

$$|\mathrm{pp}\rangle \Rightarrow |k\rangle = w_s\,|k, s\rangle + w_a |k, a\rangle, \tag{23}$$

with a part which is symmetric in the two nucleons

$$\begin{aligned} |k, s\rangle = {} & \alpha_k''' A_0^0(k)\,|\mathrm{pp}\rangle + \beta_k''' \left[\frac{1}{\sqrt{2}} A_1^0(k)\,|\mathrm{pp}\rangle - \frac{1}{\sqrt{2}} A_1^1(k) \left| \frac{\mathrm{pn}+\mathrm{np}}{\sqrt{2}} \right\rangle\right] \\ & + \gamma_k''' \left[\frac{1}{\sqrt{10}} A_2^0(k)\,|\mathrm{pp}\rangle - \sqrt{\tfrac{3}{10}}\, A_2^1(k) \left| \frac{\mathrm{pn}+\mathrm{np}}{\sqrt{2}} \right\rangle + \sqrt{\tfrac{3}{5}}\, A_2^2(k)\,|\mathrm{nn}\rangle\right] \end{aligned} \tag{24}$$

and an anti-symmetric part

$$|k, a\rangle = \alpha_k''' A_1^1(k) \left| \frac{\text{pn}-\text{np}}{\sqrt{2}} \right\rangle . \tag{25}$$

In all cases k is the index which numbers the eigen-states. In Eq. (20) we take $w_1 = -\sqrt{\frac{2}{3}}$ and $w_2 = \sqrt{\frac{1}{3}}$, i.e. equal to the Clebsch–Gordon coefficients for the $I = \frac{1}{2}$ and $I = \frac{3}{2}$ content of the initial two-particle π^-p-state. In principle, however, they could have been taken as free parameters, but with $|w_1|^2+|w_2|^2 = 1$. In the same way we take $w_s = 1$ and $w_a = 0$ in Eq. (23). The coefficients α_k, β_k and γ_k and the operators $A_I^{I_3}(k)$ — which create an indefinite number of mesons with total isospin I and third component I_3 — are all determined by the eigen-value equation for the k-th eigen-state. The technique for transforming this equation to matrix form is described in reference [5]. We obtain infinite symmetric band matrices with a width three for the $\pi^\pm$p cases and five for the pp-case. After truncation to 80×80 matrices they were diagonalized by standard numerical techniques. For the reader who has the desire but not the energy to repeat all our calculations we collected the most crucial formulae in the appendix.

3. *Results*

In this section we show some results of our calculations. The two parameters of the model occurring in the linear relation between g^2 and T are chosen so that for π^+p-scattering the dispersion of the distribution of charged particles, i.e. π^+, π^- and proton, is exactly equal to

$$\mathscr{D}(\pi^+\text{p}) = 0.53\bar{n}_c - 0.54, \tag{26}$$

which is the Wróblewski-relation as given in Ref. [6]. For π^-p the same parameters are taken. For this case we then find again a straight line for the dispersion

$$\mathscr{D}(\pi^-\text{p}) = 0.7\bar{n}_c \tag{27}$$

which should be compared with [6]

$$\mathscr{D}(\pi^-\text{p}) = (0.54\pm0.02)\bar{n}_c - (0.40\pm0.07).$$

For the pp-case the parameters were fitted again to give the correct experimental result

$$\mathscr{D}(\text{pp}) = (0.58\pm0.01)\bar{n}_c - (0.56\pm0.01). \tag{28}$$

With the theory fixed in this way we calculated the following quantities:

a) The average number of protons
For π^+p: $\bar{N}_\text{p} = 0.6$ for all but the lowest multiplicities.
For π^-p: $\bar{N}_\text{p} = \frac{17}{45} = 0.378$ for all but the lowest multiplicities.
For pp: $\bar{N}_\text{p} = 1.73$ for all but the lowest multiplicities.

The numbers of average protons differ from those assumed in Ref. [4].

b) The average number of pions

For π^+p we find that with increasing energy the ratios are given by

$$n_+ : n_- : n_0 = 2 : 2 : 1.$$

We also obtain the exact result for all energies

$$n_+ + n_- = 4n_0 + 1. \tag{29}$$

In the ϱ-model to be discussed in Section 4 this relation becomes

$$3n_0 - 2n_c = 1,$$

so that asymptotically

$$n_c : n_0 = 3 : 2.$$

For π^-p and pp the average number of neutral pions is larger than half the number of charged pions. In Fig. 1 we show for the pp-case how $\bar{n}_+$, $\bar{n}_-$ and $\bar{n}_0$ vary as a function of the average number of charged particles.

Some experimental results are also indicated. Unpublished data at ISR energies seem to be in rough agreement with our prediction.

c) The multiplicity distribution of charged particles

These are shown in Fig. 2 for pp [7] and in Fig. 3 for π^+p [9].

The dotted line in Fig. 2 corresponds to another version of our model which will be discussed in Section 4. The KNO function

$$\psi(z) = \lim_{\bar{n}_c \to \infty} \bar{n}_c \sigma(n_c)/\sigma_{\text{inel}}(\bar{n}_c)$$

with $z = \dfrac{n_c}{\bar{n}_c}$ fixed is calculated for π^+p and shown in Fig. 4.

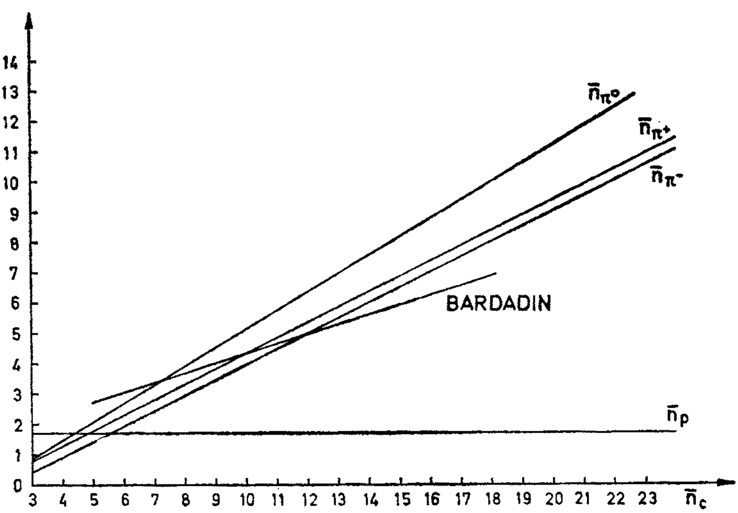

Fig. 1. Average multiplicities for pp-scattering as a function of the average number of charged particles. Also shown is the average π^0-multiplicity as measured by Bardadin et al. [4]

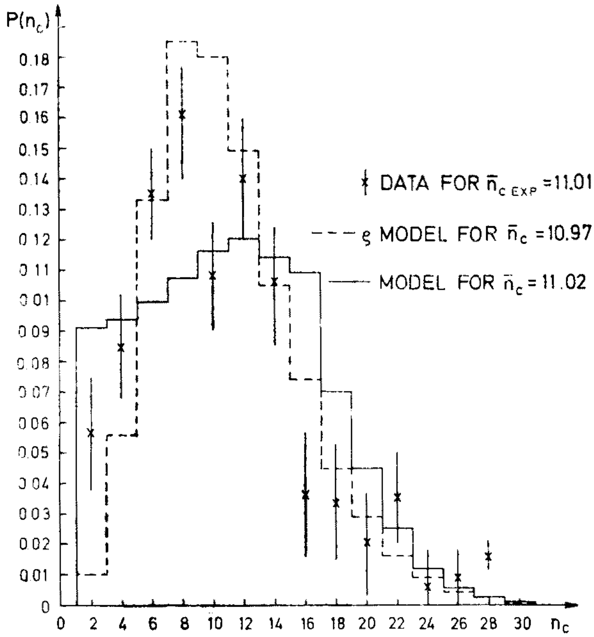

Fig. 2. Multiplicity distribution of charged particles for pp-scattering. The dotted line is our result for the ϱ-model. Data points from Ref. [7]

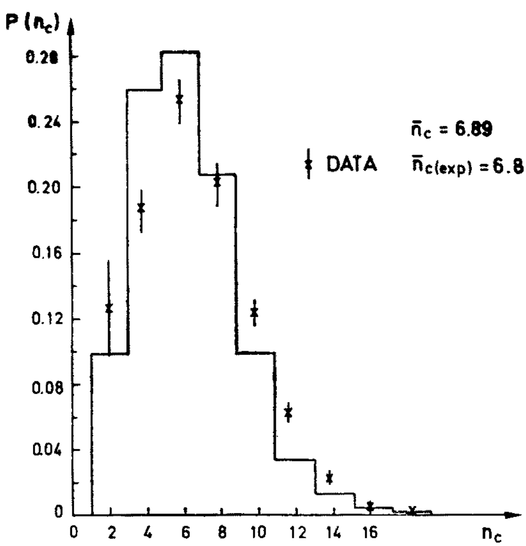

Fig. 3. Multiplicity distribution of charged particles for π^+p-scattering. Data points from Ref. [9]

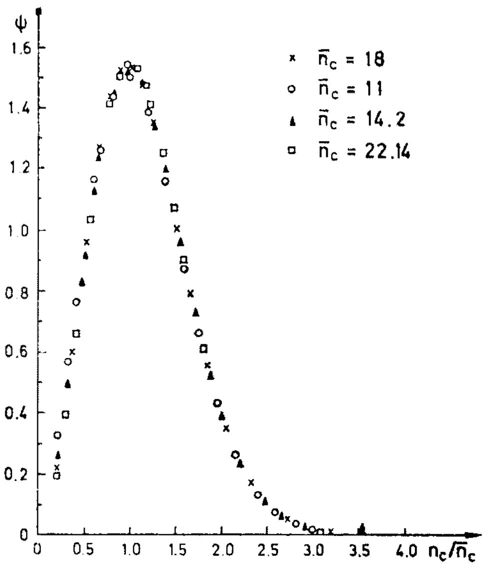

Fig. 4. KNO-function for π^+p-scattering

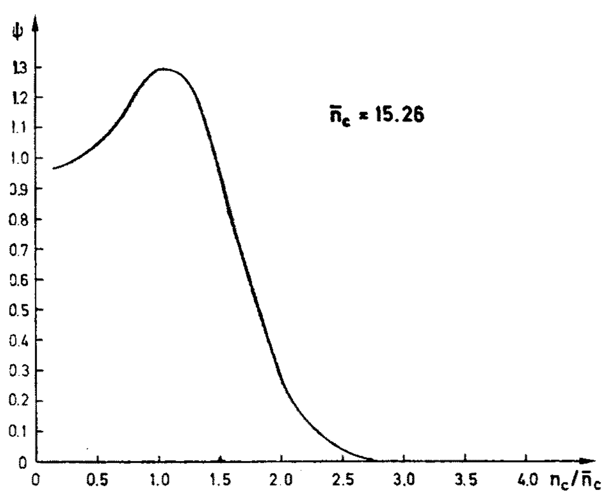

Fig. 5. KNO-function for pp-scattering

For pp plotted in Fig. 5 for only one energy corresponding to $\bar{n}_c = 15.2$. Only for low multiplicities it is still changing a little bit with increasing energy.

A way to describe the change in the multiplicity distribution with increasing energy is to give the central moments

$$D_k = [\overline{(n_c - \bar{n}_c)^k}]^{1/k} \tag{30}$$

as a function of $\bar{n}_c$.

For a distribution which shows KNO scaling these moments should increase linearly with $\bar{n}_c$.

In Fig. 6 we therefore give $\bar{n}_c/D_k$ ($k = 2, \dots 8$) for the pp-case.

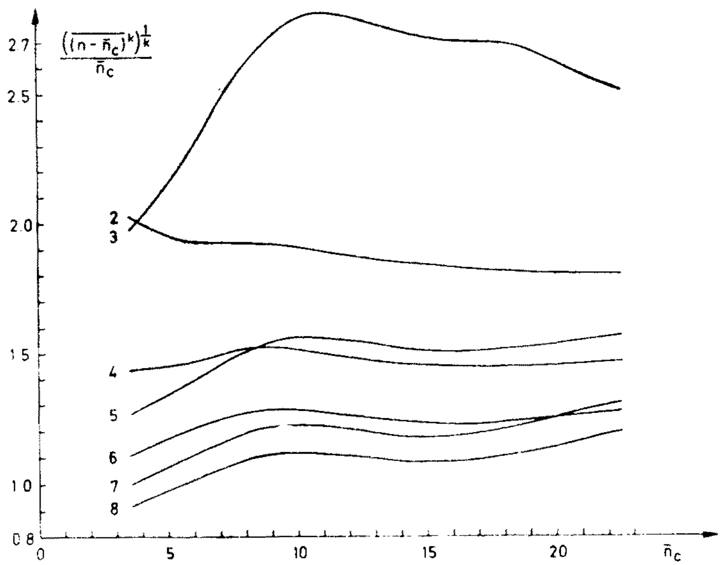

Fig. 6. Moments of distribution of charged particles for pp-scattering

A more sensitive measure of the form of the distribution is given by the moments

$$\gamma_1 = \left(\frac{D_3}{D_2}\right)^3$$

and

$$\gamma_2 = \left(\frac{D_4}{D_2}\right)^4.$$

For π^+p and π^-p we have plotted γ_1 in Fig. 7 and γ_2 in Fig. 8.

d) Correlations

The correlation function $f_{cc} = \overline{n_c(n_c - 1)} - \bar{n}_c^2$ gives no information which could not already be obtained from the dispersion. It is used, however, to measure the deviation from uncorrelated production. It agrees well with the experimental data.

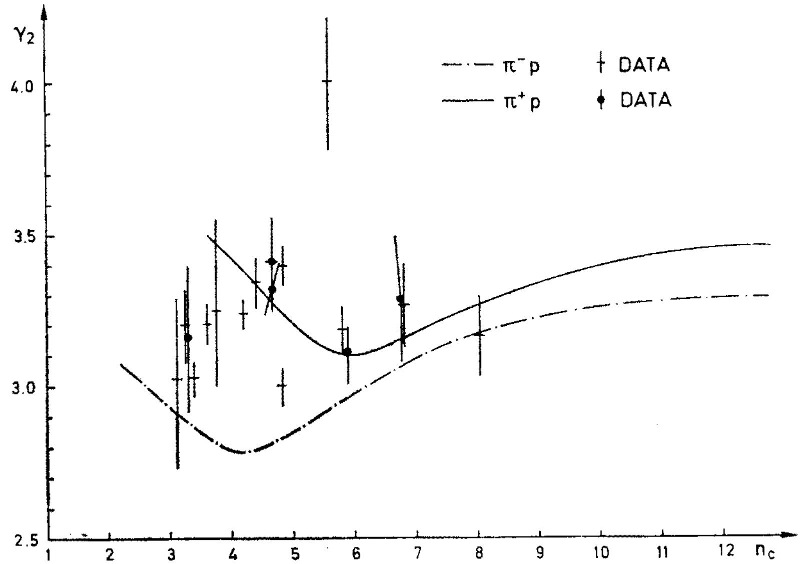

Fig. 7. Normalized third central moment of the distribution of charged particles

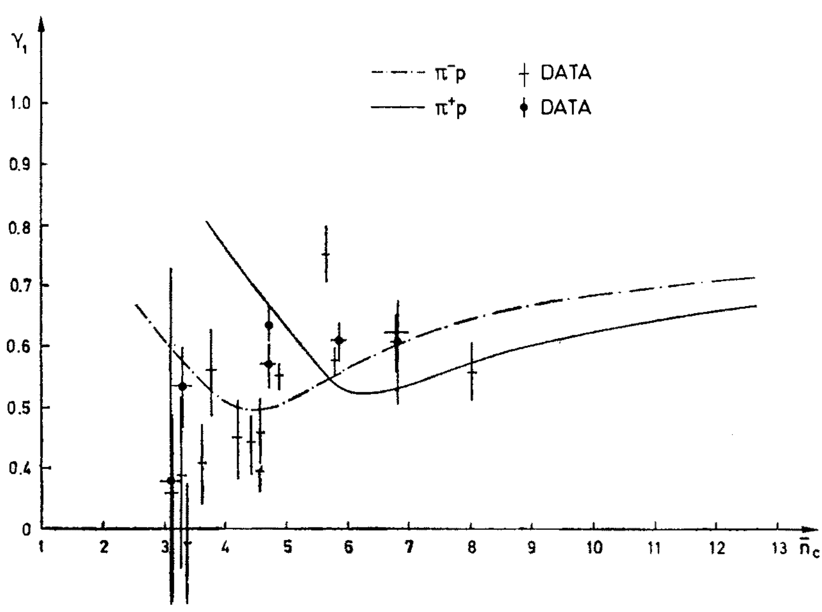

Fig. 8. Normalized fourth central moment of the distribution of charged particles

Also shown is f_{00}, as calculated from the distribution of neutral pions.

We do not show the correlation between charged particles and neutral pions, that is $f_{c0} = \overline{n_c n_0} - \bar{n}_c \bar{n}_0$.

It is negative for all cases, whereas experimentally it is positive, at least for high energies. This negative charge-neutral correlation is known to be a general property of theories in which cluster formation is not taken into account [8]. It is also seen in another way

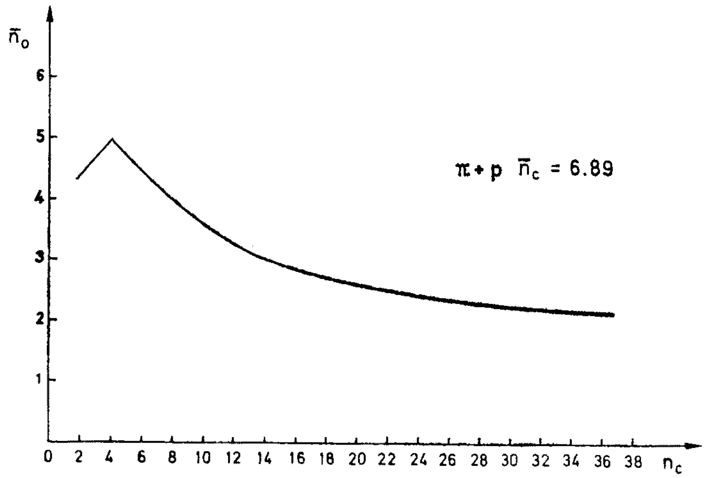

Fig. 9. The average number of neutral pions as a function of the number of charged particles

by calculating the average number of neutral pions for a given number of charged particles. Experimentally this is an increasing function of n_c, whereas we find a decreasing function, as is shown in Fig. 9 for the case of π^+p. In the next section, however, we shall show how this defect disappears in another version of our model.

4. *Discussion*

Since isospin conservation puts a very strong constraint on all multiplicity distributions we considered it worthwhile to decompose them into the contributions from the different isospins. For the pp-case this leads to the results shown in Figs 10, 11, and 12. From these pictures it can be seen that the component for which the isospin of all mesons is equal to two can be neglected completely. The $I = 0$ and $I = 1$ components are of comparable magnitude and are distinguished by an even and an odd number of neutral pions respectively. It is also seen that the form of the distributions is very different for $I = 0$ and $I = 1$ and it would be very interesting to have an experimental verification of this prediction.

Another remark we want to make is about the completely different shape of the distribution for the total number of pions and of the distribution of neutral pions. In the projection method as used by Bardadin et al. [4] one obtains charge- and neutral-distributions of the same form as the total distribution one starts from. Our calculations show

how misleading this can be. However, we do not want to hide the fact that our neutral--distribution of Fig. 11 does not have the shape as found experimentally [7]. In order to improve this situation and also to bring the charge-neutral correlations closer to the experimental value we have considered a model in which the calculations are exactly

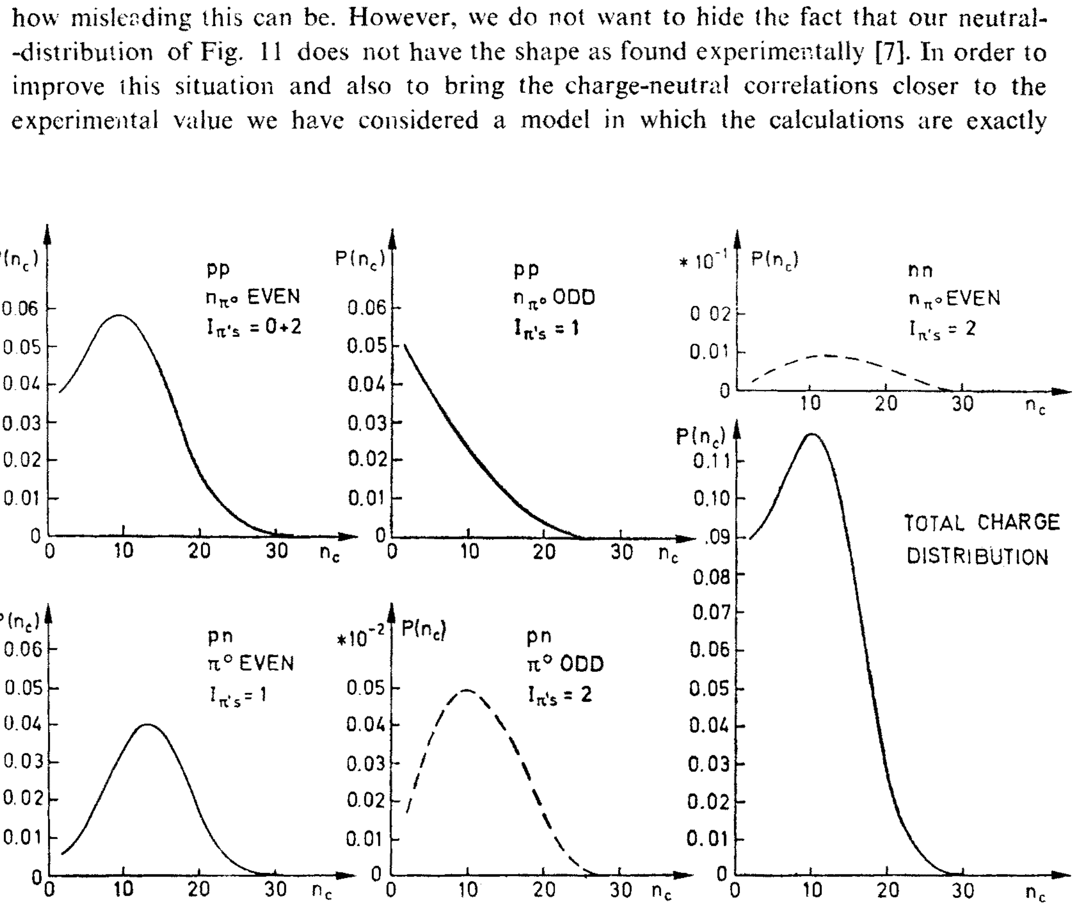

Fig. 10. Break-up of the distribution of charged particles into contributions from different values of the total mesonic isospin

the same as in our present model, but in which the produced particles are ϱ-mesons instead of pions. The pion-distributions and correlations can then be obtained from those of the ϱ-mesons by taking the decays $\varrho^{\pm} \to \pi^{\pm}\pi^0$ and $\varrho^0 \to \pi^+\pi^-$. From this follows $n_{\pi^0} = n_{\varrho^+} + n_{\varrho^-}$ and

$$n_{\pi^+} + n_{\pi^-} = n_c = n_{\varrho^+} + n_{\varrho^-} + 2n_{\varrho^0}.$$

We therefore see that the π^0-distribution in this ϱ-model is the same as the charge--distribution in the original model, which has a form which is completely different from the distribution shown in Fig. 11. The new charge-distribution must be calculated from the combined distribution of neutral and charged pions in the original model. This has been done and the result is shown as the dashed line in Fig. 2. It is clear that also in this charge-distribution there is an improvement. The correlation f_{c0} between charged and neutral pions was again determined and found to be positive and increasing with increasing energy.

These calculations for the ϱ-model could be done easily because the difficult part was the same as for the original π-model. If, however, we want to include both ϱ and π

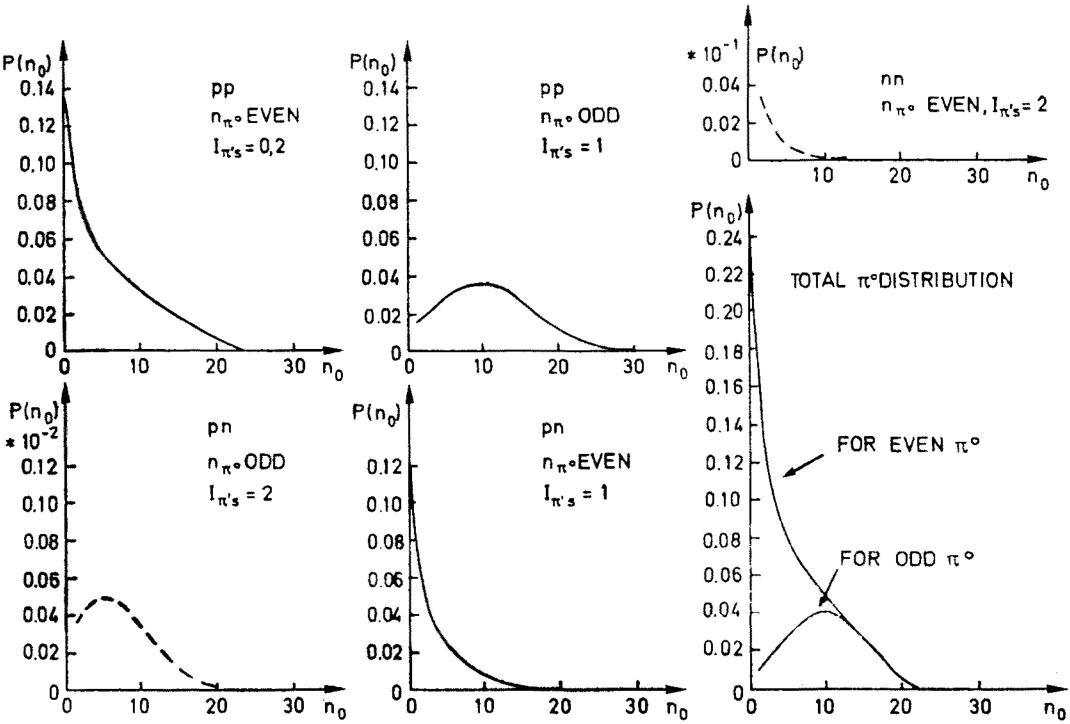

Fig. 11. Break-up of the distribution of neutral pions into contributions from different values of the total mesonic isospin

production and, perhaps, also a scalar σ-meson, we should repeat our calculations, but now for a Hamiltonian which has the form

$$H = m_\pi \vec{a}^* \cdot \vec{a} + g_\pi(\vec{a}+\vec{a}^*) \cdot \vec{\tau} + m_\varrho \vec{b}^* \cdot \vec{b} + g_\varrho(\vec{b}+\vec{b}^*) \cdot \vec{\tau} + m_\sigma c^* c + g_\sigma(c+c^*),$$

where $\vec{a}^*$, $\vec{b}^*$ and c^* create pions, ϱ-mesons and σ-mesons respectively. The calculation will be rather tedious, but hopefully worthwhile, because it will enable us to interpret the data in terms of production of some resonances.

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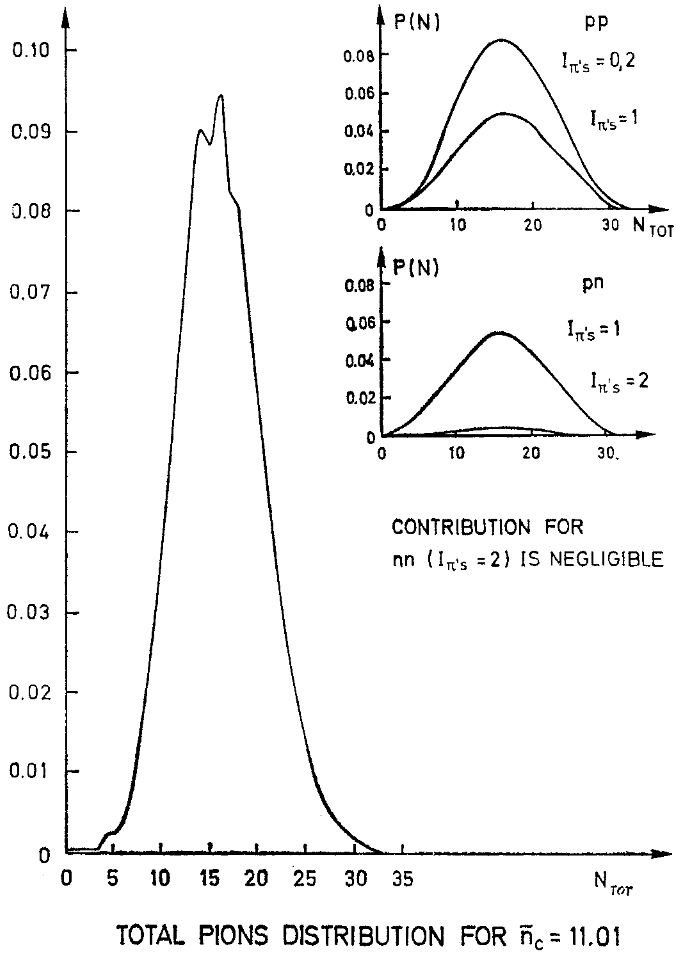

Fig. 12. Break-up of the total pions distribution into contributions from different values of the total mesonic isospin

APPENDIX

In this appendix we would like to collect a number of formulae which are important in reducing the eigen-value problem to matrix form. We will not give the proofs but for some of them the proofs can be found in the appendix of Ref. [5].

We first define the tensor operator T_k with $q = 0, \pm 1, \ldots, \pm k$ by

$$T_k^k = \frac{a_+^{*k}}{\sqrt{k!}} \tag{A1}$$

and its commutator with the components of the isospin:

$$[I_3, T_k^q] = qT_k^q,$$

$$[I_+, T_k^q] = \sqrt{(k-q)(k+q+1)}\, T_k^{q+1},$$

$$[I_-, T_k^q] = \sqrt{(k+q)(k-q+1)}\, T_k^{q-1}. \tag{A2}$$

The scalar operator which creates pairs of mesons is

$$G^* = \sum_{j=1}^{3} a_j^* a_j^*. \tag{A3}$$

For the above-mentioned reduction we now need the following formulae

$$\langle 0|G^n T_k^{q*} T_k^q G^{*n}|0\rangle = \frac{k!n!(2n+2k+1)!}{(2k+1)!(n+k)!}, \tag{A4}$$

$$a_0 T_k^k G^{*n}|0\rangle = -2n T_{k+1}^k G^{*n-1}|0\rangle, \tag{A5}$$

$$a_0 T_k^{k-1} G^{*n}|0\rangle = -\frac{2n+2k+1}{2k+1} T_{k-1}^{k-1} G^{*n}|0\rangle - 4n\sqrt{\frac{k}{2k+1}}\, T_{k+1}^{k-1} G^{*n-1}|0\rangle, \tag{A6}$$

$$a_0 T_k^{k-2} G^{*n}|0\rangle = -\frac{2}{2k+1}\sqrt{\frac{k-1}{2k-1}}(2n+2k+1) T_{k-1}^{k-2} G^{*n}|0\rangle$$

$$-2\sqrt{3}\sqrt{\frac{2k-1}{2k+1}}\, n T_{k+1}^{k-2} G^{*n-1}|0\rangle, \tag{A7}$$

$$a_0 T_k^{k-3} G^{*n}|0\rangle = -\frac{\sqrt{3}}{2k+1}\sqrt{\frac{2k-3}{2k-1}}(2n+2k+1) T_{k-1}^{k-3} G^{*n}|0\rangle$$

$$-4\sqrt{\frac{2k-2}{2k+1}}\, n T_{k+1}^{k-3} G^{*n-1}|0\rangle, \tag{A8}$$

$$a_+ T_k^k G^{*n}|0\rangle = \frac{\sqrt{k}}{2k+1}(2n+2k+1) T_{k-1}^{k-1} G^{*n}|0\rangle - \frac{2n}{\sqrt{2k+1}} T_{k+1}^{k-1} G^{*n-1}|0\rangle, \tag{A9}$$

$$a_+ T_k^{k-1} G^{*n}|0\rangle = \frac{\sqrt{k-1}}{2k+1}(2n+2k+1) T_{k-1}^{k-2} G^{*n}|0\rangle - \frac{2n\sqrt{3}}{\sqrt{2k+1}} T_{k+1}^{k-2} G^{*n-1}|0\rangle, \tag{A10}$$

$$a_+ T_k^{k-2} G^{*n}|0\rangle = \sqrt{\frac{(k-1)(2k-3)}{(2k-1)}}\,\frac{2n+2k+1}{2k+1} T_{k-1}^{k-3} G^{*n}|0\rangle$$

$$-\frac{2n\sqrt{6}}{\sqrt{2k+1}} T_{k+1}^{k-3} G^{*n-1}|0\rangle, \tag{A11}$$

$$a_+ T_k^{k-3} G^{*n}|0\rangle = \sqrt{\frac{(k-2)(2k-3)}{(2k-1)}}\,\frac{2n+2k+1}{2k+1} T_{k-1}^{k-4} G^{*n}|0\rangle$$

$$-\frac{2n\sqrt{10}}{\sqrt{2k+1}} T_{k+1}^{k-4} G^{*n-1}|0\rangle, \tag{A12}$$

$$a_- T_k^k G^{*n}|0\rangle = 2n\sqrt{k+1}\, T_{k+1}^{k+1} G^{*n-1}|0\rangle, \tag{A13}$$

$$a_- T_k^{k-1} G^{*n}|0\rangle = 2n\sqrt{k}\, T_{k+1}^{k} G^{*n-1}|0\rangle, \tag{A14}$$

$$a_- T_k^{k-2} G^{*n}|0\rangle = -\frac{2n+2k+1}{(2k+1)\sqrt{2k-1}} T_{k-1}^{k-1} G^{*n}|0\rangle$$

$$+2n\sqrt{\frac{k(2k-1)}{2k+1}}\, T_{k+1}^{k-1} G^{*n-1}|0\rangle, \tag{A15}$$

$$a_- T_k^{k-3} G^{*n}|0\rangle = -\frac{\sqrt{3}\,(2n+2k+1)}{(2k+1)\sqrt{2k-1}} T_{k-1}^{k-2} G^{*n}|0\rangle$$

$$+2n\sqrt{\frac{(k-1)(2k-1)}{2k+1}}\, T_{k+1}^{k-2} G^{*n-1}|0\rangle. \tag{A16}$$

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