JUNCTION CONDITIONS FOR THE EINSTEIN-CARTAN THEORY

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Junction conditions for the Einstein-Cartan theory are calculated from the geometry on the two sides of the hypersurface of discontinuity. Formulae in Gaussian normal coordinates are obtained for the Einstein tensor. Examples of the junction conditions are given for static spherically symmetric stars, including a constant mass and spin density Schwarzschild-like interior solution.

1. *Introduction*

Considerable interest has developed recently in the Einstein-Cartan theory of gravitation [1] in which spin and mass play equally fundamental roles in determining the geometry of space-time. An important practical part of this field theory is the set of conditions required in order to join solutions across a surface of matter and spin discontinuity such as the surface of a star bounding matter on one side and vacuum on the other.

Junction conditions for the Einstein-Cartan theory have been derived by Arkuszewski, Kopczyński and Ponomariev [4] by examining the Einstein tensor for derivatives normal to the hypersurface (in order to eliminate any δ-function behaviour) and by [5] using distribution theory to integrate the energy-momentum and spin conservation laws across the singular hypersurface. The junction conditions are derived below without the use of distribution theory by using the Ricci commutation relations [6, 7] to obtain formulae in terms of Gaussian normal coordinates for the curvature tensors on each side of the hypersurface of discontinuity. Evaluation of the corresponding Einstein tensor displays explicitly those dynamical variables that cannot experience a discontinuity at a hypersurface where the energy-momentum and spin tensors are finite, although possibly discontinuous, and yields directly the jumps in those components of the Einstein tensor needed to match solutions across the hypersurface.

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The relevant equations of the Einstein-Cartan theory are presented in Sec. 2. The continuity properties of the space-time manifold and some differential geometry are discussed in Sec. 3, with particular attention being paid to the properties of the hypersurface of discontinuity. The junction conditions are derived in Sec. 4 and this derivation is compared with that of Arkuszewski, Kopczyński and Ponomariev [4]. Finally, some applications of the junction conditions are presented in Sec. 5.

2. *The Einstein-Cartan theory*

The geometry of the Einstein-Cartan theory is characterized by a metric g_{ij} [8] and a metric connection $\Gamma^i{}_{jk}$ that is symmetric in a coordinate basis only in the absence of spin and otherwise involves the torsion

$$Q^i{}_{jk} = \Gamma^i{}_{kj} - \Gamma^i{}_{jk}. \tag{2.1}$$

This connection $\Gamma^i{}_{jk}$ differs from a metric torsion-free connection $\tilde{\Gamma}^i{}_{jk}$, the Christoffel symbol $\left\{ {i \atop jk} \right\}$ in a coordinate basis, by a tensor $\kappa^i{}_{jk}$ called either (the negative of) the contortion [2] or the defect of the connection [4]:

$$\Gamma^i{}_{jk} = \tilde{\Gamma}^i{}_{jk} + \kappa^i{}_{jk}. \tag{2.2}$$

The defect of the connection is related to the torsion by

$$\kappa^i{}_{jk} = -\tfrac{1}{2}(Q^i{}_{jk} + Q_{jk}{}^i + Q_{kj}{}^i). \tag{2.3}$$

The curvature tensor R_{ijkl} in the presence of torsion is not symmetric under interchange of the first and last pairs of indices, but it remains skew in the first two indices, because the connection is metric, and in the last two. Thus the Einstein tensor

$$G^i{}_j = R^{ik}{}_{jk} - \tfrac{1}{2}\delta^i_j R^{mn}{}_{mn} = R^i{}_j - \tfrac{1}{2}\delta^i_j R \tag{2.4}$$

is asymmetric.

The field equations are those given first by Kibble [9]:

$$G^i{}_j = \frac{8\pi G}{c^4} t^i{}_j \tag{2.5}$$

and

$$Q^k{}_{ij} - \delta^k_i Q^l{}_{lj} - \delta^k_j Q^l{}_{il} = \frac{8\pi G}{c^3} S^k{}_{ij}, \tag{2.6}$$

where $t^i{}_j$ and $S^k{}_{ij}$ are the canonical energy-momentum and spin tensors respectively. The relation between the torsion and the spin is algebraic and the torsion vanishes in the absence of spin, reducing the equations of the theory in spin-free regions to those of general relativity. The solution of (2.6) for the spin is

$$Q^k{}_{ij} = \frac{8\pi G}{c^3}(S^k{}_{ij} - \tfrac{1}{2}\delta^k_i S^l{}_{lj} - \tfrac{1}{2}\delta^k_j S^l{}_{il}). \tag{2.7}$$

The geometric sides of the field equations satisfy a set differential identities [3], the generalized Bianchi identities

$$\nabla_i G^i{}_j = Q^k{}_{jl} G^l{}_k - G^i{}_j Q^l{}_{il} - \tfrac{1}{2} R^{kl}{}_{jm} Q^m{}_{kl} + R^{kl}{}_{jk} Q^n{}_{nl}, \tag{2.8}$$

$$\nabla_m Q^m{}_{kl} + 2\nabla_{[k} Q^m{}_{l]m} = -Q^j{}_{mj} Q^m{}_{kl} + G_{lk} - G_{kl}. \tag{2.9}$$

These identities together with the field equations, but without any assumption relating the energy-momentum tensor to a Lagrangian, yield equations

$$\nabla_i t^i{}_j = Q^l{}_{li} t^i{}_j + Q^k{}_{jl} t^l{}_k - \frac{c}{2} R^{kl}{}_{jm} S^m{}_{kl} \tag{2.10}$$

and

$$\nabla_m S^m{}_{kl} = Q^j{}_{jm} S^m{}_{kl} + \frac{1}{c}(t_{lk} - t_{kl}) \tag{2.11}$$

that express the conservation of energy-momentum and spin or, alternatively, that describe their time development. Thus, these equations are called conservation laws or equations of motion.

3. *Differential geometry*

The problem of junction conditions is to match two 4-manifolds-with-boundary V^1 and V^2 across a common 3-boundary Σ to form a 4-manifold V as a model of space-time compatible with the physical theory [10]. The field equations involve no higher than second derivatives of tensors but the generalized Bianchi identities, which yield the equations of motion or the conservation laws, involve also third derivatives of tensors, and thus fourth derivatives of the coordinate transformation functions in their transformation laws. Therefore, in order to avoid the appearance outside Σ of any discontinuous behaviour whatsoever, we assume that each of V^1 and V^2 is a differentiable manifold of class C^4 [11]. A discontinuity at Σ of the second derivatives of the coordinate transformation functions would complicate the interpretation of the physical significance of the equations of motion as well as allow the introduction of spurious coordinate singularities into the connection coefficients that, together with their first derivatives, appear in the Einstein tensor. Therefore, we assume that V is a differentiable manifold of class C^2. These conditions ensure that the third and fourth derivatives of the coordinate transformation functions have definite limits as Σ is approached from either side.

The model of the space-time V thus described is called a differentiable manifold of class C^2 and piecewise of class C^4 or a piecewise differentiable manifold of class (C^2, C^4) [12]. Coordinate systems belonging to an atlas of class (C^2, C^4) are called admissible [11].

Third derivatives of the metric tensor appear in the generalized Bianchi identities so each of the metric tensors in V^1 and V^2 is assumed to be of class C^3. However, first derivatives of the metric tensor appear in the field equations in linear combination with components of the spin tensor; thus, if the energy-momentum tensor remains finite, a discontinuity in the spin tensor must be matched by corresponding discontinuities in some first derivatives of the metric tensor. Therefore, the metric tensor is assumed to be of class C^0 in V. The metric tensor is piecewise of class (C^0, C^3).

Finally, to avoid discontinuous behaviour along the hypersurface Σ, we assume that Σ is a differentiable 3-manifold of class C^4 with metrics induced by V^1 and V^2 of class C^3 and signature $(--+)$ and that on Σ each coordinate in an admissible coordinate system (x^i) of V^1 and V^2 is a function of class C^4 of admissible coordinates ξ^α of Σ:

$$x^i|_\Sigma = x^i(\xi^\alpha). \tag{3.1}$$

For the remainder of this section, unless the contrary is stated explicitly, we consider only one "side", either V^1 or V^2 and labelled V^A, of V.

The transformation coefficients $e_\alpha{}^j = \partial x^j/\partial\xi^\alpha$ define a natural basis in Σ of three linearly independent vectors tangent to Σ in V. The metric tensor induced by V^A is

$$g_{\alpha\beta} = g_{ij} e_\alpha{}^i e_\beta{}^j \tag{3.2}$$

and this metric and its inverse may be used respectively to lower and raise Greek indices.

A unit space-like normal N_i to Σ in V^A can be defined up to a sign as the solution of the equations

$$N_i e_\alpha{}^i = 0 \quad \text{and} \quad g^{ij}|_\Sigma N_i N_j = -1. \tag{3.3}$$

Choosing normals out of V^1 and into V^2 respectively ensures that the components of the two normals in V^1 and V^2 are identical.

For some calculations, it is convenient to use a normal Gaussian coordinate system [13] $\bar{x}^i = (\xi^\alpha, x)$ around Σ in V consisting of a) the coordinate x defined to be the directed distance along geodesics in V starting at each point (ξ^α) of Σ with the normal $N_i(\xi^\alpha)$ as initial direction in V^A and b) the intrinsic coordinates (ξ^α) of the starting point in Σ. Since the geodesic equations

$$\frac{d^2x^i}{dx^2} + \Gamma^i{}_{jk}\frac{dx^j}{dx}\frac{dx^k}{dx} = 0 \tag{3.4}$$

have solutions

$$x^i = x^i(\xi^\alpha) + xN^i(\xi^\alpha) - \tfrac{1}{2}x^2[\Gamma^i{}_{jk}N^jN^k](\xi^\alpha) + \ldots, \tag{3.5}$$

the vector

$$n^i = \frac{\partial x^i}{\partial\bar{x}^j}\bar{n}^j = \frac{\partial x^i}{\partial x} \tag{3.6}$$

is continuous across Σ with $n^i|_\Sigma = N^i$. Furthermore, direct calculation shows that $\partial^2 x^i/\partial x\,\partial\xi^\alpha|_\Sigma = \partial^2 x^i/\partial\xi^\alpha\partial x|_\Sigma$ so

$$\left.\frac{\partial e_\alpha{}^i}{\partial x}\right|_\Sigma = \frac{\partial N^i}{\partial\xi^\alpha}. \tag{3.7}$$

The metric of V^A is related to the intrinsic metric of Σ by

$$g^{ij} = g^{\alpha\beta}e_\alpha{}^i e_\beta{}^j - n^i n^j. \tag{3.8}$$

Components, with respect to the normal Gaussian coordinate basis, along tangential directions may be labelled by Greek letters and those along the normal direction by N so, for instance,

$$T^i{}_j n_i e_\alpha{}^j = T^N{}_\alpha = T_{N\alpha} \quad \text{and} \quad T^i{}_i = T^\alpha{}_\alpha - T^N{}_N. \tag{3.9}$$

An induced intrinsic connection may be defined on Σ, for vectors $A^j = A^\alpha e_\alpha{}^j$ tangent to Σ, by projection; if $\delta A/\delta U$ is an absolute derivative,

$$\nabla_\alpha A_\beta = e_\beta{}^i \frac{\partial x^j}{\partial \xi^\alpha} \nabla_j A_i = e_\beta{}^i \frac{\delta A_i}{\partial \xi^\alpha} = \frac{\partial A_\beta}{\partial \xi^\alpha} - e_{\gamma i} \frac{\delta e_\beta{}^i}{\delta \xi^\alpha} A^\gamma \tag{3.10}$$

so

$$\Gamma_{\gamma\beta\alpha} = e_{\gamma i} \frac{\delta e_\beta{}^i}{\delta \xi^\alpha}. \tag{3.11}$$

The intrinsic connection is metric with respect to $g_{\alpha\beta}$, although it is not torsion-free.

The manner in which Σ is attached to V^A is measured by evaluation on Σ of the extrinsic curvature $K_\alpha{}^\beta$ defined by [14]

$$\frac{\delta n^i}{\delta \xi^\alpha} = K_\alpha{}^\beta e_\beta{}^i, \tag{3.12}$$

the general form of which follows from $n_i n^i = -1$. Thus,

$$K_{\alpha\beta} = e_\beta{}^i \frac{\delta n_i}{\delta \xi^\alpha} = -n_i \frac{\delta e_\beta{}^i}{\delta \xi^\alpha}. \tag{3.13}$$

The extrinsic curvature is not symmetric as it is in the pseudo-Riemannian case:

$$K_{\alpha\beta} - K_{\beta\alpha} = Q^N{}_{\beta\alpha}. \tag{3.14}$$

The expansion of $\delta e_\alpha{}^i/\delta\xi^\beta$ corresponding to (3.12) follows from (3.11) and (3.12);

$$\frac{\delta e_\alpha{}^i}{\delta \xi^\beta} = \Gamma^\gamma{}_{\alpha\beta} e_\gamma{}^i + K_{\beta\alpha} n^i. \tag{3.15}$$

The relations (3.7), with $\delta/\delta x = n^i \nabla_i$, yield

$$\left.\frac{\delta e_\alpha{}^i}{\delta x}\right|_\Sigma = K_\alpha{}^\beta e_\beta{}^i + Q^\beta{}_{N\alpha} e_\beta{}^i + Q^N{}_{\alpha N} n^i \tag{3.16}$$

and

$$\left.\frac{\delta n^i}{\delta x}\right|_\Sigma = Q^{N\alpha}{}_N e_\alpha{}^i. \tag{3.17}$$

These equations combined with (3.2) give

$$\left.\frac{\partial g_{\alpha\beta}}{\partial x}\right|_\Sigma = 2K_{(\alpha\beta)} + 2Q_{(\beta|N|\alpha)}. \tag{3.18}$$

4. *Junction conditions*

The complete set of components of the curvature tensor can be obtained from the integrability conditions for the differential equations (3.12), (3.15), (3.16) and (3.17) for n^i and $e_\alpha{}^i$, namely

$$\left(\frac{\delta}{\delta u}\frac{\delta}{\delta v}-\frac{\delta}{\delta v}\frac{\delta}{\delta u}\right)e_\alpha{}^i = R^i{}_{jkl}e_\alpha{}^j\frac{\partial x^k}{\partial u}\frac{\partial x^l}{\partial v}, \tag{4.1}$$

for (u, v) equal to (ξ^β, ξ^γ) and (ξ^β, x) in turn. The corresponding commutators acting on n^i duplicate the $R_{\alpha Nij}$ components, since R_{ijkl} is skew in the first two indices. Evaluation of the left side of (4.1) through the use of equations (3.12), (3.15), (3.16) and (3.17) yields the components

$$R_{N\alpha\beta\gamma} = \nabla_\gamma K_{\beta\alpha}-\nabla_\beta K_{\gamma\alpha}+K_{\delta\alpha}Q^\delta{}_{\gamma\beta}, \tag{4.2a}$$

$$R_{\delta\alpha\beta\gamma} = {}^3R_{\delta\alpha\beta\gamma}-K_{\beta\alpha}K_{\gamma\delta}+K_{\gamma\alpha}K_{\beta\delta}, \tag{4.2b}$$

$$R^\gamma{}_{N\alpha N} = -\frac{\partial K_\alpha{}^\gamma}{\partial x}-K_\alpha{}^\beta K_\beta{}^\gamma-K_\alpha{}^\beta Q^\gamma{}_{N\beta}+\nabla_\alpha Q^{N\gamma}{}_N \tag{4.2c}$$

$$= -g^{\beta\gamma}\frac{\partial K_{\alpha\beta}}{\partial x}+K^{\gamma\beta}K_{\alpha\beta}+Q^{\beta N\gamma}K_{\alpha\beta}+\nabla_\alpha Q^{N\gamma}{}_N, \tag{4.2d}$$

and

$$R^\varrho{}_{\alpha N\beta} = \frac{\partial\Gamma^\varrho{}_{\alpha\beta}}{\partial x}-\nabla_\beta K_\alpha{}^\varrho-\nabla_\beta Q^\varrho{}_{N\alpha}+K_{\beta\alpha}Q^{N\varrho}{}_N-K_\beta{}^\varrho Q^N{}_{\alpha N}, \tag{4.2e}$$

where ${}^3R_{\delta\alpha\beta\gamma}$ is the intrinsic curvature tensor formed from $\Gamma_{\alpha\beta\gamma}$.

The corresponding components of the Einstein tensor follow by direct calculation. The derivative of the connection (cf. (4.2e)) can be eliminated by calculating $G_{\alpha N}$ from $G_{N\alpha}$ through use of the formula

$$R^i{}_{jik}-R^i{}_{kij} = \nabla_i Q^i{}_{kj}-\nabla_j Q^i{}_{ki}+\nabla_k Q^i{}_{ji}+Q^i{}_{li}Q^l{}_{kj}. \tag{4.3}$$

The Einstein tensor has components

$$G^N{}_N = \tfrac{1}{2}({}^3R+K^2-K_{\alpha\beta}K^{\beta\alpha}), \tag{4.4a}$$

$$G^N{}_\alpha = \nabla_\beta K_\alpha{}^\beta-\nabla_\alpha K+K_{\beta\gamma}Q^{\beta\gamma}{}_\alpha, \tag{4.4b}$$

$$G_{\alpha N} = \nabla_\beta K_\alpha{}^\beta-\nabla_\alpha K-\nabla_\beta Q^\beta{}_{\alpha N}+Q^N{}_{\alpha N}K+\frac{\partial Q^\beta{}_{\alpha\beta}}{\partial x}-Q^{N\beta}{}_N K_{\beta\alpha}-\frac{\partial Q^\beta{}_{N\beta}}{\partial\xi^\alpha}, \tag{4.4c}$$

and

$$G^\alpha{}_\beta = {}^3R^\alpha{}_\beta+K_{\beta\gamma}Q^{\alpha N\gamma}+K_\beta{}^\alpha K+\frac{\partial K_\beta{}^\alpha}{\partial x}-\nabla_\beta Q^{N\alpha}{}_N$$
$$-\tfrac{1}{2}\delta^\alpha{}_\beta\left({}^3R+K^2+2\frac{\partial K}{\partial x}+K_{\gamma\delta}K^{\delta\gamma}+2K_{\gamma\delta}Q^{\gamma N\delta}-2\nabla_\gamma Q^{N\gamma}N\right). \tag{4.4d}$$

These reduce for zero torsion to the corresponding equations [6] in general relativity.

These formulae are not expressed in a form suitable for obtaining the junction conditions by inspection since the intrinsic connection involves discontinuous functions:

$$\Gamma_{\alpha\beta\gamma} = \tilde{\Gamma}_{\alpha\beta\gamma} + \kappa_{\alpha\beta\gamma} \tag{4.5a}$$

$$= \tilde{\Gamma}_{\alpha\beta\gamma} - \tfrac{1}{2}(Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} + Q_{\gamma\beta\alpha}), \tag{4.5b}$$

where $\tilde{\Gamma}_{\alpha\beta\gamma}$ is the Christoffel symbol obtained from the intrinsic metric. Let $\tilde{\nabla}_\alpha$ denote the covariant derivative corresponding to $\tilde{\Gamma}_{\alpha\beta\gamma}$ and $\sim$ over any symbol denote the corresponding entity evaluated with that covariant derivative. Then, for example, we have

$$K_\alpha{}^\beta = \tilde{K}_\alpha{}^\beta + \kappa^\beta{}_{N\alpha}, \tag{4.6a}$$

$$K = \tilde{K} - Q^\alpha{}_{N\alpha}, \tag{4.6b}$$

and

$$^3R_{\alpha\beta} = {}^3\tilde{R}_{\alpha\beta} + \tilde{\nabla}_\beta Q^\gamma{}_{\alpha\gamma} + \tilde{\nabla}_\gamma \kappa^\gamma{}_{\alpha\beta} + \kappa^{\gamma\delta}{}_\beta \kappa_{\alpha\delta\gamma} - Q^\gamma{}_{\delta\gamma}\kappa^\delta{}_{\alpha\beta}. \tag{4.7}$$

With this notation and omitting expansion in every term of the exterior curvature by (4.6) we obtain formulae useful for the discussion of junction conditions and possibly of use in considerations on thin shells of spinning matter:

$$G^N{}_N = \tfrac{1}{2}({}^3\tilde{R} + 2\tilde{\nabla}_\alpha Q^{\beta\alpha}{}_\beta + \kappa^{\gamma\beta\alpha}\kappa_{\alpha\beta\gamma} - Q^\gamma{}_{\beta\gamma}Q^{\alpha\beta}{}_\alpha + K^2 - K_{\alpha\beta}K^{\beta\alpha}), \tag{4.8a}$$

$$G^N{}_\alpha = \tilde{\nabla}_\beta K_\alpha{}^\beta - Q^\beta{}_{\gamma\beta}K_\alpha{}^\gamma - \kappa^{N\beta\gamma}\kappa_{\beta\gamma\alpha} - \frac{\partial K}{\partial \xi^\alpha}, \tag{4.8b}$$

$$\begin{aligned} G_{\alpha N} = {} & \tilde{\nabla}_\beta K_\alpha{}^\beta - Q^\beta{}_{\gamma\beta}\tilde{K}_\alpha{}^\gamma - Q^\beta{}_{\gamma\beta}\kappa^\gamma{}_{\alpha N} - \kappa^\gamma{}_{\alpha\beta}\tilde{K}_\gamma{}^\beta - \kappa_{\alpha\beta\gamma}\kappa^{\beta\gamma N} \\ & - \frac{\partial \tilde{K}}{\partial \xi^\alpha} + Q^N{}_{\alpha N}K - \tilde{\nabla}_\beta Q^\beta{}_{\alpha N} + \frac{\partial Q^\beta{}_{\alpha\beta}}{\partial x} - Q^{N\beta}{}_N \tilde{K}_{\beta\alpha} - Q^{N\beta}{}_N \kappa_{\alpha N\beta} \end{aligned} \tag{4.8c}$$

and

$$\begin{aligned} G^\alpha{}_\beta = {} & {}^3\tilde{R}^\alpha{}_\beta + \tilde{\nabla}_\beta Q^{\gamma\alpha}{}_\gamma + \tilde{\nabla}_\gamma \kappa^{\gamma\alpha}{}_\beta + \kappa^{\gamma\delta}{}_\beta \kappa^\alpha{}_{\delta\gamma} - Q^\gamma{}_{\delta\gamma}\kappa^{\delta\alpha}{}_\beta \\ & + K_{\beta\gamma}Q^{\alpha N\gamma} + K_\beta{}^\alpha K + \frac{\partial K_\beta{}^\alpha}{\partial x} - \tilde{\nabla}_\beta Q^{N\alpha}{}_N - \kappa^\alpha{}_{\gamma\beta}Q^{N\gamma}{}_N \\ & - \tfrac{1}{2}\delta^\alpha{}_\beta \left({}^3\tilde{R} + 2\tilde{\nabla}_\varepsilon Q^{\gamma\varepsilon}{}_\gamma + \kappa^{\gamma\delta\varepsilon}\kappa_{\varepsilon\delta\gamma} - Q^\gamma{}_{\delta\gamma}Q^{\varepsilon\delta}{}_\varepsilon + K^2 + 2\frac{\partial K}{\partial x} + K_{\gamma\delta}K^{\delta\gamma} + 2K_{\gamma\delta}Q^{\gamma N\delta} - 2\nabla_\gamma Q^{N\gamma}{}_N\right). \end{aligned} \tag{4.8d}$$

The condition that the energy-momentum and spin tensors be of class C^3 in V^1 and in V^2 and experience at most a discontinuity in V at Σ implies, from the field equations and the normal derivatives in (4.8c) and (4.8d), that $Q^\beta{}_{\alpha\beta}$ and $K_\alpha{}^\beta$ are of class C^0 in V. These conditions can be written in terms of the notation

$$[f] = \lim_{x\to 0+} f(x, \xi^\alpha) - \lim_{x\to 0-} f(x, \xi^\alpha) \tag{4.9}$$

as

$$[Q^{\beta}{}_{\alpha\beta}] = 0 \tag{4.10}$$

and

$$[K_{\alpha\beta}] = 0. \tag{4.11}$$

The symmetric part of (4.11) gives, by virtue of (3.18),

$$\left[\frac{\partial g_{\alpha\beta}}{\partial x}\right] = [2Q_{(\beta|N|\alpha)}], \tag{4.12a}$$

while the skew part, combined with (4.10) and the fact that $\tilde{K}_{[\alpha\beta]} = 0$, gives

$$[S^{N}{}_{ij}] = 0. \tag{4.13}$$

The jumps in $G^{N}{}_{N}$ and $G^{N}{}_{\alpha}$, needed to match inside and outside solutions, are given by

$$[G^{N}{}_{\alpha}] = -[\kappa^{N\beta\gamma}\kappa_{\beta\gamma\alpha}] = -\tfrac{1}{4}\,[Q^{N\beta\gamma}(Q_{\beta\gamma\alpha}+2Q_{(\gamma\alpha)\beta})] \tag{4.14a}$$

and

$$[G^{N}{}_{N}] = \tfrac{1}{2}\,[\kappa^{\gamma\beta\alpha}\kappa_{\alpha\beta\gamma}] = \tfrac{1}{8}\,[Q^{\gamma\beta\alpha}(Q_{\alpha\beta\gamma}+2Q_{(\beta\gamma)\alpha})]. \tag{4.15a}$$

These junction conditions can be written in terms of general admissible coordinates x^{i} in V by introducing the projection operator $h^{i}{}_{j} = \delta^{i}{}_{j}+n^{i}n_{j}$ and defining projected component labels $\bar{j}$ by

$$T^{i}h^{j}{}_{i} = T^{-j}. \tag{4.16}$$

Equation (4.13) is already in that form and the other junction conditions are

$$[\tilde{K}_{\bar{i}\bar{j}}-Q_{(\bar{j}|N|\bar{i})}] = ,\ 0 \tag{4.12b}$$

$$[n^{j}G_{j\bar{i}}] = -[n_{j}\kappa^{jk\bar{l}}\kappa_{\bar{k}\bar{l}\bar{i}}] = -\tfrac{1}{4}\,[n_{j}Q^{jk\bar{l}}(Q_{\bar{k}\bar{l}\bar{i}}+2Q_{(\bar{l}\bar{i})\bar{k}})], \tag{4.14b}$$

and

$$[n^{i}n^{j}G_{ij}] = \tfrac{1}{2}\,[\kappa^{\bar{i}\bar{j}\bar{k}}\kappa_{\bar{k}\bar{j}\bar{i}}] = \tfrac{1}{8}\,[Q^{\bar{i}\bar{j}\bar{k}}(Q_{\bar{k}\bar{j}\bar{i}}+2Q_{(\bar{j}\bar{i})\bar{k}})]. \tag{4.15b}$$

Some of the conditions matching the components $G^{N}{}_{N}$ and $G^{N}{}_{\alpha}$ across Σ can be obtained also from the conservation laws (2.10) and (2.11), which may be written as

$$(-g)^{-1/2}\{(-g)^{1/2}t^{ij}\}_{,i} = -\Gamma^{j}{}_{li}t^{il}+Q^{kj}{}_{l}t^{l}{}_{k}-\frac{c}{2}\,R^{klj}{}_{m}S^{m}{}_{kl} \tag{4.17}$$

and

$$(-g)^{-1/2}\{(-g)^{1/2}S^{m}{}_{kl}\}_{,m} = \Gamma^{i}{}_{km}S^{m}{}_{il}+\Gamma^{i}{}_{lm}S^{m}{}_{ki}+\frac{1}{c}\,(t_{lk}-t_{kl}). \tag{4.18}$$

Integration of these expressions with the invariant volume element $d^4v = (-g)^{1/2}dx^1dx^2dx^3dx^4$ over a "pill-box" around Σ gives, since all terms on the right sides except the curvature term are finite everywhere, (4.13) and

$$[n_i t^i{}_j] = \frac{c}{2}\int R^{kl}{}_{jm} S^m{}_{kl} d^4v. \tag{4.19}$$

The integrand can involve a product of a Heaviside step function $\theta(x)$ from the spin tensor and a Dirac δ-function from the derivatives of the connection in the curvature tensor. However, the $t^N{}_\alpha$ components involve, as coefficients of the δ-function terms in the curvature tensor, only continuous components $S^N{}_{ij}$ of the spin tensor so the integration can be carried out to give, with an application of (4.11), results in agreement with (4.14). On the other hand, we obtain for the other component

$$[t^{NN}] = \frac{c}{2}\int R^{klNm} S_{mkl} d^4v = \frac{c}{2}\int [\kappa^{\alpha\beta\gamma}]\delta(x) S_{\gamma\alpha\beta}\theta(x) d^4v \tag{4.20}$$

which appears to be indeterminate.

However, Arkuszewski, Kopczyński and Ponomariev [4] noted [5] that the δ-function arises from the terms in the curvature involving derivatives of the connection linearly and that the connection is linear in the spin. Thus, one can see the origin of the $\delta(x)\theta(x)$ factor in the derivative of $[\theta(x)]^2$ so one can write here $\delta(x)\theta(x) = \frac{1}{2}d[\theta(x)]^2/dx$ and immediately integrate (4.20) to obtain a result in agreement with (4.15).

5. *Examples*

The spinning matter source of the Einstein-Cartan geometry corresponding to a perfect fluid in general relativity is a Weyssenhoff fluid [15, 16] described by the fluid velocity u^i, the rest energy density $e = u^i u^j t_{ij}$ and the vector spin density S^i defined in terms of the spin tensor $S^i{}_{jk} = u^i S_{jk}$ by

$$S^i = \tfrac{1}{2}\eta^{ijkl} u_j S_{kl} \quad \text{and} \quad u^i S_{ij} = 0. \tag{5.1}$$

The energy-momentum tensor for a Weyssenhoff fluid is [3]

$$t^i{}_j = u^i[(e+p)u_j + cu^k u^l \nabla_k S_{jl}] - p\delta^i{}_j. \tag{5.2}$$

The junction conditions from spinning matter to vacuum reduce in this case to [4]

$$[u^N S_{ij}] = 0, \quad \text{so} \quad u^N|_\Sigma = 0, \tag{5.3}$$

$$\left[\frac{\partial g_{\alpha\beta}}{\partial x}\right] = -\frac{16\pi G}{c^3} u_{(\alpha} S_{\beta)N}|_\Sigma \tag{5.4}$$

and

$$p|_\Sigma = \frac{2\pi G}{c^2} S_N{}^2|_\Sigma. \tag{5.5}$$

For a static spherically symmetric star, $n \propto \partial/\partial r$ and the only non-vanishing component of the spin is S_N. Thus, the normal derivative of $g_{\alpha\beta}$ (5.4) is zero as it is general relativity. Furthermore, the Einstein-Cartan equations for a Weyssenhoff fluid reduce to those of general relativity with the energy density e' and pressure p' in the latter theory given by [17]

$$e' = e - \frac{2\pi G}{c^2} S_N{}^2 \tag{5.6}$$

and

$$p' = p - \frac{2\pi G}{c^2} S_N{}^2. \tag{5.7}$$

Comparison of (5.5) and (5.7) shows that the Einstein-Cartan boundary condition $p'|_\Sigma = 0$ is identical to that of general relativity. Thus, known static spherically symmetric solutions of general relativity carry over directly to the Einstein-Cartan theory with the substitutions (5.6) and (5.7).

It is worth noting [5] that Equation (5.6) shows that the externally determined masses m in Tolman's solutions depend not only on the energy density e but also on the spin density S_N.

The dependence of the spin on the radial distance r is not determined, however, in the absence of a magnetic field. This dependence can therefore be chosen arbitrarily, subject only to the physical restriction of the maximum spin per relevant elementary particle. (Prasanna [17] introduced an arbitrary assumption, his equation (4.9), to determine the radial dependence of the spin.) Thus, for example, one may obtain, for various choices of $S_N(r)$, an arbitrary number of solutions corresponding to each of the solutions of Tolman [18].

A solution with constant mass and spin densities corresponding to the Schwarzschild interior solution for $r \leqslant a$ is given by

$$\begin{aligned} ds^2 = &-\left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &+ c^2[\tfrac{3}{2}(1-a^2/R^2)^{1/2} - \tfrac{1}{2}(1-r^2/R^2)^{1/2}]^2 dt^2, \end{aligned} \tag{5.8}$$

with $2G\,m/c^2 a = a^2/R$, m being the externally determined mass, and the only non-vanishing torsion components relative to an orthonormal frame being the constants

$$-Q_{423} = \frac{8\pi G}{c^3} S^1 = Q_{432}. \tag{5.9}$$

The pressure is given by

$$\begin{aligned} p(r) &= \frac{2\pi G}{c^2} S_N{}^2 + \frac{3c^4}{16\pi G R^2} \\ &\times \frac{[(1-r^2/R^2)^{1/2} - (1-a^2/R^2)^{1/2}]}{[\tfrac{3}{2}(1-a^2/R^2)^{1/2} - \tfrac{1}{2}(1-r^2/R^2)^{1/2}]} \\ &= \frac{2G\pi}{c^2} S_N{}^2 + p_{\text{eff}}(r). \end{aligned} \tag{5.10}$$

This solution arises also from Tolman's assumed relation $g_{rr} = (1-r^2/R^2)^{-1}$ for saturation of spin: $S_N = c\varrho$, for c a constant dependent on the type of matter present, implies from equation (4.3) of Tolman [18] with the substitution (5.6) that ϱ satisfies a quadratic equation with constant coefficients.

The speed of sound is infinite in this constant density model [19] but solutions with the same metric corresponding to a finite speed of sound can be obtained by selecting a radial-dependent spin. For example, choosing $(2\pi G/c^2)S_N{}^2 = p_{\text{eff}}(0)-p_{\text{eff}}(r)$ gives a zero speed of sound; the pressure is constant throughout and the density increases with increasing r.

Similar considerations apply to the remaining Tolman solutions as well as to more realistic models, like that of Kuchowicz [20].

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