A SYMPLECTIC FORMULATION OF RELATIVISTIC PARTICLE DYNAMICS

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Particle mechanics is formulated in terms of symplectic relations and infinitesimal symplectic relations. Generating functions of symplectic relations are shown to be classical counterparts of Green's functions of wave mechanics.

1. Introduction

Canonical formulations of nonrelativistic particle dynamics are based on concepts of symplectic geometry such as groups of canonical transformations and Hamiltonian vector fields. These concepts are not well suited to the purpose of describing the dynamics of relativistic particles. A generalization is obtained by noticing that canonical transformations and Hamiltonian vector fields are special cases of Lagrangian submanifolds of suitably constructed symplectic manifolds. The use of Lagrangian submanifolds in canonical descriptions of dynamics offers several advantages. A unified approach to Lagrangian and Hamiltonian formulations is obtained and the relation between these formulations is better undestood [3, 6]. Relativistic and nonrelativistic dynamics can be described within the same conceptual framework. The transition from particle dynamics to field dynamics is simplified.

In the present note we reformulate standard nonrelativistic dynamics in the general framework of Lagrangian submanifolds. Subsequently we apply this framework, to relativistic dynamics. The use of generating functions for describing Lagrangian submanifolds is emphasized. A wave mechanical interpretation of generating functions is given. Only particles moving in a given external field are considered. The note is written in the language of local coordinates. Global definitions of the concepts used in this note can be found in other publications [3–6].

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Lagrangian submanifolds are defined in Section 2 following a brief review of symplectic geometry. Generating functions of Lagrangian submanifolds are introduced in Section 3. Applications to particle dynamics are discussed in Section 4, 5 and 6. In Section 7 the relation of generating functions to objects used in wave mechanics is demonstrated.

2. *Symplectic manifolds, Lagrangian submanifolds, canonical relations*

A *symplectic manifold* (P, ω) is a manifold P with coordinates (x^κ) and a differential form

$$\omega = \tfrac{1}{2}\, \omega_{\kappa\lambda} dx^\kappa \wedge dx^\lambda \tag{2.1}$$

such that the covariant bivector field $\omega_{\kappa\lambda}$ satisfies conditions:

$$\omega_{\kappa\lambda} u^\lambda = 0 \quad \text{implies} \quad u^\lambda = 0 \tag{2.2}$$

and

$$\partial_{[\kappa}\omega_{\lambda\mu]} = 0. \tag{2.3}$$

The dimension of P is even and will be denoted by $2n$. *Darboux's theorem* states that there exist (locally) coordinates (q^i, p_j), $i, j = 1, \ldots, n$ such that

$$\omega = dp_i \wedge dq^i. \tag{2.4}$$

Due to the algebraic condition (2.2) there is a contravariant bivector field $\omega^{\kappa\lambda}$ satisfying

$$\omega_{\kappa\lambda}\omega^{\lambda\mu} = \delta^\mu_\kappa. \tag{2.5}$$

The differential condition (2.3) is equivalent to

$$\omega^{\nu[\kappa}\partial_\nu \omega^{\lambda\mu]} = 0. \tag{2.6}$$

The contravariant object $\omega^{\kappa\lambda}$ is used to define the *Poisson bracket*

$$\{f, g\} = \omega^{\kappa\lambda}\partial_\kappa f \partial_\lambda g \tag{2.7}$$

of two functions f and g on P. The *Jacobi identity*

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \tag{2.8}$$

is satisfied as a consequence of (2.6).

An active transformation

$$x'^\kappa = \phi^\kappa(x^\lambda) \tag{2.9}$$

is called a *canonical transformation* if

$$\omega_{\kappa\lambda}(\phi^\mu(x^\nu))\partial_\mu\phi^\kappa\partial_\nu\phi^\lambda = \omega_{\mu\nu}(x^\kappa). \tag{2.10}$$

A vector field X^κ is called an *infinitesimal canonical transformation* if

$$\pounds_X \omega_{\kappa\lambda} = X^\mu \partial_\mu \omega_{\kappa\lambda} + \omega_{\kappa\mu}\partial_\lambda X^\mu + \omega_{\mu\lambda}\partial_\kappa X^\mu = 0. \tag{2.11}$$

Combining (2.11) with (2.3) we obtain

$$\partial_{[\kappa}(\omega_{\lambda]\mu}X^{\mu}) = 0. \tag{2.12}$$

If f is a function on P then

$$X^{\kappa} = -\omega^{\kappa\lambda}\partial_{\lambda}f \tag{2.13}$$

is an infinitesimal canonical transformation. Conversely if X^{κ} is an infinitesimal canonical transformation then there exist (locally) functions f such that (2.13) is satisfied. If $X^{\kappa} = -\omega^{\kappa\mu}\partial_{\mu}f$ and $Y^{\kappa} = -\omega^{\kappa\mu}\partial_{\mu}g$ then

$$X^{\mu}\partial_{\mu}Y^{\kappa} - Y^{\mu}\partial_{\mu}X^{\kappa} = -\omega^{\kappa\mu}\partial_{\mu}\{f, g\}. \tag{2.14}$$

Eq. (2.13) defines a homomorphism of the Lie-algebra of functions with respect to Poisson-brackets (Poisson algebra) onto the Lie-algebra of infinitesimal canonical transformations.

A one-parameter group

$$x'^{\kappa} = \phi^{\kappa}(t, x^{\lambda}), \tag{2.15}$$

$$\phi^{\kappa}(0, x^{\lambda}) = x^{\kappa}, \tag{2.16}$$

$$\phi^{\kappa}(t, \phi^{\lambda}(t', x^{\mu})) = \phi^{\kappa}(t+t', x^{\mu}) \tag{2.17}$$

of canonical transformations defines an infinitesimal canonical transformation X^{κ} such that

$$X^{\kappa}(\phi^{\lambda}(t, x^{\mu})) = \frac{\partial}{\partial t}\phi^{\kappa}(t, x^{\mu}). \tag{2.18}$$

Conversely if equations (2.18) are integrated with the initial condition (2.16) then the result is a (local) group of (local) canonical transformations.

A submanifold N of P of dimension k described by

$$x^{\kappa} = \xi^{\kappa}(t^{\alpha}), \qquad \alpha = 1, \ldots, k \tag{2.19}$$

is called an *isotropic submanifold* of (P, ω) if

$$\omega|N = \tfrac{1}{2}\,\omega_{\kappa\lambda}\frac{\partial\xi^{\kappa}}{\partial t^{\alpha}}\frac{\partial\xi^{\lambda}}{\partial t^{\alpha}}dt^{\alpha}\wedge dt^{\beta} = 0. \tag{2.20}$$

At each point of N vectors u^{κ} satisfying

$$\omega_{\kappa\lambda}u^{\kappa}\frac{\partial\xi^{\lambda}}{\partial t^{\alpha}} = 0 \tag{2.21}$$

form a vector space of dimension $2n-k$. It follows from (2.20) that vectors tangent to N are solutions of (2.21). Vectors tangent to N form a space of dimension k. Hence $k \leqslant 2n-k$, or $k \leqslant n$. An isotropic submanifold of (P, ω) of dimension n is called a *Lagrangian submanifold* of (P, ω). It follows from Darboux's theorem that Lagrangian submanifolds exist.

Pairs of points in P form the product manifold $P \times P$. Coordinates (x^λ) and (x'^λ) of these points define coordinates (x'^κ, x^λ) of $P \times P$. The product manifold $P \times P$ and the differential form

$$\omega \ominus \omega = \tfrac{1}{2}\, \omega_{\kappa\lambda}(x'^\mu) dx'^\kappa \wedge dx'^\lambda - \tfrac{1}{2}\, \omega_{\kappa\lambda}(x^\mu) dx^\kappa \wedge dx^\lambda \tag{2.22}$$

define a symplectic manifold $(P \times P, \omega \ominus \omega)$. A map $\phi: P \to P$ is canonical if and only if its graph D is a Lagrangian submanifold of $(P \times P, \omega \ominus \omega)$; for $\dim D = 2n$ and

$$\omega \ominus \omega | D = \tfrac{1}{2}\, \omega_{\kappa\lambda}(\phi^\mu(x^\nu)) \partial_\mu \phi^\kappa \partial_\nu \phi^\lambda dx^\mu \wedge dx^\nu - \tfrac{1}{2}\, \omega_{\kappa\lambda} dx^\kappa \wedge dx^\lambda. \tag{2.23}$$

A Lagrangian submanifold of $(P \times P, \omega \ominus \omega)$ is called a *canonical relation*. The graph of a canonical transformation is a special case of a canonical relation.

Contravariant vectors at points of P form the tangent bundle TP. Components $(\dot{x}^\lambda)$ of a vector together with coordinates (x^κ) of the point at which the vector is attached define coordinates $(x^\kappa, \dot{x}^\lambda)$ of TP. The tangent bundle TP and the differential form

$$d_T\omega = \tfrac{1}{2}\, (\dot{x}^\mu \partial_\mu \omega_{\kappa\lambda} dx^\kappa \wedge dx^\lambda + \omega_{\kappa\lambda} d\dot{x}^\kappa \wedge dx^\lambda + \omega_{\kappa\lambda} dx^\kappa \wedge d\dot{x}^\lambda) \tag{2.24}$$

define a symplectic manifold $(TP, d_T\omega)$. Let X be a vector field (an infinitesimal diffeomorphism). Equations

$$\dot{x}^\kappa = X^\kappa(x^\mu) \tag{2.25}$$

define a submanifold D' of TP of dimension $2n$. It follows from (2.11) and (2.24) that X is an infinitesimal canonical transformation if and only if D' is a Lagrangian submanifold of $(TP, d_T\omega)$ since

$$d_T\omega | D' = \tfrac{1}{2}\, (X^\mu \partial_\mu \omega_{\kappa\lambda}\, \dot{x}^\kappa \wedge dx^\lambda + \omega_{\mu\lambda} \partial_\kappa X^\mu dx^\kappa \wedge dx^\lambda + \omega_{\kappa\mu} \partial_\lambda X^\mu dx^\kappa \wedge dx^\lambda). \tag{2.26}$$

A Lagrangian submanifold of $(TP, d_T\omega)$ is called an *infinitesimal canonical relation*. Infinitesimal canonical transformations define special cases of infinitesimal canonical relations.

3. The cotangent bundle, Lagrangian submanifolds generated by functions

Let Q be a manifold of dimension n with coordinates (q^i). Covariant vectors at points of Q form the cotangent bundle T^*Q. Components (p_j) of a covector together with the coordinates (q^i) of the point at which the covector is attached define coordinates (q^i, p_j) of T^*Q. The canonical structure of the cotangent bundle consists of a projection

$$\pi_Q : T^*Q \to Q : (q^i, p_j) \mapsto (q^i) \tag{3.1}$$

and the differential forms

$$\theta_Q = p_i dq^i \tag{3.2}$$

and

$$\omega_Q = d\theta_Q = dp_i \wedge dq^i \tag{3.3}$$

on T^*Q. The pair (T^*Q, ω_Q) is a symplectic manifold.

We describe three types of Lagrangian submanifolds of (T^*Q, ω_Q) generated by generating functions. Each higher type contains lower types as special cases.

I. Let $F(q^i)$ be a function on Q. The set N of elements of T^*Q with coordinates (q^i, p_j) satisfying

$$p_i dq^i = dF(q^i) \tag{3.4}$$

is a submanifold of TQ of dimension n. The equation (3.4) is equivalent to

$$p_i = \frac{\partial F}{\partial q^i}. \tag{3.5}$$

We have

$$\omega_Q|N = \frac{\partial^2 F}{\partial q^i \partial q^j} dq^j \wedge dq^i = 0. \tag{3.6}$$

Hence N is a Lagrangian submanifold of (T^*Q, ω_Q).

II. Let a submanifold C of Q of codimension k be described by

$$g^i = f^i(t^\alpha), \qquad \alpha = 1, \ldots, n-k \tag{3.7}$$

and let $F(t^\alpha)$ be a function on C. The set N of elements of T^*Q with coordinates (q^i, p_j) satisfying

$$p_i \frac{\partial f^i}{\partial t^\alpha} dt^\alpha = dF(t^\alpha), \qquad q^i = f^i(t^\alpha) \tag{3.8}$$

is a submanifold of T^*Q of dimension n. Equations (3.8) are equivalent to

$$p_i \frac{\partial f^i}{\partial t^\alpha} = \frac{\partial F}{\partial t^\alpha}, \qquad q^i = f^i(t^\alpha). \tag{3.9}$$

We have

$$\omega_Q|N = dp_i \wedge \frac{\partial f^i}{\partial t^\alpha} dt^\alpha = \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} dt^\beta \wedge dt^\alpha - p_i \frac{\partial^2 f^i}{\partial t^\alpha \partial t^\beta} dt^\beta \wedge dt^\alpha = 0. \tag{3.10}$$

Hence N is a Lagrangian submanifold of (T^*Q, ω_Q).

II′. If the submanifold C is described by

$$G^A(q^i) = 0, \qquad A = 1, \ldots, k \tag{3.11}$$

and if $\bar{F}(q^i)$ is an arbitrary continuation of $F(t^\alpha)$ from C to Q then N is the set of elements of T^*Q with coordinates (q^i, p_j) satisfying

$$p_i dq^i = d(\bar{F}(q^i) + \lambda_A G^A(q^i)) \tag{3.12}$$

for some values of $(\lambda_A) \in \boldsymbol{R}^k$. Equation (3.12) is equivalent to

$$p_i = \frac{\partial \bar{F}}{\partial q^i} + \lambda_A \frac{\partial G^A}{\partial q^i}, \qquad G^A(q^i) = 0. \tag{3.13}$$

We have again

$$\omega_Q|N = \frac{\partial^2\bar{F}}{\partial q^i \partial q^j} dq^j \wedge dq^i + \lambda_A \frac{\partial^2 G^A}{\partial q^i \partial q^j} dq^j \wedge dq^i + d\lambda_A \frac{\partial G^A}{\partial q^i} \wedge dq^i = 0 \tag{3.14}$$

confirming that N is a Lagrangian submanifold of (T^*Q, ω_Q). Functions $F(t^\alpha)$ on C and $G(\lambda_A, q^i) = \bar{F}(q^i) + \lambda_A G^A(q^i)$ on $R^k \times Q$ are two types of generating functions of N.

III. Let Λ be a manifold of dimension k with coordinates (λ_A) and let $G(\lambda_A, q^i)$ be a function on $\Lambda \times Q$ such that

$$\operatorname{rank}\left(\frac{\partial^2 G}{\partial\lambda_A \partial\lambda_B}, \frac{\partial^2 G}{\partial\lambda_A \partial q^i}\right) = k. \tag{3.15}$$

The set N of elements of T^*Q with coordinates (q^i, p_j) satisfying

$$p_i dq^i = dG(\lambda_A, q^i) \tag{3.16}$$

for some values of (λ_A) is a submanifold of T^*Q of dimension n. Equation (3.16) is equivalent to

$$p_i = \frac{\partial G}{\partial q^i}, \quad \frac{\partial G}{\partial \lambda_A} = 0. \tag{3.17}$$

We have

$$\omega_Q|N = \frac{\partial^2 G}{\partial q^i \partial q^j} dq^j \wedge dq^i + \frac{\partial^2 G}{\partial q^i \partial \lambda_A} d\lambda_A \wedge dq^i = 0. \tag{3.18}$$

Hence N is a Lagrangian submanifold of (T^*Q, ω_Q).

In the three constructions given above essential use is made of the canonical 1-form θ_Q on T^*Q. The restriction of θ_Q to the Lagrangian submanifold generated by a function is essentially equal to the differential of a lift of the function to that submanifold.

The structure of the cotangent bundle T^*Q induces similar structures in the product manifold $T^*Q \times T^*Q$ and the tangent bundle TT^*Q.

In terms of coordinates (q'^i, p'_j, p^k, p_l) the induced structure of $T^*Q \times T^*Q$ consists of the projection

$$\pi_Q \times \pi_Q : T^*Q \times T^*Q \to Q \times Q : (q'^i, p'_j, q^k, p_l) \mapsto (q'^i, q^k) \tag{3.19}$$

and the differential form

$$\theta_Q \ominus \theta_Q = p'_i dq'^i - p_i dq^i \tag{3.20}$$

and

$$\omega_Q \ominus \omega_Q = dp'_i \wedge dq'^i - dp_i \wedge dq^i. \tag{3.21}$$

The pair $(T^*Q \times T^*Q, \omega_Q \ominus \omega_Q)$ is a symplectic manifold.

In terms of coordinates $(q^i, p_j, \dot{q}^k, \dot{p}_l)$ the induced structure of TT^*Q consists of the projection

$$T\pi_Q : TT^*Q \to TQ : (q^i, p_j, \dot{q}^k, \dot{p}_l) \mapsto (q'^i, q^k) \tag{3.22}$$

and the differential forms

$$d_T\theta_Q = \dot{p}_i dq^i + p_i d\dot{q}^i \tag{3.23}$$

and

$$d_T\omega_Q = d\dot{p}_i \wedge dq^i + dp_i \wedge d\dot{q}^i. \tag{3.24}$$

A second induced structure consists of the projection

$$\tau_{T^*Q} : TT^*Q \to T^*Q : (q^i, p_j, \dot{q}^k, \dot{p}_l) \mapsto (q^i, p_j), \tag{3.25}$$

the differential form

$$i_T\omega_Q = \dot{p}_i dq^i - \dot{q}^i dp_i \tag{3.26}$$

and the form $d_T\omega_Q$. The pair $(TT^*Q, d_T\omega_Q)$ is a symplectic manifold.

The induced structures make it possible to generate Lagrangian submanifolds of $(T^*Q \times T^*Q, \omega_Q \ominus \omega_Q)$ and $(TT^*Q, d_T\omega_Q)$ from generating functions. In the simplest case a canonical relation $D \subset T^*Q \times T^*Q$ is generated by a function $S(q'^i, q^k)$ on $Q \times Q$. In this case D is described by the equation

$$d_i' dq'^i - p_i dq^i = dS(q'^i, q^j) \tag{3.27}$$

equivalent to

$$p_i' = \frac{\partial S}{\partial q'^i}, \qquad p_i = -\frac{\partial S}{\partial q^i}. \tag{3.28}$$

If D is the graph of a transformation then equations (3.28) can be written in the form

$$q'^i = \xi^i(q^k, p_l), \qquad p_i' = \eta_i(q^k, p_l). \tag{3.29}$$

Relations

$$\frac{\partial \xi^i}{\partial q^k}\frac{\partial \xi^j}{\partial p^k} - \frac{\partial \xi^j}{\partial q^k}\frac{\partial \xi^i}{\partial p_k} = 0, \tag{3.30}$$

$$\frac{\partial \xi^i}{\partial q^k}\frac{\partial \eta_j}{\partial p_k} - \frac{\partial \eta_j}{\partial q^k}\frac{\partial \xi^i}{\partial p_k} = \delta^i_j, \tag{3.31}$$

$$\frac{\partial \eta_i}{\partial q^k}\frac{\partial \eta_j}{\partial p_k} - \frac{\partial \eta_j}{\partial q^k}\frac{\partial q_i}{\partial p_k} = 0 \tag{3.32}$$

corresponding to (2.10) are satisfied as a consequence of (3.29) being a canonical transformation.

Let

$$q'^i = \xi^i(q^k) \tag{3.33}$$

be a transformation of Q. The submanifold $D \subset T^*Q \times T^*Q$ described by

$$q'^i = \xi^i(q^k), \qquad \frac{\partial \xi^i}{\partial q^k} p_i' = p_k \tag{3.34}$$

is the graph of the point transformation corresponding to (3.33). It is easily seen that D is generated by the function $F = 0$ on the graph C of the transformation (3.33). Equivalently D is generated by the function $G(\lambda_i, q'^j, q^k) = \lambda_i(q'^i - \xi^i(q^k))$ defined on $\boldsymbol{R}^n \times Q \times Q$.

Let $f(q^i, p_j)$ be a function on T^*Q and let

$$X^i = \frac{\partial f}{\partial p_i}, \qquad Y_j = -\frac{\partial f}{\partial q^j} \tag{3.35}$$

be the infinitesimal canonical transformation corresponding to (2.13). The corresponding infinitesimal canonical relation D' is described by

$$\dot{q}^i = \frac{\partial f}{\partial p_i}, \qquad \dot{p}_j = -\frac{\partial f}{\partial q^j}. \tag{3.36}$$

It follows from

$$\dot{p}_i dq^i - \dot{q}^i dp_i = -df(q^i, p_j) \tag{3.37}$$

that D' is generated by the function $-f(q^i, p_j)$. The infinitesimal canonical relation D' may also be generated by a functon $l(q^i, \dot{q}^j)$ on TQ according to the formula

$$\dot{p}_i dq^i + p_i d\dot{q}^i = dl(q^i, \dot{q}^j) \tag{3.38}$$

equivalent to

$$\dot{p}_i = \frac{\partial l}{\partial q^i}, \qquad p_i = \frac{\partial l}{\partial \dot{q}^i}. \tag{3.39}$$

4. *Nonrelativistic particle dynamics*

Let Q be the configuration manifold of a mechanical system. As usual we assume that dynamics of the system is described by a one-parameter (local) group of (local) canonical transformations

$$q'^i = \xi^i(t, q^j, p_k), \qquad p'_i = \eta_i(t, q^j, p_k) \tag{4.1}$$

of the phase manifold T^*Q.

Fixing the initial points and varying the time t in (4.1) we obtain trajectories

$$q^i = \gamma^i(t) = \xi^i(t, \mathring{q}^j, \mathring{p}_k), \qquad p_i = \chi_i(t) = \eta_i(t, \mathring{q}^j, \mathring{p}_k) \tag{4.2}$$

of the system. The graphs D_t of transformations (4.1) form a one-parameter family of Lagrangian submanifolds of $(T^*Q \times T^*Q, \omega_Q \ominus \omega_Q)$. We assume that D_t are generated by generating functions W_t of one of the types described in Section 3 and called *Hamilton principal functions.*

Dynamics can equivalently be stated as an infinitesimal canonical transformation

$$X^i(q^k, p_l), \qquad Y_j(q^k, p_l) \tag{4.3}$$

from which the group (4.1) is obtained by integration. Equations

$$\frac{\partial X^i}{\partial p_k} - \frac{\partial X^k}{\partial p_i} = 0, \tag{4.4}$$

$$\frac{\partial Y_i}{\partial p_k} + \frac{\partial X^k}{\partial q^i} = 0, \tag{4.5}$$

$$\frac{\partial Y_i}{\partial q^k} - \frac{\partial Y_k}{\partial q^i} = 0 \tag{4.6}$$

corresponding to equation (2.11) are satisfied. Equations

$$\dot{q}^i = X^i(q^k, p_l), \quad \dot{p}_j = Y_j(q^k, p_l) \tag{4.7}$$

define a Lagrangian submanifold D' of $(TT^*Q, d_T\omega_Q)$. We assume that D' is generated by a generating function $L(q^i, \dot{q}^j)$ on TQ according to the formula

$$\dot{p}_i dq^i + p_i d\dot{q}^i = dL(q^i, \dot{q}^j) \tag{4.8}$$

equivalent to

$$\dot{p}_i = \frac{\partial L}{\partial q^i}, \quad p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{4.9}$$

and also by a function $-H(q^i, p_j)$ according to the formula

$$\dot{p}_i dq^i - \dot{q}^i dp_i = -dH(q^i, p_j) \tag{4.10}$$

equivalent to

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}. \tag{4.11}$$

The functions $L(q^i, \dot{q}^j)$ and $H(q^i, p_j)$ are called the *Lagrangian* and the *Hamiltonian* respectively.

Example: the harmonic oscillator. Equations of motion

$$p = m\dot{q}, \; \dot{p} = -kq \tag{4.12}$$

define a submanifold D' of TT^*Q, where Q is the one-dimensional configuration manifold of the harmonic oscillator. From

$$\dot{p}dq + pd\dot{q} = -k\dot{q}dq + m\dot{q}d\dot{q} = d\tfrac{1}{2}(m\dot{q}^2 - kq^2) \tag{4.13}$$

and

$$\dot{p}dq - \dot{q}dp = -kqdq - \frac{1}{m}pdp = -d\,\tfrac{1}{2}\left(\frac{1}{m}p^2 + kq^2\right) \tag{4.14}$$

it follows that D' is a Lagrangian submanifold generated by the Lagrangian

$$L(q, \dot{q}) = \tfrac{1}{2}(m\dot{q}^2 - kq^2) \tag{4.15}$$

and that the Hamiltonian is

$$H(q, p) = \tfrac{1}{2}\left(\frac{1}{m} p^2 + kq^2\right). \tag{4.16}$$

The one-parameter group of canonical transformations obtained by integration is

$$q' = q \cos \omega t + \frac{p}{m\omega} \sin \omega t, \qquad p' = -qm\omega \sin \omega t + p \cos \omega t, \tag{4.17}$$

where $\omega = \sqrt{\frac{k}{m}}$. If $\sin \omega t \neq 0$ then equations (4.17) describing D_t can be written in the equivalent form

$$p = \frac{m\omega}{\sin \omega t}(q' - q \cos \omega t), \qquad p' = \frac{m\omega}{\sin \omega t}(q' \cos \omega t - q). \tag{4.18}$$

It follows from

$$\begin{aligned} p'dq' - pdq &= \frac{m\omega}{\sin \omega t}\left[(q' \cos \omega t - q)dq' - (q' - q \cos \omega t)dq\right] \\ &= d\,\frac{m\omega}{2 \sin \omega t}(q'^2 \cos \omega t - 2q'q + q^2 \cos \omega t) \end{aligned} \tag{4.19}$$

that D_t is generated by the Hamilton principal function

$$W_t(q', q) = \frac{m\omega}{2 \sin \omega t}(q'^2 \cos \omega t - 2q'q + q^2 \cos \omega t). \tag{4.20}$$

If $\sin \omega t = 0$ then $\cos \omega t = \pm 1$ and D_t is described by

$$q' = \pm q, \qquad p' = \pm p, \tag{4.21}$$

and generated by $W_t = 0$ on the submanifold C_t of $Q \times Q$ described by $q' = \pm q$.

5. *Relativistic dynamics*

The approach to dynamics emphasizing Lagrangian submanifolds and their generating functions has the advantage of being applicable to relativistic dynamics.

Let us consider the motion of a freely falling test particle of mass m (> 0) in space-time Q with an indefinite metric g_{ij}. The Lagrangian of the particle is the function

$$L_I(q^i, \dot{q}^j) = m\sqrt{g_{ij}\dot{q}^i\dot{q}^j} \tag{5.1}$$

defined for time-like vectors $\dot{q}^i$ for which $g_{ij}\dot{q}^i\dot{q}^j > 0$. Interpreted as a generating function the Lagrangian generates an infinitesimal canonical relation $D'_I \subset TT^*Q$. Equations

$$p_i = mg_{ij}\dot{q}^j(g_{kl}\dot{q}^k\dot{q}^l)^{-1/2}, \qquad g_{kl}\dot{q}^k\dot{q}^l > 0, \tag{5.2}$$

$$\dot{p}_i - \Gamma^k_{ij}p_k\dot{q}^j = 0 \tag{5.3}$$

describing D_I' are derived from

$$\dot{p}_i dq^i + p_i d\dot{q}^i = dL_I(q^i, \dot{q}^j) = m(g_{lm}\dot{q}^l\dot{q}^m)^{-1/2}(\tfrac{1}{2}\,\partial_i g_{jk}\dot{q}^j\dot{q}^k dq^i + g_{ij}\dot{q}^j d\dot{q}^i). \tag{5.4}$$

Equations (5.2), (5.3) can be written in the equivalent form

$$\dot{p}_i - \Gamma^k_{ij} p_k \dot{g}^j = 0, \tag{5.5}$$

$$g^{ij} p_i p_j = m^2 \tag{5.6}$$

and

$$\dot{q}^i = \frac{\lambda}{m} g^{ij} p_j \tag{5.7}$$

for some $\lambda > 0$. It follows from

$$\dot{p}_i dq^i - \dot{q}^i dp_i = -d(\lambda(\sqrt{g^{ij} p_i p_j} - m)) \tag{5.8}$$

that the function

$$H_I(\lambda, q^i, p_j) = \lambda(\sqrt{g^{ij} p_i p_j} - m) \tag{5.9}$$

defined on $R^+ \times TQ$ is the (generalized) Hamiltonian of D_I'.

A curve

$$q^i = \gamma^i(t), \qquad p_i = \chi_i(t) \tag{5.10}$$

will be called an *integral curve* of D_I' if for each t

$$(q^i, p_j, \dot{q}^k, \dot{p}_l) = \left(\gamma^i(t), \chi_j(t), \frac{d\gamma^k}{dt}, \frac{d\chi_l}{dt}\right) \tag{5.11}$$

belongs to D_I'. If (5.10) is an integral curve then the curve $q^i = \gamma^i(t)$ satisfies the Euler equations

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = \dot{q}^i g_{jk}\dot{q}^j(\ddot{q}^k + \Gamma^k_{lm}\dot{q}^l\dot{q}^m)(g_{rs}\dot{q}^r\dot{q}^s)^{-1},$$
$$(q^i, \dot{q}^j, \ddot{q}^k) = \left(\gamma^i(t), \frac{d\gamma^j}{dt}, \frac{d^2\gamma^k}{dt^2}\right), \qquad q_{ij}\dot{q}^i\dot{q}^j > 0 \tag{5.12}$$

easily derived from equations (5.2), (5.3). Thus $q^i = \gamma^i(t)$ is an arbitrarily parametrized geodesic. Functions $\chi_i(t)$ are obtained from (5.3). If $u^i(t)$ is the normalized tangent vector

$$u^i(t) = \frac{d\gamma^i}{dt}\left(g_{ik}\frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt}\right)^{-1/2} \tag{5.13}$$

then

$$\chi_i(t) = m g_{ij} u^j(t). \tag{5.14}$$

Equations (5.12) are invariant under arbitrary changes of parametrization

$$t \mapsto t' = \kappa(t), \qquad \frac{d\kappa}{dt} \neq 0. \tag{5.15}$$

Equations (5.14) are invariant if $\frac{d\kappa}{dt} > 0$. Consequently integral curves can be replaced by oriented one-dimensional submanifolds of TQ. An oriented one-dimensional submanifold σ of TQ will be called an *integral manifold* of D'_{I} if each vector tangent to σ and pointing in the direction of the orientation belongs to D'_{I}.

In terms of integral curves of D'_{I} we define a submanifold D_{I} of $T^*Q \times T^*Q$ An element of $TQ \times TQ$ with coordinates (q'^i, p'_j, q^k, p_l) belongs to D_{I} if there is an integral curve (5.10) of D'_{I} such that $q^k = \gamma^k(0)$, $p_l = \chi_l(0)$, $q'^i = \gamma^i(\tau)$ and $p'_j = \chi_j(\tau)$ for some $\tau > 0$. Since D'_{I} is an infinitesimal canonical relation we expect D_{I} to be a canonical relation and knowing the Lagrangian (5.1) we expect D_{I} to be generated by the function $W_{\mathrm{I}}(q'^i, g^j)$ defined for time-like separated points with coordinates (q^j) and (q'^i) and equal to the geodesic proper time interval between these points multiplied by m. Let (q'^i, p'_j, q^k, p_l) be in D_{I} and let $q^i = \gamma^i(t)$, $p_j = \chi_j(t)$ be the integral curve of D'_{I} such that $q^k = \gamma^k(0)$, $p_l = \chi_l(0)$, $q'^i = \gamma^i(\tau)$ and $p'_j = \chi_j(\tau)$. Since

$$W_{\mathrm{I}}(q'^i, q^j) = m \int_0^\tau \sqrt{g_{ij}\dot{q}^i\dot{q}^j}\, dt, \quad (q^i, \dot{q}^j) = \left(\gamma^i(t), \frac{d\gamma^j}{dt}\right) \tag{5.16}$$

and since $q^i = \gamma^i(t)$ satisfies Euler equations we have

$$dW_{\mathrm{I}}(q'^i, q^j) = mg_{ij}q'^j(g_{kl}q'^k q'^l)^{-\frac{1}{2}}dq'^i - mg_{ij}q^j(g_{kl}q^k q^l)^{-\frac{1}{2}}dq^i, \tag{5.17}$$

where

$$(q^i, \dot{q}^j) = \left(\gamma^i(0), \frac{d\gamma^j(0)}{dt}\right), \quad (q'^i \dot{q}'^j) = \left(\gamma^i(\tau), \frac{d\gamma^j(\tau)}{dt}\right). \tag{5.18}$$

It follows from (5.13) and (5.14) that

$$p_i = mg_{ij}q^j(g_{kl}\dot{q}^k\dot{q}^l)^{-\frac{1}{2}}, \quad p'_i = mg_{ij}q'^j(q_{kl}\dot{q}'^k\dot{q}'^l)^{-\frac{1}{2}}. \tag{5.19}$$

Hence

$$p'_i dq'^i - p_i dq^i = dW_{\mathrm{I}}(q'^i, q^j)$$

which confirms the expectation that D_{I} is a canonical relation generated by $W_{\mathrm{I}}(q'^i, q^j)$. The function $W_{\mathrm{I}}(q'^i, q^j)$ is called the Hamilton principal function of D_{I}. The canonical relation D_{I} is said to be the *integral* of the infinitesimal canonical relation D'_{I}.

6. *Relativistic dynamics. An alternative approach*

Relativistic dynamics has been formulated as a constraint system in the spirit of Dirac's generalized Hamiltonian dynamics [1, 2]. In our terms the dynamics of a particle of mass m is described by an infinitesimal canonical relation $D'_{\mathrm{II}} \subset TT^*Q$ generated by the zero Hamiltonian on the *mass shell* $M \subset T^*Q$ defined by

$$g^{ij}p_i p_j = m^2. \tag{6.1}$$

Equivalently D'_{II} is generated by the Hamiltonian

$$H_{\mathrm{II}}(\lambda, q^i, p_j) = \lambda(\sqrt{g^{ij}p_ip_j} - m) \tag{6.2}$$

defined on $R \times T^*Q$. The difference between $H_{\mathrm{II}}(\lambda, q^i, p_j)$ and the Hamiltonian $H_{\mathrm{I}}(\lambda, q^i, p_j)$ of D'_{I} is that λ in $H_{\mathrm{II}}(\lambda, q^i, p_j)$ is not restricted to positive values. From

$$\dot{p}_i dq^i - \dot{q}^i dp_i = -d(\lambda(\sqrt{g^{ij}p_ip_j} - m)) \tag{6.3}$$

we obtain equations

$$\sqrt{g^{ij}p_ip_j} = m, \tag{6.4}$$

$$\dot{q}^i = \frac{\lambda}{m} g^{ij}p_j, \tag{6.5}$$

$$\dot{p}_i - \Gamma^k_{ij}p_k\dot{q}^j = 0 \tag{6.6}$$

describing D'_{II}. The Lagrangian of D'_{II} is the function

$$L_{\mathrm{II}}(\lambda, \lambda_i, q^j, \dot{q}^k) = \lambda_i\dot{q}^i - \lambda(\sqrt{g^{kj}\lambda_i\lambda_j} - m) \tag{6.7}$$

defined on $\boldsymbol{R}^5 \times TQ$. From

$$\dot{p}_i dq^i + p_i d\dot{q}^i = d(\lambda_i\dot{q}^i - \lambda(\sqrt{g^{ij}\lambda_i\lambda_j} - m)) \tag{6.8}$$

we obtain equations

$$\sqrt{g^{ij}\lambda_i\lambda_j} = m, \tag{6.9}$$

$$\dot{q}^i = \frac{\lambda}{m} g^{ij}\lambda_j, \tag{6.10}$$

$$\dot{p}_i = -\frac{\lambda}{2m}\partial_i g^{jk}\lambda_j\lambda_k, \tag{6.11}$$

$$p_i = \lambda_i \tag{6.12}$$

equivalent to (6.4), (6.5), (6.6).

As in Section 5 we define integral curves

$$q^i = \gamma^i(t), \qquad p_j = \chi_j(t) \tag{6.13}$$

by requiring that

$$(q^i, p_j, q^k, p_l) = \left(\gamma^i(t), \chi_j(t), \frac{d\gamma^k}{dt}, \frac{d\chi_l}{dt}\right) \tag{6.14}$$

belongs to D'_{II} for each t. Integral curves of D'_{II} include integral curves of $D'_{\mathrm{I'}}$, curves obtained from integral curves of D'_{I} by replacing (5.14) by

$$\chi_i(t) = -mg_{ij}u^j(t) \tag{6.15}$$

and also constant curves in M (single points). Integral curves are invariant under completely arbitrary changes of parametrization. Consequently integral curves can be replaced by integral manifolds. A submanifold σ of T^*Q is called an *integral manifold* of D'_{II} if each vector tangent to σ belongs to D'_{II}. Integral manifolds are one-dimensional.

We define the integral $D_{\text{II}} \subset T^*Q \times T^*Q$ of D'_{II} by the definition that was used in Section 5 to define D_{I}. An element of $T^*Q \times T^*Q$ with coordinates (q'^i, p'_j, q^k, p_l) belongs to D_{II} if there is an integral curve (6.13) of D'_{II} such that $q^k = \gamma^k(0)$, $p_l = \chi_l(0)$, $q'^i = \gamma^i(\tau)$ and $p'_j = \chi_j(\tau)$ for some $\tau > 0$. Since D_{II} is the closure of the union of D_{I} and the canonical relation $-D_{\text{I}}$ generated by $-W_{\text{I}}(q'^i, q^j)$ it follows that D_{II} is a canonical relation. It is difficult to give the Hamilton principal function of D_{II} in full generality. If Q is flat and the metric tensor q_{ij} is constant then D_{II} is generated by the function

$$W_{\text{II}}(\lambda, \lambda_i, q'^j, q^k) = \lambda_i(q'^i - q^i) - \lambda(\sqrt{g^{ij}\lambda_i\lambda_j} - m) \tag{6.16}$$

defined on $R^5 \times Q \times Q$. If λ is restricted to positive values then formula (6.16) gives an expression for the generating function of D_{I} in the case of flat spacetime. This expression can be simplified by using stationarity conditions

$$\sqrt{g^{ij}\lambda_i\lambda_j} = m, \qquad q'^i - q^i = \frac{\lambda}{m} g^{ij}\lambda_j. \tag{6.17}$$

The result ist

$$W_{\text{I}}(q'^i, q^j) = m\sqrt{g_{ij}(q'^i - q^i)(q'^j - q^j)}\,. \tag{6.18}$$

Formula (6.16) can not be simplified without the assumption $\lambda > 0$. Formula

$$W_{\text{II}}(q'^i, q^j) = \lambda_i(q'^i - q^i) - \frac{\lambda}{2m}(g^{ij}\lambda_i\lambda_j - m^2) \tag{6.19}$$

is completely equivalent to (6.16).

There is little difference between the two approaches on the level of particle trajectories. Integral manifolds of D'_{I} are oriented submanifolds of T^*Q. Integral manifolds of D'_{II} are the same submanifolds with the orientation disregarded. The orientation is not lost in the second approach since it is a natural orientation provided by unit vectors

$$u^i = \frac{1}{m} g^{ij} p_j \tag{6.20}$$

tangent to integral manifolds.

A wave mechanical interpretation of the different Hamilton principal functions of the two approaches is given in the next section.

Vectors (6.20) distinguish a natural parametrization as well as orientation. A third approach to relativistic dynamics emphasizing this parametrization is possible. Let D'_{III} be the infinitesimal canonical relation generated by the Hamiltonian

$$H_{\text{III}}(q^i, p_j) = \frac{1}{2m} g^{ij} p_i p_j, \tag{6.21}$$

or equivalently by the Lagrangian

$$L_{\text{III}}(q^i, \dot{q}^j) = \frac{m}{2} g_{ij}\dot{q}^i\dot{q}^j. \tag{6.22}$$

The relation D'_{III} is described by equations

$$p_i = mg_{ij}\dot{q}^j, \tag{6.23}$$

$$\dot{p}_i - \Gamma^k_{ij} p_k \dot{q}^j = 0. \tag{6.24}$$

Integral curves

$$q^i = \gamma^i(s), \quad p_j = \chi_j(s) \tag{6.25}$$

of D'_{III} satisfy equations

$$\frac{d^2\gamma^i}{ds^2} + \Gamma^i_{jk} \frac{d\gamma^j}{ds}\frac{d\gamma^k}{ds} = 0, \tag{6.26}$$

$$\chi_i(s) = mg_{ij}\frac{d\gamma^j}{ds} \tag{6.27}$$

which admit only trivial changes of parametrization

$$s \leftrightarrow s + \text{const}. \tag{6.28}$$

The integral of D'_{III} is a one-parameter group of canonical transformations since D'_{III} is associated with an infinitesimal canonical transformation. The description of dynamics by D'_{III} has the disadvantage of being incomplete without the mass shell condition

$$g^{ij}p_i p_j = m^2 \tag{6.29}$$

and equations (6.23), (6.24) supplemented by (6.29) no longer define an infinitesimal canonical relation due to the loss of one dimension. The only way to increase the dimension without losing essential information on dynamics is to disregard the distinguished parametrization passing to D'_{I} or D'_{II}.

7. *Hamilton principal functions and wave mechanics*

Wave functions $\psi(t, q)$ of the harmonic oscillator satisfy the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + \frac{k}{2}q^2\psi. \tag{7.1}$$

Distributions $\Delta_t(q', q)$ defined by

$$\Delta_t(q', q) = \left(\frac{m\omega}{2\pi i \sin\omega t}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\frac{m\omega}{2\sin\omega t}(q'^2\cos\omega t - 2q'q + q^2\cos\omega t)} \tag{7.2}$$

if $\sin\omega t \neq 0$ and by

$$\Delta_t(q', q) = \delta(q' \mp q) \tag{7.3}$$

if $\sin \omega t = 0$ and $\cos \omega t = \pm 1$ lead to a one-parameter group of integral operators

$$\phi(q) \mapsto \int \Delta_t(q', q)\phi(q)dq. \tag{7.4}$$

Composition relations

$$\int \Delta_{t'}(q'', q')\Delta_t(q', q)dq' = \Delta_{t'+t}(q'', q) \tag{7.5}$$

are satisfied and the function

$$\psi(t, q) = \int \Delta_t(q, q')\phi(q')dq' \tag{7.6}$$

satisfies the Schrödinger equation (7.1) for any (sufficiently regular) function $\phi(q')$. In the exponent in the formula (7.2) we recognize the function (4.20). Also (7.3) is compatible with (4.21). We conclude that $W_t(q', q)$ is the classical counterpart of $\Delta_t(q', q)$.

Let Q be the flat space-time of special relativity with coordinates (q^i), $i = 0, 1, 2, 3$. The coordinate q^0 is interpreted as time. The metric tensor has constant components g_{ij} and g^{ij}. Wave functions $\psi(q^i)$ of a relativistic particle of mass m satisfy the Klein–Gordon equation

$$g^{ij}\frac{\partial^2\psi}{\partial q^i \partial q^j} + \frac{m^2}{\hbar^2}\psi = 0 \tag{7.7}$$

The distribution $\Delta_{\rm II}(q'^i, q^j)$ defined by

$$\begin{aligned} \Delta_{\rm II}(q'^i, q^j) &= \frac{1}{2m}\frac{\hbar}{i}\frac{1}{(2\pi\hbar)^3}\int e^{\frac{i}{\hbar}\lambda_k(q'^k - q^k)}\delta(\sqrt{g^{ij}\lambda_i\lambda_j} - m)d^4\lambda \\ &= \frac{1}{2m}\frac{\hbar}{i}\frac{1}{(2\pi\hbar)^4}\int e^{\frac{i}{\hbar}[\lambda_i(q'^i - q^i) - \lambda(\sqrt{g^{ij}\lambda_i\lambda_j} - m)]}d^4\lambda d\lambda \end{aligned} \tag{7.8}$$

satisfies the composition relation

$$\int\limits_{q^0 = \text{const}} \left(\Delta_{\rm II}(q''^i, q^j)\frac{\partial}{\partial q^0}\Delta_{\rm II}(q^j, q'^k) - \Delta_{\rm II}(q^j, q'^k)\frac{\partial}{\partial q^0}\Delta_{\rm II}(q''^i, q^j)\right)d^3q = \Delta_{\rm II}(q''^i, q'^k) \tag{7.9}$$

and the function

$$\psi(q^i) = \int\limits_{q'^0 = \text{const}} \left(\Delta_{\rm II}(q^i, q'^j)\phi_1(q'^j) + \phi_2(q'^j)\frac{\partial}{\partial q'^0}\Delta_{\rm II}(q^i, q'^j)\right)d^3q' \tag{7.10}$$

is a solution of the equation (7.7) for any (sufficiently regular) functions $\phi_1(q'^j)$ and $\phi_2(q'^j)$. The distribution $\Delta_{\rm II}(q'^i, q^j)$ can be also defined by

$$\begin{aligned} \Delta_{\rm II}(q'^i, q^j) &= \frac{\hbar}{i}\frac{1}{(2\pi\hbar)^4}\int e^{\frac{i}{\hbar}\lambda_k(q'^k - q^k)}\delta(g^{ij}\lambda_i\lambda_j - m^2)d^4\lambda d\lambda \\ &= \frac{1}{2m}\frac{\hbar}{i}\frac{1}{(2\pi\hbar)^4}\int e^{\frac{i}{\hbar}\left[\lambda_i(q'^i - q^i) - \frac{\lambda}{2m}(g^{ij}\lambda_i\lambda_j - m^2)\right]}d^4\lambda d\lambda. \end{aligned} \tag{7.11}$$

Restricting λ to positive values in the last expression we obtain the distribution

$$\Delta_{\mathrm{I}}(q'^i, q^j) = \frac{1}{2m}\frac{\hbar}{i}\frac{1}{(2\pi\hbar)^4}\int\limits_{\lambda>0} e^{\frac{i}{\hbar}\left[\lambda_i(q'^i-q^i)-\frac{\lambda}{2m}(g^{ij}\lambda_i\lambda_j-m^2)\right]}d^4\lambda d\lambda. \tag{7.12}$$

The distribution $\Delta_{\mathrm{I}}(g'^i, g^j)$ is a Green's function for the Klein–Gordon equation (7.7). The assymptotic expression

$$\Delta_{\mathrm{I}}(q'^i, q^j) \approx N(q'^i, q^j)e^{\frac{i}{\hbar}\sqrt{g_{ij}(q'^i-q^i)(q'^j-q^j)}} \tag{7.13}$$

is obtained by applying the method of stationary phase to the integral (7.12). Comparing formulae (7.11), (7.12) and (7.13) with formulae (6.16), (6.18) and (6.19) we conclude that functions $W_{\mathrm{I}}(q'^i, q^j)$ and $W_{\mathrm{II}}(q'^i, q^j)$ are classical counterparts of distributions $\Delta_{\mathrm{I}}q'^i, q^j)$ and $\Delta_{\mathrm{II}}(q'^i, q^j)$ respectively.

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REFERENCES

[1] P. A. M. Dirac, *Can. J. Math.* **2**, 129 (1950).
[2] J. Śniatycki, W. M. Tulczyjew, *Ann. Inst. H.-Poincaré* **A15**, 177 (1971).
[3] W. M. Tulczyjew, *Symposia Mathematica* **14**, 247 (1974).
[4] W. M. Tulczyjew, *CR Acad. Sci. Paris* **A283**, 15 (1976).
[5] W. M. Tulczyjew, *Les sous-variétés lagrangiennes et la dynamique lagrangienne*, to appear.
[6] W. M. Tulczyjew, *The Legendre Transformation*, to appear.