PROPERTIES OF A SOLUTION OF THE EINSTEIN EQUATIONS WITH THE COSMOLOGICAL CONSTANT

J. K. KOWALCZYŃSKI

Institute of Theoretical Physics of the Warsaw University*

J. F. PLEBAŃSKI**

Centro de Investigacion del I.P.N., Mexico

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Kerr–Schild type solutions of $R_{\alpha\beta} = \lambda g_{\alpha\beta}$ equations were sought. Some properties of the null vector appearing in such metrics when $\lambda \neq 0$ have been determined. The most general solution was found for the case of the vanishing second directional derivative of the function determining the null vector. The chosen direction of the derivative has physical meaning. Some properties of the resulting metric are given.

The results given below have been obtained from investigating the Einstein equations with the cosmological constant

$$R_{\alpha\beta} = \lambda g_{\alpha\beta}, \quad \lambda \neq 0. \tag{1}$$

These results being limited to the subclass of solutions which have the following form in the Cartesian system of co-ordinates

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 2hk_\alpha k_\beta, \tag{2a}$$

where $\eta_{\alpha\beta}$ is a flat metric tensor in the Cartesian co-ordinate system, h is an disposable real function and k_α is an disposable null vector

$$k_\alpha k^\alpha = 0. \tag{2b}$$

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

** On leave from the Institute of Theoretical Physics of the Warsaw University. Address: Centro de Investigacion del I.P.N., AP 14-740, Mexico 14, D.F., Mexico.

If a co-ordinate system $\zeta, \bar{\zeta}, u, v$ is introduced, so that the flat metric becomes $2d\zeta d\bar{\zeta} + 2dudv$ where ζ is a complex co-ordinate and u, v are real ones (the signature being $+++-$), then the metric (2) can be written as

$$ds^2 = 2d\zeta d\bar{\zeta} + 2dudv + 2h(k_\alpha dx^\alpha)^2. \tag{3}$$

As was shown in [1], for every null vector k^α in (2) there exists a complex scalar function Y and a complex vector field l^α such that $\partial_l Y = \theta + i\omega$, where θ and ω are the expansion and rotation of the vector k^α, respectively; the operator $\partial_l \underset{\text{def}}{=} l^\alpha \partial_\alpha$ being the derivative in the direction l^α. The vector k^α can then be represented by the following differential form[1]

$$k_\alpha dx^\alpha = du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv. \tag{4}$$

Two theorems have been obtained, which are given below:

Theorem 1. If the metric (2) is a solution of equations (1), then the vector k^α is geodesic, shear-free and has a non-vanishing expansion. Moreover, if that solution is not conformally flat, then k^α is a degenerate Debever–Penrose vector.

Theorem 2. Under the assumption that $\partial_l \partial_l Y = 0$, the most general metric of the form (2) resolving (1) is given by (3) and (4) with

$$h = \frac{1}{2} \frac{2mv - \lambda\left(\frac{v^4}{3} + 2a^2v^2 - a^4\right)}{v^2 + a^2}, \tag{5a}$$

$$Y = \zeta(v + ia)^{-1}, \tag{5b}$$

where m, a are arbitrary real constants.

The metric defined by (3), (4) and (5) can be either conformally flat or of the I D Petrov type. If it is of the I D type, then the second Debever–Penrose vector k'_α can be represented by the differential form: $k'_\alpha dx^\alpha = dv + hk_\alpha dx^\alpha$. For k_α we have $\theta + i\omega = (v + ia)^{-1}$. However, the constant a is not significant in (3), (4), (5). In fact, even if one takes $a \neq 0$, the constant a can be absorbed by substituting $v = a\tilde{v}$, $u = a^{-1}\tilde{u}$ and $m = a^3\tilde{m}$ (which corresponds to the renormalization of the null vectors $k_\alpha = a^{-1}\tilde{k}_\alpha$, $k'_\alpha = a\tilde{k}'_\alpha$), so that finally a solution of form (3), (4), (5) is obtained where the constant a no longer appears.

The proofs of theorems 1 and 2 have been based on long calculations.

The proof of theorem 1 consists in calculating the expansion and shear, checking if vector k^α is geodesic, and calculating the components of the Ricci tensor $R_{\alpha\beta}$. These calculations are performed taking into account (2) and (4). Subsequently it is proved that in this situation the necessary condition for equations (1) being fulfilled is that vector k^α be geodesic, shear-free and that it has a non-zero expansion. The proof of the necessity of that condition relays essentially on the assumption that $\lambda \neq 0$.

The proof of theorem 2 relays partly on properties of the function Y obtained in the proof of theorem 1. After introducing the additional assumption $\partial_l \partial_l Y = 0$, they allow

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[1] For the sake of ease of comparison we use here notation similar to that in part 5 of [1].

one to find all the explicit solutions (under this assumption) of (1) having the form (2). In the co-ordinate system used and for vector k^{α} written in the form (4) there are three apparently different metrics of form (3), of which one is identical with (5), and the remaining two can be reduced to (5) by co-ordinate transformations.

The conclusions regarding the Debever-Penrose vectors and the Petrov classification contained in theorem 1 and in the comments to theorem 2 regarding the metric determined by (3), (4) and (5) have been obtained by calculating the spinorial coefficients $C^{(i)}(i = 1, 2, 3, 4, 5)$ which determine the Weyl's conformal curvature tensor (cf. [1]).

REFERENCE

[1] G. C. Debney, R. P. Kerr, A. Schild, *J. Math. Phys.* **10**, 1842 (1969).