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SOME EXACT SOLUTIONS OF CHARGED GENERAL RELATIVISTIC FLUID SPHERES

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The note presents some exact, static spherically-symmetric solutions of the Einstein-Maxwell equations. The Kuchowicz solutions are shown to be particular members of these.

1. Introduction

In a recent series of papers, Kuchowicz (1968a, 1968b, 1970) has discussed the exact solutions of the gravitational field equations for uncharged spherically-symmetric matter distributions. He also collected an impressive number of model solutions which may be of importance in astrophysical research. In this note we obtain, for the special case in which $\exp(-\lambda)$ is a constant, some exact solutions for the charged spherically-symmetric matter distributions and demonstrate that they reduce to the Kuchowicz case in the absence of charges.

2. Solutions of the field equations

The Einstein-Maxwell equations for the charged fluid are (Adler et al. 1965)

$$G_{\mu\nu} = -8\pi T_{\mu\nu}, \tag{2.1}$$

$$[(-g)^{1/2}F^{\mu\nu}]_{,\nu} = 4\pi(-g)^{1/2}J^{\mu}, \tag{2.2}$$

$$F_{[\mu\nu,\gamma]} = 0. \tag{2.3}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor, $F^{\mu\nu}$ is the electromagnetic field tensor and J^{μ} is the current four-vector (we set c and the gravitational constant equal to unity). For a static spherically-symmetric system an appropriate metric is

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.4}$$

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It then follows that the field equations may be written in the form (Nduka 1975, Adler et al. 1965)

$$8\pi\varrho+8\pi E_0{}^0 = e^{-\lambda}\left[\frac{\lambda'}{r}-\frac{1}{r^2}\right]+\frac{1}{r^2}, \tag{2.5}$$

$$8\pi p-8\pi E_1{}^1 = e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}, \tag{2.6}$$

$$8\pi p-8\pi E_2{}^2 = e^{-\lambda}\left[\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\lambda'\nu'}{4}+\frac{\nu'-\lambda'}{2r}\right], \tag{2.7}$$

where a prime denotes differentiation with respect to r, and $E_0{}^0 = E_1{}^1 = -E_2{}^2 = -\frac{1}{8\pi} g_{00}g_{11}(F^{01})^2$. Because of the spherical symmetry, only the radial electric field $F^{10} = -F^{01}$ is non-vanishing. This choice satisfies Eq. (2.3), while Eq. (2.2) yields

$$F^{01} = e^{-\alpha}Q(r)/r^2, \quad \alpha = (\lambda+\nu)/2, \tag{2.8}$$

where $Q(r)$ is the charge up to the radius r.

$$Q(r) = 4\pi\int_0^r J^0 r^2 e^{\alpha} dr. \tag{2.9}$$

On putting $y = e^{\nu/2}$ and eliminating p from Eq. (2.7), using Eq. (2.6) we obtain the second-order differential equation for $y(r)$:

$$y''-\left(\frac{1}{r}+\frac{\lambda'}{2}\right)y'+\left(\frac{e^{\lambda}}{r^2}-\frac{\lambda'}{2r}-\frac{1}{r^2}-\frac{2Q^2}{r^4}e^{\lambda}\right)y = 0. \tag{2.10}$$

Eq. (2.10) is a generalization of Wyman's equation (Wyman 1949).

To obatin solutions of physical significance, the following boundary conditions have been imposed:

1) The functions $e^{-\lambda}$ and e^{ν} are continuous across the boundary ($r \equiv r_0$) of the fluid sphere.

2) the function $\nu' e^{\nu}$ is continuous across $r = r_0$. The line element for $r > r_0$ is given by the Reissner-Nordström metric:

$$ds^2 = (1-2M/r+Q_0^2/r^2)dt^2-(1-2M/r+Q_0^2/r^2)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{2.11}$$

where $Q_0 = Q(r_0)$ and M is the total mass of the sphere.

3. The model solution

For our model solution we take the charge distribution $Q(r)$ to be of the form $Q(r) = (\varepsilon x/\sqrt{2})\exp(-\lambda/2)$. Here $x = r/r_0$ and ε is a constant. This choice for $Q(r)$ is physically reasonable because it gives $F^{01} = (\varepsilon x/r^2\sqrt{2})\exp[-(\lambda+\nu/2)]$. The boundary condition that the geometry be euclidean at infinity implies that both v and λ go to zero as $r \to \infty$, so that the expression for F^{01} has the usual classical form, at least for large r. Thus the constant ε can be identified as the proportionality constant giving the charge at the point r. In this paper we take $e^{-\lambda} = A$, a constant. The continuity of $e^{-\lambda}$ at $r = r_0$ yields

$$A = (1-\delta)/(1-\sigma/2), \quad \delta = 2M/r_0, \; \sigma = \varepsilon^2/r_0{}^2. \tag{3.1}$$

The differential Eq. (2.10) now becomes

$$r^2 y'' + a_1 r y' + a_2 y = 0, \tag{3.2}$$

where $a_1 = -1$, $a_2 = [1/A - (1+\sigma)]$. Eq. (3.2) may be recognized as Euler's equation. On putting $y = u(z)$, $z = \log r$, this equation is transformed into one with constant coefficients

$$u''(z) + (a_1 - 1)u'(z) + a_2 u(z) = 0, \tag{3.3}$$

with characteristic roots

$$\alpha_1 = 1+\gamma, \quad \alpha_2 = 1-\gamma. \tag{3.4}$$

where

$$\gamma = \sqrt{\frac{1-2\delta+\sigma(\frac{3}{2}-\delta)}{1-\delta}}.$$

The solutions to Eq. (3.2) may now be written down — and hence obtain the metric coefficient $e^{\nu(r)}$, the density $\varrho(r)$ and the pressure $p(r)$. There are three possible cases:

(a) $\alpha_1 \neq \alpha_2$, but both real: $\delta < (1+\frac{3}{2}\sigma)/(\sigma+2)$,

$$e^{\nu} = [C_1 r^{\alpha_1} + C_2 r^{\alpha_2}]^2 \tag{3.5}$$

where C_1 and C_2 are obtained from the boundary conditions:

$$C_1 = \frac{1}{r_0^{\alpha_1}\sqrt{1-\sigma/2}} \left\{ \frac{\sqrt{1-\delta}}{2} + \frac{(3\delta-2)-\sigma(1-\delta/2)}{4\sqrt{1-2\delta+\sigma(\frac{3}{2}-\delta)}} \right\},$$

$$C_2 = \frac{1}{r_0^{\alpha_2}\sqrt{1-\sigma/2}} \left\{ \frac{\sqrt{1-\delta}}{2} - \frac{(3\delta-2)-\sigma(1-\delta/2)}{4\sqrt{1-2\delta+\sigma(\frac{3}{2}-\delta)}} \right\}, \tag{3.6}$$

$$8\pi\varrho = \frac{1}{(1-\sigma/2)r^2} \{(\delta-\sigma/2) - \sigma(1-\delta)(1-x^2/2)\}, \tag{3.7}$$

$$8\pi p = \frac{(X_2^2 - X_1^2)}{4(1-\sigma/2)\sqrt{1-2\delta+\sigma(\frac{3}{2}-\delta)}\, r^2}$$

$$\times \frac{(1-x^{2\gamma})}{\frac{\sqrt{1-\delta}}{2} - \frac{(3\delta-2)-\sigma(1-\delta/2)}{4\sqrt{(1-2\delta)+\sigma(\frac{3}{2}-\delta)}} + \left[\frac{\sqrt{1-\delta}}{2} + \frac{(3\delta-2)-\sigma(1-\delta/2)}{4\sqrt{(1-2\delta)+\sigma(\frac{3}{2}-\delta)}}\right] x^{2\gamma}}$$

$$+ \frac{(1-\delta)\sigma}{2(1-\sigma/2)r^2}(1-x^2), \tag{3.8}$$

where

$$X_1 = 2\sqrt{(1-\delta)\left[1-2\delta+\sigma(\tfrac{3}{2}-\delta)\right]},$$

$$X_2 = (3\delta-2)-\sigma(1-\delta/2).$$

(b) $\alpha_1 = \alpha_2$

The solutions are

$$e^{\nu} = [C'_1 r + C'_2 r \log r]^2, \tag{3.9}$$

with

$$C'_1 = \frac{1}{2r_0\sqrt{(\sigma+2)}}\{2+(1-\sigma/2)\log r_0\},$$

$$C'_2 = -\frac{(1-\sigma/2)}{2r_0\sqrt{(\sigma+2)}}.$$

$$8\pi\varrho = \frac{1+\dfrac{\sigma}{2}x^2}{2(1+\sigma/2)r^2}, \tag{3.10}$$

$$8\pi p = \frac{1}{(\sigma+2)r^2}\left\{\frac{\sigma-(1-\sigma/2)\log x}{2-(1-\sigma/2)\log x} - \frac{\sigma}{2}x^2\right\}. \tag{3.11}$$

(c) $\alpha_1 \neq \alpha_2$, but both complex. In this case $(1+\frac{3}{2}\sigma)/(\sigma+2) < \delta < 1$, and $\gamma = \sqrt{[(2\delta-1)-\sigma(\frac{3}{2}-\delta)]/(1-\delta)}$. It is convenient here to give the formulas in terms of dimensionless quantities. The solutions are

$$e^{\nu} = x^2[C''_1 \cos(\gamma\log x) + C''_2 \sin(\gamma \log x)]^2, \tag{3.12}$$

where

$$C''_1 = \sqrt{(1-\delta)/(1-\sigma/2)}, \qquad C''_2 = \frac{(3\delta-2)-\sigma(1-\delta/2)}{2(1-\sigma/2)\left[2\delta-1-\sigma(\frac{3}{2}-\delta)\right]},$$

$$8\pi\varrho = \frac{1}{(1-\sigma/2)x^2}\left[(\delta-\sigma/2)-\sigma(1-\delta)(1-x^2/2)\right] \tag{3.13}$$

and

$$8\pi p = \frac{1}{(1-\sigma/2)x^2}\left\{\frac{2(1-\delta)\left[\cos(\gamma\log x)(C''_1+\gamma C''_2)+\sin(\gamma\log x)(C''_2-\gamma C''_1)\right]}{\left[C''_1\cos(\gamma\log x)+C''_2\sin(\gamma\log x)\right]} + (\sigma/2-\delta)+\sigma(1-\delta)-\sigma/2(1-\delta)x^2\right\}. \tag{3.14}$$

In Eq. (3.14) we note that for certain choices of σ and δ the pressure becomes negative. This is unphysical. We therefore restrict Eq. (3.14) to apply to those values of σ and δ for which the pressure is positive. In equetions (3.7) to (3.14) the point $r \equiv 0$ is to be excluded ($r \neq 0$), because our solution is not regular at the origin.

If in Eqs. (3.5) (3.14) we set $\sigma \equiv 0$, the results coincide with those already reported by Kuchowicz (1968a). Thus our equations may be considered as the generalizations of those obtained by Kuchowicz.

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