GENERALLY COVARIANT DEFINITION OF POSITIVE FREQUENCY SOLUTIONS OF THE WAVE EQUATION FOR MASSLESS PARTICLES

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If a solution of the wave equation has a branching point at a null surface, then there are two and only two ways to extend the solution analytically across the null surface: the phase of a null coordinate, assumed zero for positive values of the coordinate, may be chosen $+\pi$ or $-\pi$ for negative values of that coordinate. It is shown that in the Minkowski space-time there exists a sufficiently general set of solutions for which the first choice gives negative frequency solutions while the second gives positive frequency solutions. Moreover, the set of solutions having this property can be defined by a construction prescription which remains meaningful in an arbitrary analytic space-time which does not have closed time-like lines and/or other global peculiarities. This allows giving a generally covariant definition to positive frequency solutions of the wave equation.

1. *Introduction*

Quantum mechanical interpretation of the wave equation becomes possible if the linear space of solutions is divided into two parts, usually called positive frequency solutions and negative frequency solutions. Unfortunately, the principle of division has no meaning at all in a curved space-time. First of all, the positive frequency solutions are in fact the positive energy solutions while energy in a curved space-time is not a well defined concept. Secondly, the positive frequency solutions can be identified as such by means of the Fourier transform which again in a curved space-time is not a well defined concept. Both these difficulties can be reduced to a single one: the distinction is based on a nonlocal property of solutions, which has no counterpart in a curved space-time.

2. *The space-time picture to be associated with the experimental counting of particles*

Some people try to resolve the difficulty by introducing an additional space-time structure, e.g. a congruence of time-like lines [1]. The idea is that these lines represent "counters" and the result of particle counting is somehow related to the state of motion of

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counters. We believe that this idea is misleading and is likely to give unphysical results, as for example creation of pairs in the empty Minkowski space-time.

To perceive the true state of affairs, we have to note that particle counting is an irreversible process. A counter which has detected a single particle is not the same counter anymore. It is true that people are clever enough to make counters which can be repeatedly used, but this is a purely technical circumstance and it tends to obscure the ideal procedure which consists in the following.

We have an unlimited supply of identical counters. Each counter can be used only once; for example each counter is a little bomb which may be triggered (or not) by a coincident particle. Counters can be made coincident with various events. Thus in the process of particle counting no idea of time-like continuity can be traced; we should think rather of a space-time pattern of isolated events.

An objection may be raised that a counter has to have some duration if it is to be "exploded". But this duration is very short; moreover, because of the Lorentz invariance of elementary interactions, the interaction of a counter with radiation does not depend on the state of motion, apart from the trivial flux factor which can be always taken into account.

3. *The phase rule*

Let P be a fixed event, L_+ and L_- resp. the future and past light cone emanating from P and s the geodesic distance from P inside L_+: The quasiclassical wave function of a particle with mass m has the form

$$\exp\,(i \text{ times classical action}) = \exp\,(-ims).$$

s has a branching point at L_+; for example in the Riemannian normal coordinates emanating from P[1]

$$s = \sqrt{(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2}.$$

To choose the phase of s outside L_+ we observe that the wave function should vanish at space-like infinity; this will be the case if

$$\arg\,(s^2) = -\pi \text{ for } s^2 < 0.$$

Inside L_- the wave function should have the form $\exp(ims)$, which will be the case if

$$\arg\,(s^2) = -2\pi$$

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[1] We assume that the space-time is analytic so that the Riemannian normal coordinates can be constructed.

inside the past light cone. Thus, the phase of the square of the geodesic distance from P has to be counted forward in time:

$$\arg(s^2) = \begin{cases} 0 & \text{inside the future light cone,} \\ -\pi & \text{outside the light cone,} \\ -2\pi & \text{inside the past light cone.} \end{cases}$$

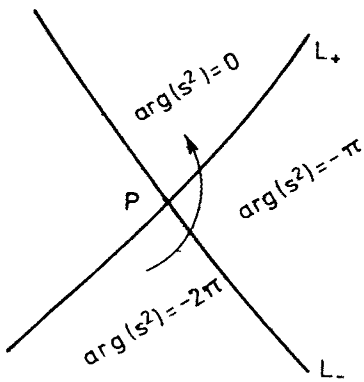

Fig. 1. The argument of s^2, where s is the geodesic distance from P, has to be counted forward in time

4. *The Kirchhoff formula*

Solution of the wave equation inside L_+ is determined by Cauchy data on L_+. In the Minkowski space-time the solution is given by the Kirchhoff formula

$$\varphi(y) = \frac{1}{\pi} \int\limits_{L_+} \frac{d^3x}{2x^0} G_R(y-x) \left[\varphi(x) + x^\mu \frac{\partial \varphi(x)}{\partial x^\mu} \right].$$

It is assumed here that P coincides with the origin of a Cartesian coordinate system x^0, x^1, x^2, x^3, y is the radius vector of a point inside L_+ and

$$G_R(x) = 2\theta(x^0)\delta(xx), \; xx = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2,$$

is the retarded Green function of the wave equation. The expression $\varphi + x^\mu \partial_\mu \varphi$ on L_+ can be calculated if φ is known on L_+. Thus the Kirchhoff formula has the form

$$\varphi(y) = \frac{1}{\pi} \int\limits_{L_+} \frac{d^3x}{2x^0} G_R(y-x) \cdot \mathrm{CD}(x),$$

where $\mathrm{CD}(x)$ is a given function on L_+.

In a curved space-time the Kirchhoff formula takes the form

$$\varphi(y) = \frac{1}{\pi} \int\limits_{L_+} d\mu(x) G_R(y; x) \cdot \mathrm{CD}(x).$$

Here

$$d\mu = \int dx^0 \ldots dx^3 \sqrt{-g}\,\delta(s^2)$$

is the invariant measure on L_+ and $G_R(y; \boldsymbol{x})$ is the retarded Green function of the wave equation, as defined e.g. by DeWitt and Brehme [2]. (Our Green's function equals 4π times that of DeWitt and Brehme.)

5. *Generally covariant definition of positive frequency solutions*

We are now able to introduce two families of solutions of the wave equation with the following properties:

1° both families are defined by a generally covariant construction prescription;

2° they contain a complete set of solutions of the wave equation;

3° in the case of the Minkowski space-time the first family spans the subspace of positive frequency solutions while the second family spans the subspace of negative frequency solutions.

Let σ, ϑ and φ denote internal coordinates on L_+ such that ϑ and φ are constant along each null ray emanating from P while σ is an affine parameter along null rays. If σ and σ' are two affine parameters, then $\sigma' = \sigma F(\vartheta, \varphi)$ where F can be an arbitrary positive function of ϑ and φ. Hence, homogeneity with respect to the affine parameter is a generally covariant property of a function on L_+.

Let us put in the Kirchhoff formula

$$\mathrm{CD}(x) = \sigma^{-(1+C)} f(\vartheta, \varphi),$$

where C is a constant (complex in general) and f an arbitrary function. The solution given by the Kirchhoff formula inside L_+ then has a branching point at L_+ (and at L_- also). To determine the solution outside L_+ we have to choose one out of two distinct possibilities: to count the phase forward or backward in time. It is shown in the Appendix that, in the case of the Minkowski space-time, we obtain positive frequency solutions when the phase is counted forward in time and negative frequency solutions if the phase is counted backward in time. Thus, the difference between positive and negative frequency solutions is reduced to the purely local and geometrically meaningful difference between the two ways of counting phase.

To avoid a possible misunderstanding we have to say the following. We do not mean to say that each solution obtained by application of the phase rule is a positive frequency one. This is not the case even in the flat space-time since, given a positive frequency solution obtained by application of the phase rule, we can always add to it a holomorphic one, e.g. a plane cosinus wave, and the sum, in general, would not be a positive frequency solution anymore. The point we try to make is that there exists a sufficiently general set of solutions for which the nonlocal and noncovariant concept of positive frequency can be replaced by a local and covariant concept, namely the behaviour of discontinuity across a null surface. Moreover, the set of solutions with this property can be defined by a generally covariant construction prescription.

6. *Discussion*

Positive frequency solutions have been treated by several authors, notably by Lichnerowicz [3] and Menski [4]. The most characteristic feature of their treatment is that the positive frequency solutions are defined once and for all, which means physically that the creation of pairs by the gravitational field is not expected. Menski, in fact, states this point of view explicitly.

Hawking [5] in his celebrated paper on pair creation handles a somewhat different problem, similar to the Klein paradox in electrodynamics. On the problem considered here he says that a time dependent gravitational field will create pairs but he does not give any details.

We agree with Hawking's point of view. Granted, however, that the distinction between positive and negative frequencies must be time dependent, it is extremely difficult to see how — in a generally covariant theory — it can fail to be space dependent. We push this point of view to its logical limit making the distinction purely local and event dependent. We think that our treatment is in agreement with the physical nature of particle counting, as described previously. Event dependent distinction is a sort of probabilistic snapshot of the universe, as taken by an observer located at a definite place and in a definite epoch.

7. *The case of massive particles*

Unfortunately, our definition is not applicable for the Klein–Gordon equation. The particular Cauchy data on L_+, chosen by us because of their generally covariant dependence on the affine parameter, would give solutions rising exponentially at space-like infinity. To avoid this, we have to prescribe Cauchy data on L_+ and L_- and to demand exponential vanishing at space-like infinity. However, such Cauchy data cannot be chosen arbitrarily but have to satisfy a consistency condition similar to the one used in the exact theory of diffraction. The difficulty in handling this condition prevented us from obtaining any results for the Klein–Gordon equation. We believe, however, that the phase rule is still the clue to the solution.

APPENDIX

We consider solutions of the wave equation, which can be represented inside L_+ by the integral

$$\varphi(y) = \frac{1}{\pi} \int\limits_{x^0 = |x|} \frac{d^3x}{x^0} \, \delta[(y-x)(y-x)] \, |x|^{-1-C} f\left(\frac{x}{|x|}\right), \qquad y^0 > |y|. \tag{1}$$

For $f = 1$

$$\varphi(y) = \frac{2^C}{C} \, \frac{(y^0-|y|)^{-C}-(y^0+|y|)^{-C}}{|y|}. \tag{2}$$

Consider also the integral

$$\int\limits_{k^0=|\boldsymbol{k}|} \frac{d^3k}{k^0}\, e^{-ikx}|\boldsymbol{k}|^{C-1}, \qquad kx = k^0x^0 - \boldsymbol{k}\cdot\boldsymbol{x}, \tag{3}$$

which is to be understood as the limit of the integral

$$\int\limits_{k^0=|\boldsymbol{k}|} \frac{d^3k}{k^0}\, e^{-ik(x-i\varepsilon)}|\boldsymbol{k}|^{C-1}, \tag{4}$$

where ε is an infinitesimal future oriented time-like vector. The last integral is convergent for $\operatorname{Re} C > -1$ and equal to (we put $x = y$)

$$2\pi\Gamma(C)e^{-i\frac{\pi}{2}(C+1)}\frac{(y^0-|\boldsymbol{y}|)^{-C}-(y^0+|\boldsymbol{y}|)^{-C}}{|\boldsymbol{y}|} \qquad \text{for } y^0 > |\boldsymbol{y}|. \tag{5}$$

Thus we have inside L_+

$$\varphi(y) \overset{\mathrm{df}}{=} \frac{1}{\pi}\int \frac{d^3x}{x^0}\,\delta[(y-x)(y-x)]\,|\boldsymbol{x}|^{-C-1}$$

$$= \frac{2^C e^{i\frac{\pi}{2}(C+1)}}{2\pi C\Gamma(C)} \int\limits_{k^0=|\boldsymbol{k}|} \frac{d^3k}{k^0}\, e^{-iky}|\boldsymbol{k}|^{C-1}. \tag{6}$$

But the integral on the right-hand side is an analytic positive frequency solution of the wave equation. It will be identical with $\varphi(y)$ everywhere provided the latter is extended analytically outside L_+ in accordance with the phase rule. Thus we have established the equivalence of the phase rule and positivity of frequency for $f = 1$.

The case of an arbitrary f can be reduced to the previous one. The operator

$$\frac{1}{i}M_{0i} = y_0\frac{\partial}{\partial y^i} - y_i\frac{\partial}{\partial y^0}, \qquad i = 1, 2, 3,$$

acts on $\varphi(y)$ in the following way

$$\frac{1}{i}M_{0i}\,\frac{1}{\pi}\int \frac{d^3x}{x^0}\,\delta[(y-x)(y-x)]\mathrm{CD}(x)$$

$$= \frac{1}{\pi}\int \frac{d^3x}{x^0}\,\delta[(y-x)(y-x)]x^0\frac{\partial}{\partial x^i}\mathrm{CD}(x). \tag{7}$$

Thus, by repeated application of the operator we can obtain Cauchy data of the form $|\boldsymbol{x}|^{-1-C}S_{i_1 i_2 \ldots i_n}$

where

$$S_{i_1 i_2 \dots i_n} = |x|^{C+1}|x|\partial_{i_1} \dots |x|\partial_{i_n}|x|^{-C-1}.$$

It is easy to prove by induction that for $C \neq 0$ all spherical functions

$$Y_{i_1 i_2 \dots i_n} \stackrel{\text{df}}{=} \partial_{i_1 i_2 \dots i_n} \frac{1}{|x|} \cdot |x|^{n+1}$$

can be expressed by means of $S_{i_1 \dots i_n}$. This means that Cauchy data of the form $|x|^{-C-1}f(x/|x|)$ can be obtained by repeated application of the operator $i^{-1}M_{0i}$ to the spherically symmetric function $|x|^{-C-1}$. Now it is clear that the operator $i^{-1}M_{0i}$ preserves both positivity of frequency and the phase rule. Thus, it is seen that the phase rule is equivalent to the positivity of frequency for the whole class $\mathrm{CD}(x) = |x|^{-C-1}f(x/|x|)$.

A remark on completeness

C can be any complex number for which calculations given in this Appendix are valid. The integral (1) is convergent for any C because the δ-function cuts out both potentially troublesome end points $|x| = 0$ and $|x| = \infty$. In a curved space-time, however, there will be in general a "tail" of Green's function (see the paper by De Witt and Brehme); to make the integral convergent when there is the tail we have to assume that

$$\operatorname{Re} C < 1.$$

The integral (4) is convergent for

$$\operatorname{Re} C > -1.$$

The inequality

$$-1 < \operatorname{Re} C < 1$$

has a simple physical meaning. Suppose that C is real. The functions $\varphi(y)$ are then the eigenfunctions of the operator $\frac{1}{2} M_{\mu\nu}M^{\mu\nu}$, where

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

and eigenvalues are easily seen to be C^2-1:

$$\tfrac{1}{2} M_{\mu\nu}M^{\mu\nu}\varphi(y) = (C^2-1)\varphi(y).$$

Now, we have classically for a massless particle

$$\tfrac{1}{2} M_{\mu\nu}M^{\mu\nu} = [xxpp-(px)^2] = -(px)^2 \leqslant 0.$$

Therefore

$$C^2-1 \leqslant 0$$

and

$$-1 \leqslant C \leqslant 1.$$

It is not necessary, however, to take C real. We shall obtain a complete set of functions taking C, for example, along any straight line

$$-1 < \operatorname{Re} C = \text{const} \neq 0 < 1.$$

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