BEL-ROBINSON SUPERENERGY TENSOR AND THE TETRAD DESCRIPTION OF GRAVITATIONAL FIELD

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(Received October 5, 1976)

A choice of tetrad field corresponding to the normal Riemannian coordinates (NRC) and insertion of the resulting expressions for the Ricci rotation coefficients etc. into conserved vector densities from the Noether theorem (in its tetrad representation) enables the recovery of the full set of the Bel-Robinson superenergy tensor components in vacuum as the only lowest order terms in decompositions of these vector densities in NRC. Hence the Bel-Robinson tensor as a whole can be interpreted as a relative energy-momentum-stress quantity of gravitational field.

1. Introduction

The gravitational field can be treated similarly to the electromagnetic one. However, two approaches exist, differently reflecting the nature of this analogy. The first one is based upon the description of gravitational-and-inertial fields relative to reference frames (possibly, expressed directly in terms of bi-metrical formalism or, in its simplest modification in the weak field approximation). The second approach uses a generally covariant description of gravitation without any explicit introduction of kinematic quantities (kinematic "fields"). In the latter case the role of the gravitational field strength is played by the Riemann-Christoffel tensor $R_{\mu\nu\lambda\sigma}$, but in contrast to the electromagnetic field where $F_{\mu\nu}$ is the "absolute" field strength, one can regard the Riemann-Christoffel tensor only as a quantity characterizing the relative strength of the gravitational field. Existence or absence of the "true" gravitational field can be stated indeed when the geodesic equation is considered (world lines of two freely falling particles). Geodesic deviation equation has the form

$$\frac{D^2\eta^\mu}{du^2} = R^{\mu}_{\cdot\,\lambda\sigma\kappa}\tau^\lambda\tau^\sigma\eta^\kappa, \tag{1}$$

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where η^{μ} is the deviation vector, τ^{μ} the tangent vector to the world line of one of the particles. The interpretation of $R_{\mu\nu\lambda\sigma}$ as (relative) gravitational field strength justifies the use of quasi Maxwellian equations for this tensor [1]. This analogy between gravitational and electromagnetic fields can be traced still further, on the level of energy-momentum tensors. In the case of the electromagnetic field the energy-momentum tensor is of the form

$$T^{\mu\nu}_{\text{em}} = -\frac{1}{8\pi}(F^{\mu\alpha}F^{\nu}_{\cdot\alpha}+F*^{\mu\alpha}F*^{\nu}_{\cdot\alpha}) \equiv -\frac{1}{4\pi}(F^{\mu\alpha}F^{\nu}_{\cdot\alpha}-\tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}), \tag{2}$$

where $F_*{}^{\mu\nu} = (1/2)E^{\mu\nu\lambda\varrho}F_{\lambda\varrho}$, $E^{\mu\nu\lambda\varrho}$ being the axial tensor of Levi-Cività. For the gravitational field a quantity quadratic in the (relative) field strength also arises, and this is the Bel--Robinson superenergy tensor [2],

$$T^{\alpha\beta\lambda\mu} = \tfrac{1}{2}(R^{\alpha\varrho\lambda\sigma}R^{\beta\cdot\mu\cdot}_{\cdot\varrho\cdot\sigma}+R*^{\alpha\varrho\lambda\sigma}R*^{\beta\cdot\mu\cdot}_{\cdot\varrho\cdot\sigma}) \equiv \tfrac{1}{2}(R^{\alpha\varrho\lambda\sigma}R^{\beta\cdot\mu\cdot}_{\cdot\varrho\cdot\sigma}+R^{\beta\varrho\lambda\sigma}R^{\alpha\cdot\mu\cdot}_{\cdot\varrho\cdot\sigma}-\tfrac{1}{8}g^{\alpha\beta}g^{\lambda\mu}$$

$$\times R^{\kappa\varrho\sigma\tau}R_{\kappa\varrho\sigma\tau}), \qquad R*^{\alpha\varrho\lambda\sigma} = \tfrac{1}{2}E^{\alpha\varrho\gamma\delta}R_{\gamma\delta\cdot\cdot}^{\cdot\cdot\lambda\sigma}. \tag{3}$$

The superenergy tensor has the properties

$$g_{\alpha\beta}T^{\alpha\beta\lambda\mu} = 0; \qquad T^{\alpha\beta\lambda\mu} = T^{(\alpha\beta\lambda\mu)}; \qquad T^{\alpha\beta\lambda\mu}{}_{;\alpha} = 0, \tag{4}$$

similar to those of $T^{\alpha\beta}_{\text{em}}$ (where the conservation is considered in the absence of sources too). The Bel–Robinson superenergy tensor was connected with the Einstein pseudotensor by Garecki [3] and with the Papapetrou pseudotensor by Yefremov [4]. In Refs [3–5] the fact was noticed that the superenergy tensor is a relative characterization of the gravitational field energy.

2. *Normal Riemannian coordinates and the tetrad formalism*

Normal Riemannian coordinates (NRC) with the origin at a point P_0 are defined as follows. $x^{\mu} = \xi^{\mu}u$ for an arbitrary given point $P = P(x^{\mu})$, where ξ^{μ} is a tangent vector of the geodesic connecting the point P with the origin, u is a canonical parameter along the geodesic. The definition of these coordinates presumes that every point P can be connected with the origin at P_0 by a unique geodesic. Hence the Riemannian coordinates fail to be definite outside the region of the boundaries where geodesics begin crossing. At the origin of NRC the relations hold:

$$\overset{0}{g}_{\mu\nu} = \delta_{\mu\nu}, \qquad \overset{0}{\Gamma}{}^{\alpha}_{\mu\nu} = 0,$$

$$\delta_{\mu\nu} = \operatorname{diag}(+1, -1, -1, -1), \tag{5}$$

the superscript "0" meaning that the corresponding quantity is taken at the origin. Moreover, in the applicability region of NRC the relation

$$\Gamma^{\alpha}_{\mu\nu}x^{\mu}x^{\nu} = 0 \tag{6}$$

holds [6]. The metric tensor and Christoffel symbols can be represented in the neighbourhood of the origin in the form [6],

$$g_{\mu\nu} = \delta_{\mu\nu} + \tfrac{1}{3} \overset{0}{R}_{\mu\alpha\beta\nu} x^{\alpha} x^{\beta} + \ldots,$$

$$\Gamma^{\alpha}_{\mu\nu} = -\tfrac{2}{3} \overset{0}{R}{}^{\alpha}_{\cdot(\mu\nu)\lambda} x^{\lambda} + \ldots . \tag{7}$$

We need now expansions of the Ricci rotation coefficients $\Phi_{\mu\nu\lambda} = -\Phi_{\mu\lambda\nu}$ in NRC. As usual,

$$\Phi_{\mu\nu\lambda} = g^{(\alpha)}{}_{\lambda} g_{(\alpha)\nu;\mu}, \tag{8}$$

where the four vectors $g_{(\alpha)}{}^{\lambda}$ form an orthonormal basis,

$$g_{\mu\nu} = g_{(\alpha)\mu} g^{(\alpha)}{}_{\nu}, \qquad g_{(\alpha)}{}^{\nu} g_{(\beta)\nu} = \delta_{\alpha\beta}. \tag{9}$$

It is easy to check that

$$\Phi_{\mu\nu\lambda;\kappa} - \Phi_{\kappa\nu\lambda;\mu} = R_{\lambda\nu\mu\kappa} + 2\Phi_{[\kappa|\lambda|}{}^{\alpha}\Phi_{\mu]\nu\alpha}, \tag{10}$$

where $[\kappa|\lambda|\cdot\mu]$ means antisymmetrization in the indices κ, μ. Since ξ^{μ} satisfies the parallel transport equation,

$$(D/du)\xi^{\mu} = 0, \tag{11}$$

the Ricci rotation coefficients are submitted to relations

$$\xi^{\mu}\Phi_{\mu\nu\lambda} = 0; \quad \frac{D}{du}(\xi^{\mu}\Phi_{\mu\nu\lambda}) = 0;$$

$$\frac{D^2}{du^2}(\xi^{\mu}\Phi_{\mu\nu\lambda}) = 0; \quad \ldots . \tag{12}$$

It follows from (12) that at the origin the relations

$$\underset{(\alpha_1\alpha_2\ldots\alpha_n)}{\mathrm{S}} \overset{0}{\Phi}_{\alpha_1\nu\lambda;\alpha_2;\ldots;\alpha_n} = 0 \tag{13}$$

hold (the vector ξ^{μ} was arbitrary). Here the semicolon denotes covariant differentiation and $\underset{(\alpha_1\alpha\ldots\alpha_n)}{\mathrm{S}}$ a sum of cyclic permutations of the indices shown in the brackets. From the relations (13) obviously follows the formula

$$(D^n/du^n)\overset{0}{\Phi}_{\mu\nu\lambda} + (D^{n-1}/du^{n-1})(\xi^{\kappa}\overset{0}{\Phi}_{\kappa\nu\lambda;\mu})$$

$$+ \sum_{i=2}^{n} (D^{n-i}/du^{n-i})(\xi^{\kappa}\overset{0}{\Phi}_{\kappa\nu\lambda;\alpha_1;\ldots;\alpha_{i-1};\mu}\xi^{\alpha_1}\ldots\xi^{\alpha_{i-1}}) = 0. \tag{14}$$

With the help of (12) we get from (10)

$$(D^n/du^n)\overset{0}{\Phi}_{\mu\nu\lambda} - (D^{n-1}/du^{n-1})(\xi^{\mu}\overset{0}{\Phi}_{\kappa\nu\lambda;\mu}) = (D^{n-1}/du^{n-1})(\overset{0}{R}_{\lambda\nu\mu\kappa}\xi^{\kappa}). \tag{15}$$

Consider the derivative $(D^{n-i}/du^{n-i})(\xi^\kappa \overset{0}{\Phi}_{\kappa\nu\lambda;\alpha_1;\ldots;\alpha_{i-1};\mu}\xi^{\alpha_1}\ldots\xi^{\alpha_{i-1}})$. From (10) and (12) it follows that

$$(D^{n-i}/du^{n-i})(\xi^\kappa \overset{0}{\Phi}_{\kappa\nu\lambda;\alpha_1;\ldots;\alpha_{i-1};\mu}\xi^{\alpha_1}\ldots\xi^{\alpha_{i-1}})$$

$$= (D^{n-i+1}/du^{n-i+1})(\xi^\kappa \overset{0}{\Phi}_{\kappa\nu\lambda;\alpha_1;\ldots;\alpha_{i-2};\mu}\xi^{\alpha_1}\ldots\xi^{\alpha_{i-1}})$$

$$+(D^{n-i}/du^{n-i})[\underset{(\alpha_1\ldots\alpha_{i-1}\mu)}{\mathfrak{S}}(\overset{0}{\Phi}_{\alpha_1\nu\lambda;\alpha_2;\ldots;\alpha_{j-1};\hat{\alpha}_j;\ldots;\alpha_{i-1}}\overset{0}{R}{}^{\alpha_{j-1}}{}_{\alpha_j\varrho\mu}\xi^\varrho\xi^{\alpha_1}\ldots\xi^{\alpha_{i-1}})]$$

$$= (D^{n-i+1}/du^{n-i+1})(\xi^\kappa \overset{0}{\Phi}_{\kappa\nu\lambda;\alpha_1;\ldots;\alpha_{i-2};\mu}\xi^{\alpha_1}\ldots\xi^{\alpha_{i-2}})+\sum_{k=0}^{n-i}C^k_{n-i}\underset{(\mu\alpha_1\ldots\alpha_{i+k-1})}{\mathfrak{S}}$$

$$\times(\overset{0}{\Phi}_{\alpha_1\nu\lambda;\alpha_2;\ldots;\alpha_{j-1};\hat{\alpha}_j;\ldots;\alpha_{i+k-1}}\xi^{\alpha_1}\ldots\xi^{\alpha_{i+k-1}})(D^{n-i-k}/du^{n-i-k})$$

$$\times(\overset{0}{R}{}^{\alpha_{j-1}}{}_{\alpha_j\varrho\mu}\xi^\varrho\xi^{\alpha_j}) = (D^{n-i+1}/du^{n-i+1})(\xi^\kappa \overset{0}{\Phi}_{\kappa\nu\lambda;\alpha_1;\ldots;\alpha_{i-2};\mu}\xi^{\alpha_1}\ldots\xi^{\alpha_{i-2}}), \tag{16}$$

so that the formula (14) can be rewritten as follows,

$$(D^n/du^n)\overset{0}{\Phi}_{\mu\nu\lambda}+(D^{n-1}/du^{n-1})(\xi^\kappa\overset{0}{\Phi}_{\kappa\nu\lambda;\mu})+(n-1)(D^{n-2}/du^{n-2})(\xi^\kappa\xi^\varrho\overset{0}{\Phi}_{\kappa\nu\lambda;\varrho;\mu}) = 0. \tag{17}$$

A further rearrangement of (17) gives

$$(D^n/du^n)\overset{0}{\Phi}_{\mu\nu\lambda}+n(D^{n-1}/du^{n-1})(\xi^\kappa\overset{0}{\Phi}_{\kappa\nu\lambda;\mu})$$

$$+(n-1)(D^{n-2}/du^{n-2})(\overset{0}{\Phi}_{\alpha\nu\lambda}\overset{0}{R}{}^{\alpha}{}_{\kappa\varrho\mu}\xi^\kappa\xi^\varrho) = 0. \tag{18}$$

Now formulas (18) and (15) lead to the recurrent relation

$$(D^n/du^n)\overset{0}{\Phi}_{\mu\nu\lambda} = \frac{n}{n+1}(D^{n-1}/du^{n-1})(\overset{0}{R}_{\lambda\nu\mu\kappa}\xi^\kappa)$$

$$-\frac{n-1}{n+1}(D^{n-2}/du^{n-2})(\overset{0}{\Phi}_{\alpha\nu\lambda}\overset{0}{R}{}^{\alpha}{}_{\kappa\varrho\mu}\xi^\kappa\xi^\varrho). \tag{19}$$

In the neighbourhood of the origin in NRC the coefficients $\Phi_{\mu\nu\lambda}$ can be represented as series,

$$\Phi_{\mu\nu\lambda}(\xi^\mu, u) = \overset{0}{\Phi}_{\mu\nu\lambda}+u\,d\overset{0}{\Phi}_{\mu\nu\lambda}/du+\tfrac{1}{2}u^2d^2\overset{0}{\Phi}_{\mu\nu\lambda}/du^2+\ldots \tag{20}$$

which can be trivially rewritten, since $x^\mu = \xi^\mu u$ and $d/du = \xi^\mu\partial/\partial x^\mu$, as

$$\Phi_{\mu\nu\lambda}(x) = \overset{0}{\Phi}_{\mu\nu\lambda}+x^\varrho\overset{0}{\Phi}_{\mu\nu\lambda,\varrho}+\tfrac{1}{2}x^\varrho x^\sigma\overset{0}{\Phi}_{\mu\nu\lambda,\varrho,\sigma}+\ldots. \tag{21}$$

The representation (20) is sometimes more convenient. We connect ordinary derivatives (d/du) and absolute derivatives (D/du) with the help of the covariant differentiation definition,

$$(D/du)\Phi_{\mu\nu\lambda} = (d/du)\Phi_{\mu\nu\lambda}-\Phi_{\alpha\nu\lambda}\Gamma^\alpha_{\mu\varrho}\xi^\varrho-2\Phi_{\mu\alpha[\lambda}\Gamma^\alpha_{\nu]\varrho}\xi^\varrho. \tag{22}$$

From the relations (19) we get the following expressions of absolute derivatives of the Ricci rotation coefficients, up to the 3rd order,

$$(D/du)\overset{0}{\Phi}_{\mu\nu\lambda} = \tfrac{1}{2}\overset{0}{R}_{\lambda\nu\mu\kappa}\xi^{\kappa}; \quad (D^2/du^2)\overset{0}{\Phi}_{\mu\nu\lambda} = \tfrac{2}{3}\overset{0}{R}_{\lambda\nu\mu\kappa;\varrho}\xi^{\kappa}\xi^{\varrho};$$

$$(D^3/du^3)\overset{0}{\Phi}_{\mu\nu\lambda} = \tfrac{3}{4}\overset{0}{R}_{\lambda\nu\mu\kappa;\varrho;\sigma}\xi^{\kappa}\xi^{\varrho}\xi^{\sigma} - \tfrac{1}{2}\overset{0}{R}_{\lambda\nu\alpha\kappa}\overset{0}{R}{}^{\alpha}_{\cdot\varrho\sigma\mu}\xi^{\kappa}\xi^{\varrho}\xi^{\sigma}. \tag{23}$$

Now from (7), (23), and (20) we finally get the desired expansion of the Ricci rotation coefficients in NRC (up to terms cubic in x^{μ})

$$\Phi_{\mu\nu\lambda} = \left(\frac{1}{2!}\right)\overset{0}{R}_{\lambda\nu\mu\kappa}x^{\kappa} + \left(\frac{2}{3!}\right)\overset{0}{R}_{\lambda\nu\mu\kappa;\varrho}x^{\kappa}x^{\varrho}$$

$$+\left(\frac{1}{3!}\right)(\tfrac{3}{4}\overset{0}{R}_{\lambda\nu\mu\kappa;\varrho;\sigma} + \overset{0}{R}_{\mu\kappa\cdot[\lambda}^{\cdot\cdot\alpha}\overset{0}{R}_{\nu]\sigma\alpha\varrho} - \tfrac{1}{4}\overset{0}{R}_{\lambda\nu\alpha\varrho}\overset{0}{R}{}^{\alpha}_{\cdot\kappa\sigma\mu})x^{\kappa}x^{\varrho}x^{\sigma}. \tag{24}$$

3. The Noether theorem and the Bel-Robinson superenergy tensor

The Noether theorem can be based on "invariance" (strictly speaking, the property to be a scalar density) of the Lagrangian density $\mathfrak{L}$ under infinitesimal transformations [1] $x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}$. Here ξ^{μ} is an arbitrary vector, and ε an infinitesimal parameter. In order to get conservation laws with proper physical sense, one has to connect ξ^{μ} with some meaningful operation (e.g., time translation relative to a given reference frame or symmetry property of space time, etc.). This condition leads to a weak differential conservation law

$$\mathfrak{w}^{\alpha}{}_{,\alpha} = 0 \tag{25}$$

where

$$\mathfrak{w}^{\alpha} = \mathfrak{t}_{g\sigma}^{\ \alpha}\xi^{\sigma} - \mathfrak{M}_{g\sigma}^{\ \alpha\tau}\xi^{\sigma}{}_{,\tau}$$

(in absence of gravitational field sources), $\mathfrak{t}_{g\sigma}^{\ \alpha}$ being the standard canonical pseudotensor density of the gravitational field,

$$\mathfrak{M}_{g\sigma}^{\ \alpha\tau} = \frac{\partial\mathfrak{L}_g}{\partial\mathscr{A}_{B,\alpha}}\mathscr{A}_B|_{\sigma}^{\tau}, \quad \mathscr{A}_B|_{\sigma}^{\tau} = \frac{\partial}{\partial\varepsilon\xi^{\sigma}{}_{,\tau}}(\mathscr{A}'_B(x') - \mathscr{A}_B(x)), \tag{26}$$

the "generalized spin density" of the gravitational field [1] and $\mathfrak{L}_g$ the gravitational Lagrangian density. We take the latter in the form

$$\mathfrak{L}_{g,\,\mathrm{tetr.}} = \frac{\sqrt{-g}}{2\kappa}[\Phi_{\sigma\cdot\cdot}^{\cdot\nu\tau}\Phi_{\nu\cdot\tau}^{\cdot\sigma\cdot} - \Phi_{\nu\cdot\cdot}^{\cdot\nu\tau}\Phi_{\sigma\cdot\tau}^{\cdot\sigma\cdot}] \tag{27}$$

which follows from the Lagrangian density $\mathfrak{L}_g = \sqrt{-g}\,R/2\kappa$ when the second derivatives of potentials (tetrad vectors) are covariantly excluded in a divergence term. Inserting $\xi^{\mu} = g_{(\beta)}{}^{\mu}$ we get [7]

$$\mathfrak{w}_{(\beta)}{}^{\alpha} = \frac{\sqrt{-g}}{2\kappa}[\Phi_{\nu\cdot\cdot}^{\cdot\alpha\sigma}\Phi_{\mu\cdot\sigma}^{\cdot\nu\cdot} - \Phi_{\mu\cdot\cdot}^{\cdot\alpha\sigma}\Phi_{\nu\cdot\sigma}^{\cdot\nu\cdot} + \Phi_{\nu\cdot\cdot}^{\cdot\tau\alpha}\Phi_{\tau\cdot\mu}^{\cdot\nu\cdot} - \Phi_{\nu\cdot\cdot}^{\cdot\nu\alpha}\Phi_{\tau\cdot\mu}^{\cdot\tau\cdot} + \Phi_{\nu\cdot\cdot}^{\cdot\nu\tau}\Phi_{\tau\cdot\mu}^{\cdot\alpha\cdot}$$

$$-\tfrac{1}{2}\delta^{\alpha}_{\mu}(\Phi_{\sigma\cdot\cdot}^{\cdot\nu\tau}\Phi_{\nu\cdot\tau}^{\cdot\sigma\cdot} - \Phi_{\nu\cdot\cdot}^{\cdot\nu\tau}\Phi_{\sigma\cdot\tau}^{\cdot\sigma\cdot})]g_{(\beta)}{}^{\mu}. \tag{28}$$

In NRC we have

$$\mathfrak{w}_{(\beta)}{}^{\alpha} = \overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha} + u d\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha}/du + \tfrac{1}{2}u^2 d^2\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha}/du^2 + \dots .$$

In accordance with the expansion (24) we find that the 0th and 1st order terms vanish, i.e.

$$\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha} = 0, \quad d\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha}/du = 0.$$

The further application of (24) and the expression of the superenergy tensor (3 up to term cubic in x^{μ},

$$\mathfrak{w}_{(\beta)}{}^{\alpha} = (1/2\kappa)\left(\tfrac{1}{2}\overset{0}{T}{}^{\alpha}_{\cdot\beta\varrho\sigma}x^{\varrho}x^{\sigma} + \tfrac{1}{3}\overset{0}{T}{}^{\alpha}_{\cdot\beta\varrho\sigma;\tau}x^{\varrho}x^{\sigma}x^{\tau}\right).$$

A study of the recurrent relation (19) and the definition of the contravariant vector $\mathfrak{w}^{\alpha}$ results in the expression

$$(D^n/du^n)\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha} = \frac{n-1}{2}\left(\overset{0}{T}{}^{\alpha}_{\cdot\beta\nu_1\nu_2;\nu_3;\dots;\nu_n} + \overset{0}{\theta}{}^{\alpha}_{\cdot\beta\nu_1\nu_2\nu_3\dots\nu_n}\right)\xi^{\nu_1}\xi^{\nu_2}\xi^{\nu_3}\dots\xi^{\nu_n},$$

where the last term in the brackets contains linear combinations of products of the Ri Christoffel tensor components and their absolute derivatives up to the order $(n-$ expressible through the Bel–Robinson tensor or its derivatives,

$$\overset{0}{\theta}{}^{\alpha}_{\cdot\beta\nu_1\dots\nu_n} = \begin{cases} = 0, & n < 4, \\ \neq 0, & n \geqslant 4. \end{cases}$$

The transition from absolute derivatives (D^n/du^n) to the ordinary ones, (d^n/du^n), formed with the help of relation

$$(D/du)\mathfrak{w}_{(\beta)}{}^{\alpha} = (d/du)\mathfrak{w}_{(\beta)}{}^{\alpha} + \mathfrak{w}_{(\beta)}{}^{\varrho}\Gamma_{\varrho\sigma}{}^{\alpha}\xi^{\sigma} - \mathfrak{w}_{(\beta)}{}^{\alpha}\,\frac{d\sqrt{-g}}{du}\Big/\sqrt{-g}\,,$$

since $\mathfrak{w}_{(\beta)}{}^{\alpha}$ is a vector density.

Hence, by means of using expressions (31) and (32), $\mathfrak{w}_{(\beta)}{}^{\alpha}$ can be represe NRC by the series

$$\mathfrak{w}_{(\beta)}{}^{\alpha} = \frac{1}{2\kappa}\Bigg[(\tfrac{1}{2})\overset{0}{T}{}^{\alpha}_{\cdot\beta\varrho\sigma}x^{\varrho}x^{\sigma} + (\tfrac{1}{3})\overset{0}{T}{}^{\alpha}_{\cdot\beta\varrho\sigma;\tau}x^{\varrho}x^{\sigma}x^{\tau}$$

$$+ \sum_{n=4}^{\infty}\frac{n-1}{n!}\left(\overset{0}{T}{}^{\alpha}_{\cdot\beta\nu_1\nu_2;\nu_3;\dots;\nu_n} + \overset{0}{\tilde{\theta}}{}^{\alpha}_{\cdot\beta\nu_1\nu_2\nu_3\dots\nu_n}\right)x^{\nu_1}\dots x^{\nu_n}\Bigg],$$

$$\overset{0}{\tilde{\theta}}{}^{\alpha}_{\cdot\beta\nu_1\dots\nu_n} = \overset{0}{\theta}{}^{\alpha}_{\cdot\beta\nu_1\dots\nu_n} + \overset{0}{\psi}{}^{\alpha}_{\cdot\beta\nu_1\dots\nu_n},$$

where $\overset{0}{\psi}{}^{\alpha}_{\cdot\beta\nu_1\ldots\nu_n}$ consists of summands which we get by exchange of $(d^n/du^n)\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha}$ by $(D^n/du^n)\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha}$ as a consequence of (32) and formula

$$(d^n/du^n)\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha} = (D^n/du^n)\overset{0}{\mathfrak{w}}_{(\beta)}{}^{\alpha} + \overset{0}{\psi}{}^{\alpha}_{\cdot\beta\nu_1\ldots\nu_n}\xi^{\nu_1}\ldots\xi^{\nu_n}$$

resulting from (32). Here $\overset{0}{\psi}{}^{\alpha}_{\cdot\beta\nu_1\ldots\nu_n}$ differ from zero only if $n \geqslant 4$.

The expansion we derived here for the conserved vector density $\mathfrak{w}_{(\beta)}{}^{\alpha}$ enables one to interpret the Bel-Robinson superenergy tensor as a relative characteristics (meaning comparison of situations existing in nearby points) of the energy, momentum and stress densities for gravitational field in vacuum. It is remarkable that this conclusion concerns not a part of the superenergy tensor components, as it was previously claimed (see [3]), but all of them simultaneously.

REFERENCES

[1] N. V. Mitskiévič, *Physical Fields in General Relativity*, Nauka, Moscow 1969 (in Russian).
[2] V. D. Zakharov, *Gravitational Waves in the Einsteinian Theory of Gravity*, Nauka, Moscow 1972 (in Russian).
[3] J. Garecki, *Acta Phys. Pol.* **B4**, 347 (1973).
[4] A. P. Yefremov, *Acta Phys. Pol.* **B6**, 667 (1975).
[5] F. A. E. Pirani, *Phys. Rev.* **105**, 1089 (1957).
[6] A. Z. Petrov, *New Methods in General Relativity Theory*, Nauka, Moscow 1965 (in Russian).
[7] N. V. Mitskiévič, Theodoro M. Ribeiro, *Zh. Eksp. Teor. Fiz.* **56**, 954 (1969).