A GAUGE ON THE QUARK MASS SHELL*

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Some divergent integrals encountered in the Kondo–Yang–Mills model of strong interactions are analyzed in n dimensions. This is to argue that the gauge which was used by Stuller in order to determine the mass of quarks can be characterized at $n = 4$ explicitly by the gauge parameter $\xi = 2$. The possible existence of such values of the coupling constant for which ξ will be undetermined is indicated.

1. *Introduction*

This paper is thought as a comment in addendum to recent work by Stuller [1] on Kondo like models of strong interactions. The underlying physical idea is that perhaps a mechanism responsible for quark confinement may be analogous to that causing the Kondo effect [2] in solid state physics. The analogy is set by identification of the solid with physical vacuum, spin $\frac{1}{2}$ impurity with a colored quark, spin waves with colored Yang–Mills gluons and excitons with color singlet hadrons.

The first task in this Kondo–Yang–Mills picture was to find a viable equation for the quark mass. This was to see what sort of dynamics is to be exhibited by the Yang–Mills theory as to yield an infinite value of the quark mass (not as a constituent) which could explain the (total) confinement. This equation was formulated by fixing the gauge that does not affect positions of the quark pole (not so for the residuum) and quark-antiquark singularities. The gauge group is assumed to be a $G \times SU(3)$ color, but here we are not referring to the strong group G at all.

In Section 2 we will recall under what assumptions the eigenvalue equation for the quark mass was derived by Stuller [3]. Thereby the gauge to be used throughout was defined quite indirectly, and in Section 3 we will evaluate the gauge parameter explicitly. This will allow us to see under what additional assumptions a possibly infinite (or even undetermined) value of this parameter might be excluded.

The methods we are using are borrowed from the dimensional regularization in practice. We could not fully appreciate the original methods of 't Hooft and Veltman [4]

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as these are not well adapted for a regularization of integrals which are both infrared and ultraviolet divergent as will be encountered here. Therefore our results are borne with an arbitrariness due to our way of defining these integrals.

2. *The equation for the quark mass*

The nonperturbative form of the eigenvalue equation for the quark mass

$$m(k^2)|_{k^2=-m^2} = m \tag{2.1}$$

is to be deduced from the Schwinger–Dyson equation for the quark propagator $G(k)$

$$G^{-1}(k) = \gamma k + m_0 + ig^2\gamma^\mu \frac{\lambda_a}{2}\int d^4p G(k-p)\Gamma^\nu_b(k-p, k)\mathscr{G}^{ab}_{\mu\nu}(p) \tag{2.2}$$

where $a = 1, 2, \ldots, 8$, $\lambda_a/2$ is the fundamental representation of the SU(3) color and m_0 is a bare mass. The vector gluon propagator $\mathscr{G}^{ab}(p)$ was decomposed, due to the first Slavnov identity [5], into gauge dependent and gauge independent parts

$$\mathscr{G}^{ab}_{\mu\nu}(p) = \delta_{ab}\xi \frac{p_\mu p_\nu}{(p^2)^2} + \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{d_{ab}\ p^2)}{p^2} \tag{2.3}$$

Here $d_{ab}(p^2)$ is the function responsible for the dynamics and ξ is a gauge parameter ($\xi = 0$ in the Landau gauge, $\xi = 1$ in the Feynman gauge, and $\xi = 3$ in the Yennie gauge). The assumption motivated by the Kondo mechanism was

$$d_{ab}(p^2) = \delta_{ab}d(p^2) \tag{2.4}$$

and for the vertex function it was simply assumed that

$$\Gamma^\nu_b(k-p, k) = \gamma^\nu \frac{\lambda_b}{2}. \tag{2.5}$$

Defining

$$G^{-1}(k) \equiv A(k^2)(\gamma k + m(k^2)) \tag{2.6}$$

equation (2.2) becomes a system of two highly coupled equations for $A(k^2)$ and $m(k^2)$

$$A(k^2)m(k^2) = m_0 - ic_0g^2\int d^4p \frac{m[(k-p)^2]}{m^2[(k-p)^2]+(k-p)^2}\, g^{\mu\nu}\mathscr{G}_{\mu\nu}(p)\, \frac{1}{A[(k-p)^2]}, \tag{2.7}$$

$$(A(k^2)-1)\,(-k^2) = \frac{ic_0g^2}{4}\,\mathrm{Tr}\int d^4p \frac{\gamma k\gamma^\mu\gamma(p-k)\gamma^\nu}{m^2[(k-p)^2]+(k-p)^2}\,\mathscr{G}_{\mu\nu}(p)\,\frac{1}{A[(k-p)^2]}, \tag{2.8}$$

with $c_0 = \sum_{a=1}^{8} (\lambda_a/2)^2$.

The crucial assumption was that $d(p^2)$ behaves in such a way that the main contribution to the above integrals is coming from small p^μ and $m[(k-p)^2]$ and $A[(k-p)^2]$ were simply replaced by $m(k^2)$ and $A(k^2)$. At this point it becomes clear that fixing the gauge on the quark mass shell (which is the region we are after) by setting

$$A(k^2)|_{k^2=-m^2} = 1 \tag{2.9}$$

will provide the desired eigenvalue equation which reads

$$m = m_0 - ic_0 g^2 m \int d^4p \frac{1}{m^2+(k-p)^2} g_{\mu\nu}\mathcal{G}^{\mu\nu}(p)|_{k^2=-m^2}. \tag{2.10}$$

Further (and for consistency) the infrared parametrization of the function $d(p^2)$ was introduced

$$d(p^2) = \left(\frac{\Omega^2}{p^2}\right)^{\lambda} + 1, \tag{2.11}$$

while the values of the parameter λ were subjected to an ansatz. The Ω in (2.11) is a constant with the mass dimension. Let us notice that it is only for quantum electrodynamics when $\lambda = 0$, and that possible logarithms in the ultraviolet region are ignored. This parametrization reflects the fundamental physical assumption about vector mass spectrum extending down to zero.

In order to characterize solutions to (2.10) there were ultraviolet and infrared cutoffs used and it was argued that these two are intertwined in such an intriguing way as to leave behind finite m for $\lambda < \frac{1}{2}$ and infinite m for $\lambda > \frac{1}{2}$ (the m_0 was taken to be zero). Hence, it is the $\lambda > \frac{1}{2}$ ansatz which assures the (total) quark confinement. In spite of $G = 0$ for $\lambda > \frac{1}{2}$. Stuller has also derived a Bethe–Salpeter equation for mesons with small quark masses and this equation has turned out to be three dimensional (!). Further, it was noticed that $\frac{1}{2} < \lambda < 1$ must be rejected as this would lead to an infinity of low lying meson states.

The disadvantages of this model are that it predicts Regge trajectories rising not infinitely, and it is this total confinement which sounds rather too hard. It is clear that the theory of strong interactions will require an input of more physical ideas, e.g., these from Kondo models and superconductor models in one turn. Nevertheless, the results which were recalled above, together with other interesting phenomenological consequences, are in our opinion worth further investigations on their own right. First of all they are calling for a thorough examination of the consistency of the $\lambda > 1$ ansatz. This is because in general there can be $\lambda = \lambda(g^2)$ and the vector gluon propagator has to satisfy its Schwinger--Dyson equation as well. This problems, however, will be pursued elsewhere and here we will only try to evaluate the gauge parameter explicitely.

3. The evaluation of the gauge parameter

In the Stuller approach the gauge was defined quite implicitly by (2.9). This is, from (2.8), equivalent to

$$\mathrm{Tr} \int d^4p \frac{\gamma k \gamma^{\mu}(p-k)\gamma^{\nu}}{m^2+(p-k)^2} \mathcal{G}_{\mu\nu}(p)|_{k^2=-m^2} = 0, \tag{3.1}$$

where the gauge parameter is hidden in

$$g^{\mu\nu}\mathcal{G}_{\mu\nu}(p) = \frac{3}{p^2}\left(\frac{\Omega^2}{p^2}\right)^{\lambda} + (3+\xi)\frac{1}{p^2}. \tag{3.2}$$

The evaluation of traces in (3.1) gives

$$\xi(\lambda) = 1 + \frac{I_1(\lambda)}{I_2(0)} - \frac{I_2(\lambda)}{I_2(0)} - \frac{I_1(0)}{I_2(0)}, \tag{3.3}$$

where

$$I_1(\lambda) = 16\Omega^{2\lambda}\int d^4p\, \frac{k_\mu(p-k)^\mu}{[m^2+(k-p)^2]}\,\frac{1}{(p^2)^{1+\lambda}}\bigg|_{k^2=-m^2}, \tag{3.4}$$

$$I_2(\lambda) = 4\Omega^{2\lambda}\int d^4p\, \frac{p^2p^\nu k_\nu - 2k^\mu p_\mu k^\nu p_\nu + k^2p^2}{[m^2+(k-p)^2]}\,\frac{1}{(p^2)^{2+\lambda}}\bigg|_{k^2=-m^2}. \tag{3.5}$$

This expression for $\xi(\lambda)$ is only formal since it involves the integrals (3.4) and (3.5) which are plagued with both ultraviolet and infrared divergencies. Note that the condition on the mass shell implements an additional infrared divergence.

In order to give a meaning to $\xi(\lambda)$ we propose evaluating (3.4) and (3.5) as n-dimensional integrals, treating n, appearing in the results $I_{1,2}(\lambda, n)$, as a continuous complex parameter, taking the corresponding ratios still for generic n and then defining

$$\xi(\lambda) = \lim_{n\to 4}\xi(\lambda, n). \tag{3.6}$$

The results are

$$I_1(\lambda, n) = 16i\Omega^{2\lambda}\pi^{n/2}m^{n-2-2\lambda}\,\frac{\Gamma\left(\lambda+\dfrac{4-n}{2}\right)}{\Gamma(\lambda+1)}$$

$$\times\left\{\int_0^1 dx\,x^{n-4-2\lambda}(1-x)^\lambda - \int_0^1 dx\,x^{n-3-2\lambda}(1-x)^\lambda\right\}, \tag{3.7}$$

$$I_2(\lambda, n) = -4i\Omega^{2\lambda}\pi^{n/2}m^{n-2-2\lambda}\Gamma\left(\lambda+\frac{4-n}{2}\right)$$

$$\times\left\{\frac{1}{\Gamma(\lambda+1)}\left(\int_0^1 dx\,x^{n-4-2\lambda}(1-x)^\lambda - \int_0^1 dx\,x^{n-3-2\lambda}(1-x)^\lambda\right)\right.$$

$$\left. + \frac{2}{\Gamma(\lambda+2)}\left[\left(\lambda+\frac{4-n}{2}\right)\int_0^1 dx\,x^{n-6-2\lambda}(1-x)^\lambda - \int_0^1 dx\,x^{n-4-2\lambda}(1-x)^\lambda\right]\right\}. \tag{3.8}$$

The results for $\lambda = 0$ to be read off from (3.7) and (3.8) were also checked by other methods. In getting the above answers the technical problem was that for physically reasonable values of λ (noninteger $\lambda > 1$) the Feynman parametric integration could not be applied in its simplest form and we were using formulae like

$$\frac{1}{(p^2)^{\lambda+1}} = \frac{1}{\Gamma(\lambda+1)} \int_0^1 dx x^{\lambda} e^{-p^2 x}. \tag{3.9}$$

Now, as promised, we are still keeping n complex and write the remaining (otherwise divergent) integrals over the Feynman parameter x as β-functions which we further express by Γ-functions. This gives

$$I_1(\lambda, n) = 16 i \Omega^{2\lambda} \pi^{n/2} m^{n-2-2\lambda} \Gamma\left(\lambda + \frac{4-n}{2}\right)\left(\frac{\Gamma(n-3-2\lambda)}{\Gamma(n-2-\lambda)} + \frac{\Gamma(n-2-2\lambda)}{\Gamma(n-1-\lambda)}\right), \tag{3.10}$$

$$I_2(\lambda, n) = -4 i \Omega^{2\lambda} \pi^{n/2} m^{n-2-2\lambda} \Gamma\left(\lambda + \frac{4-n}{2}\right)\left\{\frac{\Gamma(n-3-2\lambda)}{\Gamma(n-2-\lambda)} + \frac{\Gamma(n-2-2\lambda)}{\Gamma(n-1-\lambda)}\right.$$

$$\left. +2\left(\lambda + \frac{4-n}{2}\right)\frac{\Gamma(n-5-2\lambda)}{\Gamma(n-1-\lambda)} - \frac{\Gamma(n-3-2\lambda)}{\Gamma(n-1-\lambda)}\right\}. \tag{3.11}$$

We obviously wish to interprete (3.10) and (3.11) as analytic continuations of (3.7) and (3.8), respectively, and now we see that we can really do so only if λ takes values in the interval (1, 3/2) or only in (3/2, 2), etc. For values of λ varying in just one of such intervals it is seen that

$$\lim_{n\to 4} \frac{I_1(\lambda, n)}{I_2(0, n)} = \lim_{n\to 4} \frac{I_2(\lambda, n)}{I_2(0, n)} = 0. \tag{3.12}$$

To get the final answer we still need to evaluate the last ratio in (3.3). This is found to be equal -1 from

$$I_{1,2}(0, n) = \mp 8 i \pi^{n/2} m^{n-2} \Gamma\left(\frac{4-n}{2}\right), \tag{3.13}$$

and we get that the value of the gauge parameter in the physical case $n = 4$ is

$$\xi = 2. \tag{3.14}$$

To summarize, we have shown that if the infrared behaviour of the theory would be such as expressed by (2.11), but with values of the parameter λ lying in the specific interval $((2+\alpha)/2,\ (3+\alpha)/2)$ $(\alpha = 0, 1, 2, \ldots)$, then it is possible to define the gauge parameter corresponding to (2.9) in such a way that it does not depend on the (assumed) dynamics anymore and is finite. The value $\xi = 2$, which we have obtained, characterizes the gauge defined by the condition on the mass shell (2.9) which is something in between the well known Feynman and Yennie gauges. All this, however, has been achieved by paying perhaps the cost of admitting the existence of a sequence of some strange "critical points". These points could be just this values of the coupling constant g^2 at which the function $\lambda(g^2)$ would take the boundary values equal to 3/2, 2, 5/2, etc. and, when meeting them the gauge parameter could not be determined by the method proposed here. Thus, it is also the example of the problem of the gauge parameter which brings out how welcome it would be to know the general $\lambda = \lambda(g^2)$ dependence for studying the consistency of the Stuller-Kondo model of strong interactions.

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REFERENCES

[1] R. L. Stuller, *Acta Phys. Pol.* **B6**, 907 (1975).
[2] Kondo, *Adv. Solid State Phys.* **23**, 183 (1969); K. Wilson, CLNS preprint 271 (1974); P. W. Anderson, TCM/7 preprint (1975).
[3] R. L. Stuller, *Phys. Rev.* **D13**, 513 (1976); R. L. Stuller, ICTP/74/14 preprint (1974).
[4] G. 't Hooft, M. Veltman, *Nucl. Phys.* **B44**, 189 (1972); G. 't Hooft, M. Veltman, *Diagrammar*, CERN 73/9 (1973).
[5] A. Slavnov, *Theor. Math. Phys.* **10**, 99 153 (1972).