SINGLE VARIABLE DESCRIPTION IN A GENERALIZED WEIZSÄCKER WILLIAMS METHOD

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(*Received November 3, 1976*)

A generalization of the Weizsäcker Williams method for massive fields with arbitrary tensor structure is considered. The resulting distribution of "equivalent quanta" has a particularly simple form, which is related to the field's energy momentum tensor in the rest frame.

1. *Introduction*

One of the earliest models for multiparticle production in hadronic reactions was constructed in analogy to electromagnetic bremsstrahlung by Heisenberg [1] and subsequently elaborated by a number of authors [2, 3]. The essential experimental ingredient in these considerations is the large number of produced particles in high energy collision. This allows a classical treatment of the relevant fields. Only in a final step the distribution of the field energy with respect to wave vectors is identified with the inclusive particle distribution

$$\frac{dE}{d^3k} = \omega\frac{dN}{d^3k}\,. \tag{1}$$

In a recent paper [4] Białas and Stodolsky argued, that the distribution of "equivalent quanta" for the field of a fast moving rotation invariant system (calculated similar to the distribution of "equivalent photons" in the Weizsäcker Williams method [5]) should be of the form

$$\omega\frac{dN}{d^3k} = f(\vec{k}^{*2}), \quad \text{where} \quad k_\perp^* \equiv k_\perp, \quad k_3^* \equiv k_3/\gamma$$

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and γ is an "effective" $\sqrt{1-\beta^2}$ for the incident projectile[1]. Such a form was suggestive, because the distribution of a fast moving field, which is rotation symmetric in the rest frame, is of the form $f(X_\perp^2+\gamma^2 X_3^2)$ and the Fourier transform depends on $(k_\perp^2+k_3^2/\gamma^2)$ only. Therefore one might expect a corresponding variable dependence of the energy distribution.

In a generalization of the method of Weizsäcker and Williams to arbitrary fields we shall calculate the energy distribution in the modes of an "equivalent" free field, whose energy density is that of the moving static system of interacting fields. In part 2 we shall give the arguments in detail for the scalar field — in part 3 the method will be generalized to arbitrary fields.

For free fields the energy momentum tensor $T_{\mu\nu}(X)$ is bilinear and a sum of terms of the form[2]

$$t_{\mu\nu}^{\alpha\beta} A_\alpha(X) A_\beta(X).$$

The energy distribution in modes is given through the corresponding sum

$$\frac{dE}{d^3k} = \sum_t (2\pi)^{-3} t_{\mu\nu}^{\alpha\beta} A_\alpha^+(\vec{k}) A_\beta^-(\vec{k}) + \text{h.c.} \tag{2}$$

where

$$A_\alpha^\pm(\vec{k}) \equiv \int\limits_{X_0=0} d\vec{X} e^{\mp ikX} \frac{\pm \overleftrightarrow{\partial}_0}{2i\omega} A_\alpha(X).$$

In the "generalized Weizsäcker Williams method" we calculate the energy distribution of an interacting field *as it if were a free field*, i.e. we use Eqs (2) also for the interacting field. We shall show, that the result is related in a simple manner to the energy momentum tensor in the rest frame. Furthermore from rotation invariance in the rest frame we find

$$\frac{dE}{d^3k} = f_1(\vec{k}^{*2}) + k_3^* f_2(\vec{k}^{*2}) + k_3^{*2} f_3(\vec{k}^{*2}) \tag{3}$$

leading to a corresponding distribution for the "equivalent quanta".

The argument is solely based on transformation properties of the energy momentum tensor and rotation invariance in the rest frame, but independent of the Lorentz structure of the field. We will conclude with some remarks on the non uniqueness of the relation between the energy distribution in momentum and configuration space, and comment on the impact parameter density of the flux of "equivalent quanta" and the relation of this picture to the usual bremsstrahlung calculation.

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[1] A_α may denote fields and derivatives of fields with arbitrary Lorentz structure. α, β are Lorentz multi indices $\alpha_1, \dots \alpha_n$; $\beta_1, \dots \beta_m$; (including possible derivatives), with n and m arbitrary. $t_{\mu\nu\alpha\beta}$ is a constant Lorentz tensor, like $g_{\mu\nu} g_{\alpha_1\alpha_2}{}^* \dots g_{\beta_{n-1}\beta_m}$. $k_0 = \omega = \sqrt{m^2+\vec{k}^2}$ where m is the mass of the free fields under considerations.

[2] k_3^* plays the role of $X_{\text{Feynman}} \equiv k_3/\gamma M$, where M is the mass of the projectile.

2. The scalar case

To illustrate our latter arguments, we restate some standard calculations for the classical scalar field with an external source. We first calculate the density of modes (which transforms into particle density in the quantized form) directly and then through the corresponding energy density. The latter procedure will easily be generalized in Section 3 to the case of arbitrary fields.

Consider the (classical or quantum) field A, coupled to a static classical source

$$(\Box + m^2)A = j.$$

We then define an "equivalent" free field through[3]

$$A^0(x) \equiv \int_{y_0=0} d\vec{y}\left(D(x-y)\frac{\overleftrightarrow{\partial}}{\partial y_0}A(y)\right).$$

Apparently $A^0(x)$ and $A(x)$ coincide for $x_0 = 0$ together with their first time derivatives. Since the field A is static in its rest frame, it depends on the coordinates through $X_\perp$ and $\gamma(X_3-\beta t)$ only

$$A(X) = A^{\mathrm{R}}(X_\perp, \gamma(X_3-\beta t)).$$

We decompose the field A^0 into modes[4] according to

$$A^0(X) = (2\pi)^{-3}\int d\vec{k}(A^+(\vec{k})e^{ikX} + A^-(\vec{k})e^{-ikX}), \tag{4a}$$

where

$$A^{\pm}(\vec{k}) \equiv \int_{X_0=0} d\vec{X}e^{\pm ikX}\frac{\pm\overleftrightarrow{\partial}_0}{2i\omega}A^0(X) \tag{4b}$$

and we get for the density of modes

$$\frac{dN}{d^3k} \equiv (2\pi)^{-3}(2\omega)A^+(\vec{k})A^-(\vec{k}) = (2\pi)^{-3}(2\omega)\frac{1}{\gamma^2}\left(\frac{\omega+\beta k_3}{2\omega}\right)^2|\tilde{A}^{\mathrm{R}}(\vec{k}^*)|^2, \tag{5}$$

where

$$\tilde{A}^{\mathrm{R}}(\vec{k}) \equiv \int d\vec{X}e^{i\vec{k}\vec{X}}A^{\mathrm{R}}(\vec{X}).$$

The corresponding energy distribution for $\beta \to 1$

$$\omega\frac{dN}{d^3k} = (2\pi)^{-3}2k_3^{*2}|\tilde{A}^{\mathrm{R}}(\vec{k}^*)|^2\theta(k_3^*) \tag{6}$$

shows Feynman scaling, but it does not lead to a rising multiplicity

$$\int dN \xrightarrow[\gamma\to\infty]{} \text{const.} \tag{7}$$

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[3] It is easy to see, that for $\beta \to 1$ the resulting Eqs (5) and (6) are independent of the choice of the spacelike hyperplane over which we are integrating.

[4] In the notation of Bogoliubov and Shirkov $A^{\pm} = \sqrt{\frac{(2\pi)^3}{2\omega}}\,a^{+}$.

This is in contrast to the case of a vector field (cf. Eq. (18)). The result can be phrased by saying, that for a scalar source the equivalent quanta are moving together with the source or — speaking classically — that the energy density for the small wave vectors is negligible.

Instead of calculating the mode density, we could have calculated directly the energy density. For the free real scalar field A the energy momentum tensor in configuration space is given by

$$T_{\mu\nu}(X) = \partial_\mu A \partial_\nu A + \tfrac{1}{2}\, g_{\mu\nu}(-\partial_\alpha A \partial^\alpha A + m^2 A^2). \tag{8}$$

The energy distribution with respect to wave vectors k is then[5]

$$\frac{dE}{d^3k} = (2\pi)^{-3}(A^+_{,0}(\vec{k})A^-_{,0}(\vec{k}) + \tfrac{1}{2}\, g_{00}(-A^+_{,\alpha}(\vec{k})A^{-,\alpha}(\vec{k}) + m^2 A^+(\vec{k})A^-(\vec{k}))) + \text{h.c.} \tag{9a}$$

where $A^\pm(\vec{k})$ is given through Eq. (4b) and similar

$$A^\pm_{,\mu}(\vec{k}) \equiv \int_{X_0=0} d\vec{X} e^{\mp ikX} \frac{\pm\overleftrightarrow{\partial}_0}{2i\omega}\, \partial_\mu A(X). \tag{9b}$$

The main assumption now is, that we can calculate the energy density of the interacting field, as if it were a free field: We use Eq. (4b) and (9b) to calculate $A^\pm$ and $A^\pm$ despite the fact, that the field A is no longer a free field.

We now want to express the fields in the boosted[6] frame A (and the derivatives $A_{,\mu}$) by the corresponding fields in the rest frame A^{R} (and $A^{\text{R}}_{,\mu}$)

$$A(X)|_{t=0} = A^{\text{R}}(X_\perp, \gamma X_3),$$

$$A^-(\vec{k}) \to \tilde{A}^{\text{R}}(\vec{k}^*)\theta(k_3^*), \qquad A^-_{,\mu}(\vec{k}) \to 1/\gamma \Lambda^{\mu'}_{\mu} \tilde{A}^{\text{R}}_{,\mu'}(\vec{k}^*)\theta(k_3^*). \tag{10}$$

from Eq. (9) we get

$$\begin{aligned}\frac{dE}{d^3k} &= 1/\gamma^2 \Lambda_0{}^\mu \Lambda_0{}^\nu (2\pi)^{-3}\{\tilde{A}^{\text{R}*}_{,\mu}(\vec{k}^*)\tilde{A}^{\text{R}}_{,\nu}(\vec{k}^*)\\ &\quad + \tfrac{1}{2}\, g_{\mu\nu}(-\tilde{A}^{\text{R}*}_{,\alpha}(\vec{k}^*)\tilde{A}^{\text{R},\alpha}(\vec{k}^*) + m^2 \tilde{A}^{\text{R}*}\tilde{A}^{\text{R}})\}\theta(\vec{k}_3^*) + \text{h.c.}\\ &= 1/\gamma^2 \Lambda_0{}^\mu \Lambda_0{}^\nu T^{\text{R}}_{\mu\nu}(\vec{k}^*)\theta(k_3^*)\end{aligned} \tag{11}$$

in an obvious notation.

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[5] Eq. (9) and the later Eq. (14) can be verified by calculating (with box normalisation) the contribution of one mode

$$A_{\vec{K}}(X) = (A^+_{\vec{K}} e^{ikX} + A^-_{\vec{K}} e^{-ikX})/V$$

to the energy:

$$E_{\vec{K}} = \int d\vec{X}\, T_{00}(A_{\vec{K}}(X) = 1/V(A^+_{,0\vec{K}} A^-_{,0\vec{K}} + \ldots),$$

using

$$1/V \sum_{\vec{k}} \to (2\pi)^{-3} \int d\vec{k}.$$

[6] $\Lambda_{\mu\nu}$ denotes the Lorentz transformation, which connects the rest frame of the static field with the rest frame of the observer.

Therefore

$$\frac{dE}{d^3k} = [T^R_{00}(\vec{k}^*)+T^R_{03}(\vec{k}^*)+T^R_{30}(\vec{k}^*)+T^R_{33}(\vec{k}^*)]\theta(k_3^*).$$

Since

$$T^R_{00} = (2\pi)^{-3} * 2 * \{\underbrace{|\tilde{A}^R_{,0}|^2}_{=0}+\tfrac{1}{2}\, g_{00}(\ldots)\},$$

$$T^R_{33} = (2\pi)^{-3} * 2 * \{\underbrace{|\tilde{A}^R_{,3}|^2}_{=k_3^{*2}|\tilde{A}^R|^2}+\tfrac{1}{2}\, g_{33}(\ldots)\}, \qquad T^R_{03} = T_{30} = 0,$$

we get the same result as Eq. (6) — as expected.

3. *Arbitrary fields*

The above line of argument can easily be generalized to the case of arbitrary fields: let us consider a set of interacting fields $A_\alpha(X)$, where α denotes some multi Lorentz index, (A_α may also be the derivative of the field). A_α is assumed to be of the form

$$A_\alpha(X) = \Lambda_\alpha{}^{\alpha'} A^R_{\alpha'}(X_\perp, \gamma(X_3-\beta t)) \tag{12}$$

where A^R_α, the interacting field in the rest frame, is rotation invariant and time independent. β is close to 1. $\Lambda_\alpha{}^{\alpha'} \equiv \Lambda_{\alpha_1}{}^{\alpha_1'} * \ldots * \Lambda_{\alpha_n}{}^{\alpha_n'}$.

We then assume, that the energy density in a mode $\vec{K}$ can be calculated, as if A_α were a free field. For a free field the energy momentum tensor $T_{\mu\nu}$ is a sum of terms of the form

$$t_{\mu\nu}{}^{\alpha\beta} A_\alpha(X) A_\beta(X)$$

and the energy density in modes is given through the corresponding sum

$$\frac{dE}{d^3k} = \sum_t (2\pi)^{-3} t_{\mu\nu}{}^{\alpha\beta} A^+_\alpha(\vec{k}) A^-_\beta(\vec{k}) + \text{h.c.} \tag{13}$$

$A^\pm$ is defined trough

$$A^\pm(\vec{k}) \equiv \int_{X_0=0} d\vec{X} e^{\mp ikX} \left(\frac{\pm\partial_0}{2i\omega}\right) A(X).$$

Analogous to the case of the scalar field we conclude, that the energy distribution in modes is given by

$$\frac{dE}{d^3k} = 1/\gamma^2 \Lambda_0{}^\mu \Lambda_0{}^\nu T^R_{\mu\nu}(\vec{k}^*)\theta(k_3^*)$$

where

$$T^{R}_{\mu\nu}(\vec{k}^*) \equiv \sum_t (2\pi)^{-3} t_{\mu\nu}{}^{\alpha\beta} \tilde{A}^{R*}_{\alpha}(\vec{k}^*) \tilde{A}^{R}_{\beta}(\vec{k}^*) + \text{h.c.}$$

and therefore

$$\frac{dE}{d^3k} = [T^{R}_{00}(\vec{k}^*) + T^{R}_{03}(\vec{k}^*) + T^{R}_{30}(\vec{k}^*) + T^{R}_{33}(\vec{k}^*)]\theta(k^*_3). \tag{15}$$

We only have to verify

$$t_{\mu\nu}{}^{\alpha'\beta'} \Lambda_{\alpha'}{}^{\alpha} \Lambda_{\beta'}{}^{\beta} = \Lambda_{\mu}{}^{\mu'} \Lambda_{\nu}{}^{\nu'} t_{\mu'\nu'}{}^{\alpha\beta}$$

which is obvious, since t is Lorentz tensor and independent of X. From rotation invariance in the rest frame we conclude, that

$$T^{R}_{00}(\vec{k}) = f_1(\vec{k}^2), \quad T^{R}_{03} + T^{R}_{30} = k_3 f_2(\vec{k}^2), \quad T^{R}_{33} = f_3(\vec{k}^2) + k_3^2 f_4(\vec{k}^2), \tag{16}$$

which proves Eq. (2).

4. *Examples and remarks*

For convenience of the reader we list some examples of fields, which are coupled to static external sources

1) Scalar field

$$(\square + m^2) A^R = j,$$

$$\tilde{A}^R = \tilde{j}/(\vec{k}^2 + m^2),$$

$$T_{\mu\nu}(X) = A_{,\mu} A_{,\nu} + \tfrac{1}{2} g_{\mu\nu}(-A_{,\alpha} A^{,\alpha} + m^2 A^2).$$

The energy density for the field of a fast moving source according to Eq. 6 is

$$\frac{dE}{d^3k} = (2\pi)^{-3} 2k_3^{*2}/(\vec{k}^{*2} + m^2)^2 |\tilde{j}(\vec{k}^*)|^2 \theta(k^*_3).$$

2) Massive vector field

$$(\square + m^2) U^R_\mu = j_\mu, \quad \partial^\mu U^R_\mu = 0,$$

$$j_0(\vec{X}) \equiv j(\vec{X}), j_i = 0,$$

$$\tilde{U}^R_0 = \tilde{j}/(\vec{k}^2 + m^2), \quad \tilde{U}^R_i = 0, \tag{17}$$

$$T_{\mu\nu} = -H_{\mu\alpha} H^{\alpha}_{\nu} + m^2 U_\mu U_\nu + \tfrac{1}{2} g_{\mu\nu}(\tfrac{1}{2} H_{\alpha\beta} H^{\alpha\beta} - m^2 U_\alpha U^\alpha),$$

$$H_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu.$$

The energy density for the field of a fast moving source according to Eq. (15) is

$$\frac{dE}{d^3k} = (2\pi)^{-3} 2(m^2 + k_\perp^2)/(\vec{k}^{*2} + m^2)^2 |j(\vec{k}^*)|^2 \theta(k^*_3). \tag{18}$$

This form of the inclusive distribution is non zero for $K_3^* \to 0$ and leads to a logarithmically rising multiplicity in contrast to the case of the scalar field

$$\int dN \sim \ln \gamma.$$

3) For $m \to 0$, one gets the "equivalent photons" calculated by Weizsäcker and Williams. This is made more explicit by calculating the distribution with respect to the impact parameter

$$\frac{dE}{dk_3 d^2 X_\perp} = 2/(2\pi)\left|\int \partial_\perp U_0^{\mathrm{R}}(|X_\perp| = b, X_3) e^{-ik_3 * X_3} dX_3\right|^2$$

$$= 1/\pi k_3^{*2} K_1^2(bk_3^*) \approx 1/(\pi b^2)$$

for a pointlike source.

5. *Remarks*

1) In calculating the energy distribution through formula (15), we notice that the only difference between scalar and vector fields lies in their opposite sign of T_{33}. This difference (corresponding to attractive/repulsive forces in the scalar/ vector case) leads to the different forms of $dE/d\vec{k}$. It is interesting that a system of charges which are coupled with equal strength to a scalar and vector field simultaneously, has no self stress ($T_{ik} = 0$) and leads to $dE/d\vec{k} = f(\vec{k}^{*2})$.

2) Eq. (15) and (1) show clearly the connection between the field's part of the energy momentum tensor and the inclusive particle distribution. In particular one might conclude from the lack of selfstress and momentum flow for a system of fields — i.e. $T_{ij}^{\mathrm{R}}(\vec{X}) = 0$ and $T_{0i}^{\mathrm{R}}(\vec{X}) = 0$ — that $T_{ij}^{\mathrm{R}}(\vec{k}) = 0$ and $T_{0i}^{\mathrm{R}}(\vec{k}) = 0$ and therefore $dE/d\vec{k} = T_{00}^{\mathrm{R}}(\vec{k}) = f(\vec{k}^{*2})$. This is the case in simple examples but not generally true. The reason is that there is no uniqueness relation between the absolute square of a function and the absolute square of its Fourier transform[7].

3) As mentioned in the introduction, a necessary condition for the applicability of Eq. (1) is high-particle number in the problem under consideration. This is always satisfied for small k_3^* and large γ in the vector case, but doubtfull for scalar fields.

If we calculate the energy density as a function of the impact parameter[8] for scalar (S) and vector (V) Yukawa fields ($m \neq 0$), we get for small k_3 and large $b \cdot m$

$$dE_{\mathrm{V}} \propto ce^{-2bm} dk_3 d^2 b,$$

$$dE_{\mathrm{S}} \propto k_3^{*2} e^{-2bm} dk_3 d^2 b.$$

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[7] Note, that $T^R(\vec{X})$ and $T^R(\vec{k})$ are bilinear in the fields and the Fourier transformed fields respectively.

[8] To get $dE/dk_3 d^2 b$, we again have to use Eq. (13), but without Fourier transformation of the fields with respect to $X_\perp$.

Hence we have for fixed b a (no) plateau in the vector (scalar) case, leading to rising (constant) multiplicity — multiplied by e^{-2bm} in both cases

$$dN_V \propto \ln \gamma e^{-2bm} d^2 b,$$

$$dN_S \propto \quad e^{-2bm} d^2 b.$$

In the scalar case the increasing available field energy — as we boost the source — goes solely into the kinetic energy of the equivalent quanta, whereas in the vector case a small fraction is used to increase the number of particles.

The quasi classical picture applies only up to those impact parameters where the particle density is high enough, such that the fluctuations are small. If we require that the number of quanta in a region of the size of their Compton wave length should be larger than one:

$$\frac{dN}{d^2b} m^{-2} \gtrsim 1,$$

then b_{max} rises only in the first case: $b_{max} \sim \ln \ln \gamma$.

A logarithmically rising cross section (as predicted by Heisenberg) can in this type of picture only be achieved, if the multiplicity rises like γ^{α} ($\alpha > 0$). This was in fact a feature of the "nonlinear strong coupling model" of Heisenberg.

4) The distributions (6) and (7) of equivalent quanta are identical (up to the θ-function) to the inclusive distribution of real quanta, which are radiated from a source which is moving with velocity $\beta \approx 1$ for $t < 0$ and $-\beta$ for $t > 0$.

5) With respect to the applicability of Eq. (2) to present data, we refer to Białas and Stodolsky [4].

The author is grateful to L. Stodolsky for suggesting this problem, for a number of helpfull and stimulating discussions and for critical reading the manuscript. He also thanks T. Jaroszewicz for useful discussions.

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