THE SUPERENERGY TENSORS

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In this paper we introduce the notion of the superenergy supertensors and the notion of superenergy tensors. Every physical field which possesses the energy-momentum tensor (canonical or dynamical) will also have a corresponding superenergy supertensor (canonical or dynamical respectively) and a corresponding superenergy tensor (canonical or dynamical respectively). A superenergy supertensor and a superenergy tensor can also be assigned to physical fields which do not possess any energy-momentum tensor, e. g., to the gravitational field $\left\{{\alpha \atop \beta\gamma}\right\}$ and to the curvature tensor field $R_{\mu\nu\lambda}^{\cdot\,\cdot\,\cdot\,\kappa}$. In this paper we construct, among others, the superenergy supertensor for the tensor field $R_{\mu\nu\lambda}^{\cdot\,\cdot\,\cdot\,\kappa}(\Gamma)$ of the Einstein-Cartan Theory. In the special case of the General Theory of Relativity this tensor is proportional to the Bel-Robinson tensor.

1. *The superenergy supertensors and the superenergy tensors of a physical field Φ which possesses the energy-momentum tensor or pseudotensor*

In the following ECT means the Einstein-Cartan Theory, GRT — the General Theory of Relativity and SRT — the Special Theory of Relativity. We restrict ourselves temporarily to the space-time of the GRT, i. e., to the differentiable manifold V_4 [1].

Let us suppose that in this space-time a physical field Φ is given. Its invariant Lagrangian density is Λ and the corresponding to it dynamical ($\equiv$ metric) or canonical energy-momentum tensor is ${}^{\Phi}T_{\mu}^{\cdot\,\nu}$. Following Pirani's paper [2] we will define the dynamical or, respectively, canonical superenergy tensor ${}^{\Phi}_{S}T_{\mu}^{\cdot\,\nu}(P)$ of the field Φ and at the point $P \in V_4$ in the following manner. Let us introduce, at the point P, the normal coordinate system NCS(P) for the connection $\left\{{\alpha \atop \beta\gamma}\right\}$. In the "hyperplane" $y^{\circ} = 0$ of this coordinate system we consider "sphere" $\mathscr{K}: (y^1)^2+(y^2)^2+(y^3)^2 \leqslant \mathscr{R}_{\mathscr{K}}^2$ with $\mathscr{R}_{\mathscr{K}}$ being sufficiently small. The point P is the center

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of this sphere. Then we define[1]

$$ {}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P) := \lim_{\mathscr{R}_{\mathscr{K}}\to 0} \frac{\iiint\limits_{\mathscr{K}} ({}^{\Phi}T_{\mu}^{\cdot\,\nu} - {}^{\Phi}\mathring{T}_{\mu}^{\cdot\,\nu})d^3y}{\frac{4}{15}\pi\mathscr{R}_{\mathscr{K}}^5} = \lim_{\mathscr{R}_{\mathscr{K}}\to 0} \frac{\oint\!\!\oint\limits_{\partial\mathscr{K}} ({}^{\Phi}T_{\mu}^{\cdot\,\nu} - {}^{\Phi}\mathring{T}_{\mu}^{\cdot\,\nu})d^2S}{\frac{4}{3}\pi\mathscr{R}_{\mathscr{K}}^4}. \tag{1} $$

We see that the superenergy tensor ${}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P)$ of the field Φ is constructed by means of some kind of averaging of the differences of energy-momentum. In other words, it is a tensor constructed from a sort of relative energy and relative momentum.

Let the field ${}^{\Phi}T_{\mu}^{\cdot\,\nu}$ be of the class[2] C^r, $r \geqslant 3$ in the neighbourhood $U: K \subset U$ of the point P. Then

$$ \lim_{\mathscr{R}_{\mathscr{K}}\to 0} \frac{\iiint\limits_{\mathscr{K}} ({}^{\Phi}T_{\mu}^{\cdot\,\nu} - {}^{\Phi}\mathring{T}_{\mu}^{\cdot\,\nu})d^3y}{\frac{4}{15}\pi\mathscr{R}_{\mathscr{K}}^5} = \lim_{\mathscr{R}_{\mathscr{K}}\to 0} \frac{\oint\!\!\oint\limits_{\partial\mathscr{K}} ({}^{\Phi}T_{\mu}^{\cdot\,\nu} - {}^{\Phi}\mathring{T}_{\mu}^{\cdot\,\nu})d^2S}{\frac{4}{3}\pi\mathscr{R}_{\mathscr{K}}^4} $$

$$ = \tfrac{1}{2}\sum_{k\to 1}^{3} {}^{\Phi}\mathring{T}_{\mu\,\cdot\,,kk}^{\cdot\,\nu} = \tfrac{1}{2}(\mathring{v}^{\alpha}\mathring{v}^{\beta} - \mathring{g}^{\alpha\beta}){}^{\Phi}\mathring{T}_{\mu\,\cdot\,,\alpha\beta}^{\cdot\,\nu} = {}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P, v^{\varrho}), $$

$$ {}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P, \mathrm{v}^{\varrho}) := \tfrac{1}{2}(\mathring{v}^{\alpha}\mathring{v}^{\beta} - \mathring{g}^{\alpha\beta}){}^{\Phi}\mathring{T}_{\mu\,\cdot\,,\alpha\beta}^{\cdot\,\nu}. \tag{2} $$

where

$$ \mathring{v}^{\alpha} \underset{\mathrm{NCS}(P)}{\overset{*}{=}} \delta_0^{\alpha}, \qquad \mathring{g}^{\alpha\beta} \underset{\mathrm{NCS}(P)}{\overset{*}{=}} \eta^{\alpha\beta}. $$

The sign "∘" above the tensor field denotes the value of this field at the point P and a comma "," denotes partial differentiaition. The ${}^{\Phi}\mathring{T}_{\mu\,\cdot\,,\alpha\beta}^{\cdot\,\nu}$ is a true tensor [1, 3]:

$$ {}^{\Phi}\mathring{T}_{\mu\,\cdot\,,\alpha\beta}^{\cdot\,\nu} = \overset{*}{\nabla}_{(\beta}\overset{*}{\nabla}_{\alpha)}{}^{\Phi}\mathring{T}_{\mu}^{\cdot\,\nu} + \tfrac{1}{3}\mathring{R}^{\lambda}_{\cdot\,(\alpha|\mu|\beta)}{}^{\Phi}\mathring{T}_{\lambda}^{\cdot\,\nu} $$

$$ + \tfrac{1}{3}\mathring{R}^{\lambda\,\cdot\,\nu}_{\cdot\,(\alpha\,\cdot\,\beta)}(\{\}){}^{\Phi}\mathring{T}_{\mu\lambda}^{\cdot\,\cdot} - \tfrac{2}{3}(\mathring{R}_{(\mu|\alpha|\varrho)\beta}(\{\}) + \mathring{R}_{(\mu|\beta|\varrho)\alpha}(\{\})){}^{\Phi}\mathring{T}^{\varrho\nu}_{\cdot\,\cdot}. $$

$\overset{*}{\nabla}$ denotes the covariant derivative with respect to the Riemannian connection $\left\{ {\alpha \atop \beta\gamma} \right\}$. We will call this four-index tensor the dynamical or respectively, canonical supertensor of the superenergy of the field Φ and denote it by ${}_{\mathrm{S}}^{\Phi}T_{\mu\,\cdot\,\alpha\beta}^{\cdot\,\nu\,\cdot\,\cdot}$. This tensor is more fundamental than the two-index tensor ${}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P, v^{\varrho})$.

The superenergy tensor ${}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P, v^{\varrho})$ is a local construction which explicitly depends on the form of the energy-momentum tensor ${}^{\Phi}T_{\mu}^{\cdot\,\nu}$ and on the four-velocity v^{ϱ} of the observer O which is at rest in the origin of the NCS(P). For the given energy-momentum tensor ${}^{\Phi}T_{\mu}^{\cdot\,\nu}$ the supertensor ${}_{\mathrm{S}}^{\Phi}T_{\mu\,\cdot\,\alpha\beta}^{\cdot\,\nu\,\cdot\,\cdot}$ may be uniquely determined. The same can be done for the tensor ${}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P, v^{\varrho})$ provided a vector field v^{ϱ} is given.

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[1] We may also use the cube C: $-a \leqslant y^b \leqslant a$ $(b = 1, 2, 3)$ in NCS(P) instead of the sphere $\mathscr{K}$ and define

$$ {}_{\mathrm{S}}^{\Phi}T_{\mu}^{\cdot\,\nu}(P) := \lim_{a\to 0} \frac{\iiint\limits_{C} ({}^{\Phi}T_{\mu}^{\cdot\,\nu} - {}^{\Phi}\mathring{T}_{\mu}^{\cdot\,\nu})d^3y}{\frac{8}{3}a^5} = \lim_{a\to 0} \frac{\oint\!\!\oint\limits_{\partial C} ({}^{\Phi}T_{\mu}^{\cdot\,\nu} - {}^{\Phi}\mathring{T}_{\mu}^{\cdot\,\nu})d^2S}{\frac{40}{3}a^4}. $$

The results will be the same.

[2] In the following we will assume that the all considered fields are of the class C^r, $r \geqslant 3$.

Further we define the density, ε_S, of the superenergy of the field Φ and Poynting's supervector P_ϱ of this field for an observer O

$$\varepsilon_S := {}^{\Phi}_{S}\mathring{T}_{\mu\nu}^{\cdot\cdot}\mathring{v}^\mu\mathring{v}^\nu \underset{\mathrm{NCS}(P)}{\overset{*}{=}} {}^{\Phi}\mathring{T}_{00}^{\cdot\cdot}, \qquad P_\varrho := (\delta^\mu_\varrho - \mathring{v}^\mu\mathring{v}_\varrho){}^{\Phi}_{S}\mathring{T}_{\mu\nu}^{\cdot\cdot}\mathring{v}^\nu \underset{\mathrm{NSC}(P)}{\overset{*}{\longrightarrow}} P_0 = 0, \qquad P_K = {}^{\Phi}_{S}T_{0K}^{\cdot\cdot}.$$

${}^{\Phi}_{S}T_{\mu\nu}^{\cdot\cdot}(P, v^\varrho) := \frac{1}{2}(\mathring{v}^\alpha\mathring{v}^\beta - \mathring{g}^{\alpha\beta}){}^{\Phi}\mathring{T}_{\mu\nu,\alpha\beta}^{\cdot\cdot}$ is here the tensor obtained from ${}^{\Phi}T_{\mu\nu}^{\cdot\cdot} = g_{\nu\varrho}{}^{\Phi}T_{\mu}^{\cdot\varrho}$ by means of the same procedure as above. We would like to emphasize that the tensor ${}^{\Phi}_{S}T_{\mu\nu}^{\cdot\cdot}(P, v^\varrho)$ is an object different from $g_{\alpha\nu}{}^{\Phi}_{S}T_{\mu}^{\cdot\alpha}(P, v^\varrho)$. The same is true for the tensor ${}^{\Phi}_{S}T^{\mu\nu}_{\cdot\cdot}(P, v^\varrho) := \frac{1}{2}(\mathring{v}^\alpha\mathring{v}^\beta - \mathring{g}^{\alpha\beta}){}^{\Phi}\mathring{T}^{\mu\nu}_{\cdot\cdot,\alpha\beta}$ obtained from ${}^{\Phi}T^{\mu\nu}_{\cdot\cdot}$. ${}^{\Phi}_{S}T^{\mu\nu}_{\cdot\cdot}(P, v^\varrho)$ and ${}^{\Phi}_{S}T^{\mu\nu}_{\cdot\cdot}(P, v^\varrho)$ have the same symmetry properties as ${}^{\Phi}T_{\mu\nu}^{\cdot\cdot}(P)$ and ${}^{\Phi}T^{\mu\nu}_{\cdot\cdot}(P)$ respectively and, additionally, if the trace ${}^{\Phi}T_{\mu}^{\cdot\mu}(P) = 0$ then ${}^{\Phi}_{S}T_{\mu}^{\cdot\mu}(P, v^\varrho) = 0$ also. The components of the superenergy tensors ${}^{\Phi}_{S}T_{\mu}^{\cdot\nu}(P, v^\varrho)$, ${}^{\Phi}_{S}T_{\mu\nu}^{\cdot\cdot}(P, v^\varrho)$ and ${}^{\Phi}_{S}T^{\mu\nu}_{\cdot\cdot}(P, v^\varrho)$ have the dimension equal to the cm^{-2} × dimension of the components of the energy-momentum tensor. The same dimension have also the components of the superenergy supertensors ${}^{\Phi}_{S}T_{\mu\cdot\alpha\beta}^{\cdot\nu\cdot\cdot}$, ${}^{\Phi}_{S}T_{\mu\nu\alpha\beta}^{\cdot\cdot\cdot\cdot}$ and ${}^{\Phi}_{S}T^{\mu\nu\cdot\cdot}_{\cdot\cdot\alpha\beta}$.

The construction of the superenergy tensors of the field Φ is also possible in the framework of the SRT. We must do it in the same way as described above and in the Carthesian coordinate system. In this case ${}^{\Phi}_{S}T^{\mu\nu}(P, v^\varrho) = g^{\mu\varrho}{}^{\Phi}_{S}T_{\varrho}^{\cdot\nu}(P, v^\varrho)$, ${}^{\Phi}_{S}T_{\mu\nu}^{\cdot\cdot}(P, v^\varrho) = g_{\nu\varrho}{}^{\Phi}_{S}T_{\mu}^{\cdot\varrho}(P, v^\alpha)$ and the tensor ${}^{\Phi}_{S}T_{\mu}^{\cdot\nu}(P, v^\varrho)$ possesses the same algebraic properties as the tensor ${}^{\Phi}T_{\mu}^{\cdot\nu}(P)$. Moreover, in a very crude approximation and for the infinitesimal, fixed sphere $\mathscr{K}$ in SRT we have

$$\frac{\partial}{\partial t}\iiint\limits_{\mathscr{K}} {}^{\Phi}T_{0}^{\cdot 0}d^3y \cong \oint\!\!\oint\limits_{\partial\mathscr{K}} {}^{\Phi}_{S}T_{0}^{\cdot K}n_K d^2S$$

because, now

$$\partial_\nu({}^{\Phi}T_{\mu}^{\cdot\nu}) = 0.$$

If we construct, in the same way, the supertensors and the tensors of the superenergy from so-called "pseudotensors of the energy-momentum" which exist in the GRT, then we will also get intrinsic tensors. These tensors may have more symmetry properties than the pseudotensors from which they were obtained, e. g., the canonical superenergy tensor ${}_{P}t_{\mu}^{\cdot\nu}(P, v^\varrho)$ of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$ obtained from the canonical pseudotensor ${}_{E}t_{\mu}^{\cdot\nu}$ is symmetric. Moreover, in this case, ${}_{P}t_{\mu\nu}^{\cdot\cdot}(P, v^\varrho) = g_{\nu\varrho}\,{}_{P}t_{\mu}^{\cdot\varrho}(P, v^\varrho)$ where the ${}_{P}t_{\mu\nu}^{\cdot\cdot}(P, v^\varrho)$ is obtained from ${}_{E}t_{\mu\nu}^{\cdot\cdot} := g_{\nu\varrho}\,{}_{E}t_{\mu}^{\cdot\varrho}$.

In the space-time of the ECT, i. e., in U_4 following Schouten's notation [1], we will define the superenergy supertensors and the superenergy tensors of a field Φ which possesses an energy-momentum tensor or pseudotensor in the same way as in the space-time of the GRT. We are choosing that way, because:

1. For the full connection

$$\Gamma^\lambda_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + K_{\mu\nu}^{\cdot\cdot\lambda}$$

there does not exist any NCS(P); here

$$K_{\mu\nu\cdot}^{\cdot\cdot\lambda} := S_{\mu\nu\cdot}^{\cdot\cdot\lambda} - S_{\nu\cdot\mu}^{\cdot\lambda\cdot} + S_{\cdot\mu\nu}^{\lambda\cdot\cdot}$$

is the (—) contorsion tensor [4].

2. In this theory the Christoffel part of the connection, $\left\{ {\alpha \atop \beta\gamma} \right\}$, is physically privileged [5].

3. In the normal coordinate system NCS(P) for the Christoffel part of the connection, $\left\{ {\alpha \atop \beta\gamma} \right\}$, the calculation of the explicit form of the superenergy supertensors and the superenergy tensors for the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$ is easier than in e. g., connected with $\Gamma_{(\mu\nu)}{}^{\lambda}$.

Obviously, in the framework of U_4 geometry it is most natural to express the superenergy supertensors and the superenergy tensors with the help of the full connection $\Gamma_{\mu\nu}{}^{\lambda}$. We will be able to do that if we use the identities which connect $\overset{*}{\nabla}$ with ∇ and $R_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}(\{\,\})$ with $R_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}(\Gamma)$. These identities are given e. g., in [1].

2. The canonical superenergy supertensor and the canonical superenergy tensor of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$ in the Einstein-Cartan Theory

In the following we will work in the framework of the ECT geometry. In this section we will use the canonical energy-momentum pseudotensor of the pseudoeinsteinian (or combined) version of the ECT [4]. Formally, this pseudotensor is constructed from Christoffel's part of the connection $\left\{ {\alpha \atop \beta\gamma} \right\}$ in exatly the same way as the Einstein pseudotensor ${}_{E}t_{\mu\cdot}^{\cdot\nu}$.

In the GRT the field of the Christoffel connection $\left\{ {\alpha \atop \beta\gamma} \right\}$ has the usual physical interpretation of the gravitational field.

In the framework of the ECT we can interpret the Christoffel part of the connection $\left\{ {\alpha \atop \beta\gamma} \right\}$ also as the gravitational field [5].

Let us construct, in the way described in Section 1, the canonical superenergy tensor of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$ in the ECT. We will denote this tensor by ${}_{P}t_{\mu\cdot}^{\cdot\nu}(P, v^{\varrho})$. As a result of a long calculation, we get

$$\begin{aligned}
{}_{P}t_{\mu\cdot}^{\cdot\nu}(P, v^{\varrho}) = \frac{\kappa}{2}\,(\overset{\circ}{v}{}^{\alpha}\overset{\circ}{v}{}^{\beta} - \overset{\circ}{g}{}^{\alpha\beta})\,\{\tfrac{2}{9}\,[\overset{\circ}{T}{}_{\cdot\mu\alpha\beta}^{\nu\cdots} - \tfrac{1}{2}\,\delta_{\mu}^{\nu}\overset{\circ}{R}{}^{\lambda\sigma\varepsilon\cdot}_{\cdots\alpha}(\Gamma)\overset{\circ}{R}{}_{\lambda\sigma\varepsilon\beta}^{\cdots\cdot}(\Gamma)]\\
+\tfrac{2}{9}\,[\overset{\circ}{\tilde{T}}{}_{\cdot\mu\alpha\beta}^{\nu\cdots} - \tfrac{1}{2}\,\delta_{\mu}^{\nu}\overset{\circ}{R}{}^{\lambda\sigma\varepsilon\cdot}_{\cdots\alpha}(\Gamma)\overset{\circ}{R}{}_{\lambda\varepsilon\sigma\beta}^{\cdots\cdot}(\Gamma)]\\
+\bar{\kappa}\sum[R(\overset{*}{\nabla}K) + R\cdot E] + \bar{\kappa}^{2}\sum[K\cdot K\cdot R + (\overset{*}{\nabla}K)(\overset{*}{\nabla}K)
\end{aligned}$$

$$+(\overset{*}{\nabla}K)\cdot E+E\cdot E]+\overline{\kappa}^{3}\sum[(\overset{*}{\nabla}K)K\cdot K+E\cdot K\cdot K]$$

$$+\overline{\kappa}^{4}\sum(K\cdot K\cdot K\cdot K)\} = \frac{\kappa}{2}\,(\overset{\circ}{v}{}^{\alpha}\overset{\circ}{v}{}^{\beta}-\overset{\circ}{g}{}^{\alpha\beta})\phi^{\nu\cdots}_{\cdot\mu\alpha\beta}; \tag{3}$$

$\Phi^{\nu\cdots}_{\cdot\mu\alpha\beta}$ is the tensor within the bracket $\{\}$ and

$$\kappa = \frac{c^4}{16\pi G} = \frac{1}{2\overline{\kappa}}.$$

E denotes here the modified canonical energy-momentum tensor ${}_{E}T^{\cdot\cdot}_{\mu\nu} := {}_{K}T^{\cdot\cdot}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\,{}_{K}T^{\cdot\cdot}_{\alpha\beta}g^{\alpha\beta}$[4], R — the curvature tensor $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\Gamma)$ of the full connection $\Gamma^{\alpha}_{\beta\gamma}$; K — the conspin tensor $K^{\cdot\cdot\lambda}_{\mu\nu\cdot} := \tau^{\cdot\cdot\lambda}_{\mu\nu\cdot}-\tau^{\cdot\lambda\cdot}_{\nu\cdot\mu}+\tau^{\lambda\cdot\cdot}_{\cdot\mu\nu}+\delta^{\lambda}_{\mu}\tau^{\cdot\cdot\alpha}_{\nu\alpha\cdot}-g_{\mu\nu}\tau^{\lambda\cdot\alpha}_{\cdot\alpha\cdot}$ and $\overset{*}{\nabla}$ means the covariant derivative with respect to the Christoffel part of the connection $\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}$. $\tau^{\cdot\cdot\lambda}_{\mu\nu\cdot}$ is the spin-angular momentum tensor [4]. Σ denotes the sum of expressions the detailed form of which is not important for our present purpose and is symbolically shortened as indicated in the square brackets. These expressions represent the interaction terms.

The tensor $\kappa\Phi^{\nu\cdots}_{\cdot\mu\alpha\beta}$ is the canonical superenergy supertensor of the gravitational field $\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}$. The $T^{\nu\cdots}_{\cdot\mu\alpha\beta}$ is the tensor

$$T^{\nu\cdots}_{\cdot\mu\alpha\beta} := R^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\alpha}(\Gamma)R^{\cdot\cdot\cdot\cdot}_{\mu\lambda\sigma\beta}(\Gamma)+R^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\beta}(\Gamma)R^{\cdot\cdot\cdot\cdot}_{\mu\lambda\sigma\alpha}(\Gamma)-\tfrac{1}{2}\,\delta^{\nu}_{\mu}R^{\lambda\sigma\varepsilon\cdot}_{\cdot\cdot\cdot\alpha}(\Gamma)R^{\cdot\cdot\cdot\cdot}_{\lambda\sigma\varepsilon\beta}(\Gamma) \tag{4}$$

and

$$\overline{T}^{\nu\cdots}_{\cdot\mu\alpha\beta} := R^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\alpha}(\Gamma)R^{\cdot\cdot\cdot\cdot}_{\mu\sigma\lambda\beta}(\Gamma)+R^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\beta}(\Gamma)R^{\cdot\cdot\cdot\cdot}_{\mu\sigma\lambda\alpha}(\Gamma)-\tfrac{1}{2}\,\delta^{\nu}_{\mu}R^{\lambda\sigma\varepsilon\cdot}_{\cdot\cdot\cdot\alpha}(\Gamma)R^{\cdot\cdot\cdot\cdot}_{\lambda\varepsilon\sigma\beta}(\Gamma). \tag{5}$$

We have

$$T^{\nu\cdots}_{\cdot\nu\alpha\beta} = 0, \qquad T^{\cdot\cdot\cdot\cdot}_{\nu\mu\alpha\beta} = T^{\cdot\cdot\cdot\cdot}_{\mu\nu\alpha\beta} = T^{\cdot\cdot\cdot\cdot}_{\nu\mu\beta\alpha};$$

$$\overline{T}^{\nu\cdots}_{\cdot\nu\alpha\beta} = 0, \qquad \overline{T}^{\cdot\cdot\cdot\cdot}_{\nu\mu\alpha\beta} = \overline{T}^{\cdot\cdot\cdot\cdot}_{\mu\nu\alpha\beta} = \overline{T}^{\cdot\cdot\cdot\cdot}_{\nu\mu\beta\alpha}.$$

In vacuum ${}_{P}t^{\cdot\nu}_{\mu\cdot}(P, v^{\varrho})$ possesses a simpler form

$${}_{P}t^{\cdot\nu}_{\mu\cdot}(P, v^{\varrho}) = \frac{\kappa}{9}\,(\overset{\circ}{v}{}^{\alpha}\overset{\circ}{v}{}^{\beta}-\overset{\circ}{g}{}^{\alpha\beta})\,\{{}_{BR}\overset{\circ}{T}{}^{\nu\cdots}_{\cdot\mu\alpha\beta}+\overset{\circ}{W}{}^{\nu\cdots}_{\cdot\mu\alpha\beta}\}. \tag{6}$$

${}_{BR}T^{\nu\cdots}_{\cdot\mu\alpha\beta}$ denotes here the vacuum Bel-Robinson tensor

$${}_{BR}T^{\cdot\cdot\cdot\cdot}_{\nu\mu\alpha\beta} := C^{\cdot\cdot\cdot\cdot}_{\nu\lambda\sigma\alpha}C^{\cdot\lambda\sigma\cdot}_{\mu\cdot\cdot\beta}+{}^{*}C^{\cdot\cdot\cdot\cdot}_{\nu\lambda\sigma\alpha}{}^{*}C^{\cdot\lambda\sigma\cdot}_{\mu\cdot\cdot\beta} \equiv C^{\cdot\lambda\sigma\cdot}_{\nu\cdot\cdot\alpha}C^{\cdot\cdot\cdot\cdot}_{\mu\lambda\sigma\beta}+C^{\cdot\lambda\sigma\cdot}_{\nu\cdot\cdot\beta}C^{\cdot\cdot\cdot\cdot}_{\mu\lambda\sigma\alpha}-\tfrac{1}{2}\,g_{\mu\nu}C^{\lambda\sigma\gamma\cdot}_{\cdot\cdot\cdot\alpha}C^{\cdot\cdot\cdot\cdot}_{\lambda\sigma\gamma\beta},$$

$${}^{*}C_{\nu\lambda\sigma\alpha} := \tfrac{1}{2}\,\eta_{\nu\lambda\varrho\delta}C^{\varrho\delta\cdot\cdot}_{\cdot\cdot\sigma\alpha}, \qquad \eta_{0123} = \sqrt{-g}\,.$$

The $W^{\nu\cdots}_{\cdot\mu\alpha\beta}$ is the tensor

$$W^{\nu\cdots}_{\cdot\mu\alpha\beta} = C^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\alpha}C^{\cdot\cdot\cdot\cdot}_{\mu\sigma\lambda\beta}+C^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot\beta}C^{\cdot\cdot\cdot\cdot}_{\mu\sigma\lambda\alpha}-\tfrac{1}{4}\,g_{\alpha\beta}\delta^{\nu}_{\mu}C^{\lambda\delta\varepsilon\sigma}_{\cdot\cdot\cdot\cdot}C^{\cdot\cdot\cdot\cdot}_{\lambda\delta\varepsilon\sigma}$$

and $C^{\cdot\cdot\cdot\cdot}_{\mu\nu\alpha\beta}$ denotes the Weyl tensor.

In vacuum, the tensor ${}_{P}t_{\mu\varrho}^{\cdot\cdot} := g_{\nu\varrho}\, {}_{P}t_{\mu}^{\cdot\nu}(P, v^{\varrho})$ is the symmetric tensor: ${}_{P}t_{\mu\varrho}^{\cdot\cdot} = {}_{P}t_{\varrho\mu}^{\cdot\cdot}$.

We will call the tensor (4) the generalized Bel-Robinson tensor and we will shortly denote it by GBRT. This tensor is a natural generalization of the standard Bel-Robinson tensor on the tensor field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$. This can be seen from the following facts:

1°. $T_{\nu\mu\alpha\beta}^{\cdot\cdot\cdot\cdot} = R_{\nu\varrho\sigma\alpha}^{\cdot\cdot\cdot\cdot}(\Gamma)\, R_{\mu\cdot\cdot\beta}^{\cdot\varrho\sigma\cdot}(\Gamma) + {}^{*}R_{\nu\varrho\sigma\alpha}^{\cdot\cdot\cdot\cdot}(\Gamma)\, {}^{*}R_{\mu\cdot\cdot\beta}^{\cdot\varrho\sigma\cdot}(\Gamma)$ and by the formula of the same form, the standard Bel-Robinson tensor (denoted by BRT shortly from now on) is defined in the framework of the GRT [6, 7].

2°. The GBRT closely corresponds to the BRT physically, because both of them follow from the analogical, quasimaxwellian system of equations (see Section 3).

It is obvious that in the framework of the ECT we can represent the explicite form of the ${}_{P}t_{\mu}^{\cdot\nu}(P, v^{\varrho})$ in one of the three privileged modes:

1° With the use of the tensor $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$ as in (3).

2° With the use of the tensor $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\{\,\})$, i. e., with the use of the Riemannian part of the full curvature.

3° With the use of the Weyl tensor $C_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\{\,\})$.

It is a consequence of the identities [1] which exist between the components of the tensors $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$, $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\{\,\})$ and $C_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\{\,\})$ in the framework of the U_4 geometry. In every representation the contribution to the ${}_{P}t_{\mu}^{\cdot\nu}(P, v^{\varrho})$ arising only from the curvature tensor that we have used has the same structure as the contribution arising from the $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$ in (3). Only the contribution which depends on the presence of matter will be changed.

In the framework of the GRT the canonical superenergy supertensor and the canonical superenergy tensor of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$ possess formally the same form as $\kappa\Phi_{\cdot\,\mu\alpha\beta}^{\nu\cdot\cdot\cdot}$ and ${}_{P}t_{\mu}^{\cdot\nu}(P, v^{\varrho})$ respectively in which terms depended of the spin have been neglected. The full expressions are constructed in this case from the tensor $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\{\,\})$ (or $C_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\{\,\})$) and from the dynamical energy-momentum tensor of matter.

The BRT enters explicitly in these expressions.

3. *The Maxwellian superenergy supertensor of the tensor field* $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$ *in the* ECT

Physically, the field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$ is connected with the differences of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$. Thus, in the framework of the ECT, we cannot expect the field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$ to possess the Lagrangian and the energy-momentum tensor. However, we might hope that this field possesses the supertensors and the tensors of the superenergy. This point of view is supported by the explicit form of the expression (3) in which one can easily see the contribution to the canonical superenergy supertensor of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$ arising only from the field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$. This contribution is quadratic in $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$. The superenergy supertensors of the field $\left\{ {\alpha \atop \beta\gamma} \right\}$ calculated from other energy-momentum pseudotensors contain analogical contributions.

Obviously, we must define the superenergy supertensors of the tensor field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\varkappa}(\Gamma)$

in a different way than in Section 1. First of all, in this case no averaging is needed to obtain the four-index tensor whose components have the dimension of the components of the superenergy supertensor. We may define the superenergy supertensor and the superenergy tensor of the field $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$ in a pure algebraic way. It is known that in the framework of ECT the field $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$ satisfies the following system of equations, possessing the Maxwellian structure

$$\nabla_{[\mu}R_{\nu\omega]\lambda}{}^{\varrho} \equiv 2S_{[\mu\nu}{}^{\sigma}R_{\omega]\sigma\lambda}{}^{\varrho}, \tag{7}$$

$$\nabla_{\kappa}R^{\kappa\lambda\nu\mu} = J^{\lambda\nu\mu}. \tag{8}$$

Here

$$\begin{aligned} J^{\lambda\nu\mu} := {} & 2\bar{\kappa}\nabla^{[\mu}{}_{E}T^{\nu]\lambda} + 4\bar{\kappa}\bar{\tau}_{\kappa}{}^{[\mu|\varrho|}R^{\nu]}{}_{\varrho}{}^{\lambda\kappa} \\ & + 2\bar{\kappa}^{2}\bar{\tau}^{\mu\nu\varrho}{}_{E}T_{\varrho}{}^{\nu} - 3\bar{\kappa}\nabla_{\kappa}(\nabla^{[\kappa}\bar{\tau}^{\nu\lambda]\mu} - 2\bar{\kappa}\bar{\tau}^{[\lambda\kappa|\varrho|}\bar{\tau}^{\nu]}{}_{\varrho}{}^{\mu} \\ & + \nabla^{[\kappa}\bar{\tau}^{\nu\mu]\lambda} - 2\bar{\kappa}\bar{\tau}^{[\mu\kappa|\varrho|}\bar{\tau}^{\nu]}{}_{\varrho}{}^{\lambda} + \nabla^{[\mu}\bar{\tau}^{\nu\lambda]\kappa} \\ & - 2\bar{\kappa}\bar{\tau}^{[\lambda\mu|\varrho|}\bar{\tau}^{\nu]}{}_{\varrho}{}^{\kappa} + \nabla^{[\lambda}\bar{\tau}^{\mu\kappa]\nu} - 2\bar{\kappa}\bar{\tau}^{[\kappa\lambda|\varrho|}\bar{\tau}^{\mu]}{}_{\varrho}{}^{\nu}). \end{aligned} \tag{9}$$

In the formulae (8) and (9) $\bar{\kappa} = \frac{8\pi G}{c^4}$, ${}_{E}T_{\mu\beta}$ denotes the modified canonical energy-momentum tensor of matter and $\bar{\tau}_{\mu\nu}{}^{\lambda} := \tau_{\mu\nu}{}^{\lambda} - \frac{1}{2}\delta^{\lambda}_{\mu}\tau_{\kappa\nu}{}^{\kappa} - \frac{1}{2}\delta^{\lambda}_{\nu}\tau_{\mu\kappa}{}^{\kappa}$ is the modified spin-tensor.

The identities (7) are known as the Bianchi identities [1]. We can obtain from them new identities [1] which combined with the ECT system of equations give equation (8). The equations (7) and (8) characterize the field $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$ in the framework of the ECT. Consequently, we propose to determine the superenergy supertensors of the field $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$ from these equations. We define here the Maxwellian superenergy supertensors of the field $R_{\mu\nu\lambda}{}^{\kappa}(\Gamma)$ as the tensor.

$$S^{\nu}{}_{\mu\alpha\beta} := \kappa T^{\nu}{}_{\mu\alpha\beta} \tag{10}$$

where $T^{\nu}{}_{\mu\alpha\beta}$ is the tensor obtained from the system of equations (7) and (8) in the following way [10]:

1° We transvect (7) with the tensor $R^{\nu\omega\lambda}{}_{\delta}(\Gamma)$ and get the new identities

$$R^{\nu\omega\lambda}{}_{\delta}\nabla_{\nu}R_{\mu\omega\lambda\varrho} - \tfrac{1}{2}R^{\nu\omega\lambda}{}_{\delta}\nabla_{\mu}R_{\nu\omega\lambda\varrho} \equiv 3R^{\omega\nu\lambda}{}_{\delta}S_{[\mu\nu}{}^{\sigma}R_{\omega]\sigma\lambda\varrho} \tag{11}$$

2° We add, side by side, to the identities (11) the identities obtained from (11) by transposition of the indices (δ, ϱ) and get, after a simple calculation with the help of (8) and ECT equations

$$\nabla_{\nu}(R^{\nu\omega\lambda}{}_{\delta}R_{\mu\omega\lambda\varrho} + R^{\nu\omega\lambda}{}_{\varrho}R_{\mu\omega\lambda\delta} - \tfrac{1}{2}\delta^{\nu}_{\mu}R^{\sigma\omega\lambda}{}_{\varrho}R_{\sigma\omega\lambda\delta}) = 6\bar{\kappa}R^{\omega\nu\lambda}{}_{(\delta}\bar{\tau}_{|[\mu\nu}{}^{\sigma}R_{\omega]\sigma\lambda|\varrho)} + 2J^{\omega\lambda}{}_{(\varrho}R_{|\mu\omega\lambda|\delta)}. \tag{12}$$

3° We put

$$T^{\nu}{}_{\mu\delta\varrho} := R^{\nu\omega\lambda}{}_{\delta}R_{\mu\omega\lambda\varrho} + R^{\nu\omega\lambda}{}_{\varrho}R_{\mu\omega\lambda\delta} - \tfrac{1}{2}\delta^{\nu}_{\mu}R^{\sigma\omega\lambda}{}_{\varrho}R_{\sigma\omega\lambda\delta}. \tag{13}$$

It is seen that the tensor $T^{\nu}{}_{\mu\delta\varrho}$ is identical with the GBRT.

The multiplier κ in (10) guarantees the proper dimension of the components of the tensor $S^{\nu\cdot\cdot\cdot}_{\cdot\,\mu\alpha\beta}$. The necessity of this multiplier is obviously seen from (3).

When the identical procedure, as above, is applied to the full system of Maxwell equations in ECT, it gives

$$\nabla_{\varrho}\,{}_{\mathrm{em}}T^{\varrho\cdot}_{\cdot\,\mu} = -\tfrac{3}{4}\,\kappa\bar{\tau}_{[\mu\nu\cdot}^{\cdot\cdot\lambda}\,F^{\cdot}_{\varrho]\lambda}F^{\varrho\nu}_{\cdot\cdot}+\frac{1}{c}\,J^{\nu}F^{\cdot\cdot}_{\nu\mu}-\frac{1}{2\pi}\,\kappa F^{\cdot\cdot}_{\nu\mu}\bar{\tau}^{\cdot\cdot\lambda}_{\lambda\varrho\cdot}F^{\nu\varrho}_{\cdot\cdot}. \tag{14}$$

${}_{\mathrm{em}}T^{\varrho\cdot}_{\cdot\,\mu}$ denotes here the dynamical (or Maxwellian) energy-momentum tensor of the electromagnetic field $F_{\mu\nu}$ [11]. Thus, the method of construction of the Maxwellian superenergy supertensor of the field $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\Gamma)$ is identical with the first step of construction of the dynamical superenergy supertensor of the electromagnetic field $F^{\cdot\cdot}_{\mu\nu}$ with the help of Maxwell equations, i.e., with the construction of the dynamical energy-momentum tensor of this field from Maxwell equations. Consequently, with respect to the indices $(^{\nu\cdot}_{\cdot\mu})$ the supertensor $S^{\nu\cdot\cdot\cdot}_{\cdot\,\mu\alpha\beta}$ possesses the same algebraic and analytic properties as ${}_{\mathrm{em}}T^{\nu\cdot}_{\cdot\,\mu}$. No other possible superenergy supertensor of the field $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\Gamma)$ will have such properties.

It is obviously seen that working still in the framework of ECT we may define, in a similar way as above, the Maxwellian superenergy supertensors of the tensor fields $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$ and $C^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$. The Maxwellian superenergy supertensor of the Riemannian part $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\})$ of the full curvature is formally identical with the BRT of the GRT multiplied by κ. We may call this tensor the (multiplied by κ) Bel-Robinson tensor of the ECT. The geometric identities which connect the components of the tensors $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\Gamma)$, $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$ and $C^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$ lead to the corresponding relationships between Maxwellian superenergy supertensors of these tensors. The Maxwellian superenergy supertensor of the Weyl tensor is the irreducible component of the Maxwellian superenergy supertensors of the field $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\Gamma)$ and the field $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$.

All the considerations that we have made up to now may be repeated, with a small simplification, in the framework of the standard GRT. Especially, in the framework of GRT we will get that the Maxwellian superenergy supertensor of the tensor field $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$ determined from the system of equations corresponding to the equations (7) and (8) is identical with the BRT multiplied by κ. Thus, on the superenergy level, we may interpret the $\kappa\times$GBRT and the $\kappa\times$BRT to be the Maxwellian superenergy supertensors of the fields $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\Gamma)$ and $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$ respectively. In consequence of such an interpretation, the presence of these tensors in the superenergy supertensors of the gravitational field $\left\{{\alpha \atop \beta\gamma}\right\}$ can be explained as the contribution to the superenergy supertensors of the field $\left\{{\alpha \atop \beta\gamma}\right\}$ arising from the Maxwellian superenergy of the field $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\Gamma)$ or from the Maxwellian superenergy of the field $R^{\cdot\cdot\cdot\kappa}_{\mu\nu\lambda\cdot}(\{\,\})$. In our opinion, the physical interpretation of the tensors $\kappa\times$GBRT and $\kappa\times$BRT given above is more appropriate than the attempts to interpretate these tensors by means of expansions of pseudotensors in the NCS(P) [12, 13]. This opinion follows from the fact that these tensors are a natural consequence of the quasimaxwellian system of equations (7) and (8) and they are not a natural consequence of the structure of the pseudotensors.

Finally, we shall construct from the components of the tensor $S^{\nu\cdot\cdot\cdot}_{\cdot\mu\alpha\beta}$ the expressions for the superenergy density ε_S of the field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}(\Gamma)$ and for the Poynting supervector P_ϱ of this field. Let us consider an observer 0 using the orthonormal tetrad $\{\vec{e}_{(\alpha)}(P)\}$. Following to (3) we form the two-index tensor

$$M^{\nu}_{\cdot\mu}(P, v^{\varrho}) := \tfrac{1}{2}(\overset{\circ}{v}{}^{\alpha}\overset{\circ}{v}{}^{\beta} - \overset{\circ}{g}{}^{\alpha\beta})S^{\nu\cdot\cdot\cdot}_{\cdot\mu\alpha\beta}. \tag{15}$$

The tensor $M^{\nu}_{\cdot\nu}(P, v^{\varrho})$ is the Maxwellian superenergy tensor of the field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}(\Gamma)$ for the observer 0. Thus for the observer 0

$$\varepsilon_S = M_{\mu\nu}^{\cdot\cdot}\overset{\circ}{v}{}^{\mu}\overset{\circ}{v}{}^{\nu} \underset{0T}{\overset{*}{=}} M_{00}^{\cdot\cdot},$$

$$P_\varrho = (\delta^{\mu}_{\varrho} - \overset{\circ}{v}{}^{\mu}\overset{\circ}{v}_{\varrho})M_{\mu\nu}^{\cdot\cdot}\overset{\circ}{v}{}^{\nu} \xrightarrow[0T]{*} P_0 = 0, \qquad P_K = M_{K0}^{\cdot\cdot}, \tag{16}$$

are the superenergy density and the Poynting supervector connected with the Maxwellian superenergy tensor of the field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}(\Gamma)$.

In vacuum

$$\varepsilon_S \underset{0T}{\overset{*}{=}} \frac{\kappa}{2}\,{}_{\mathrm{BR}}T^{\cdot\cdot\cdot\cdot}_{0000}, \qquad P_0 \underset{0T}{\overset{*}{=}} 0, \qquad P_K \underset{0T}{\overset{*}{=}} \frac{\kappa}{2}\,{}_{\mathrm{BR}}T^{\cdot\cdot\cdot\cdot}_{K000}. \tag{17}$$

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