MULTIPLE EXCHANGES OF NONLINEAR REGGEONS

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It is shown that the conventional expression for the n-reggeon branch point trajectory $\alpha_n(t) = n\alpha(t/n^2)-n+1$ does not apply in general to nonlinear Regge pole trajectories. When the form of a trajectory is somewhat restricted, a reconstruction of the J-plane occurs and a new branch point emerges to the right of the classical branch point. If pole trajectories rise as $|t|^x$ where $x \leqslant \frac{1}{2}$, the momentum distribution on reggeons becomes asymmetric when the momentum transfered by one reggeon reaches some critical value.

1. *Introduction*

Experiment shows that Regge trajectories are approximately linear in the scattering range $0 \geqslant t \geqslant -1$ GeV^2. Resonances belonging to a Regge-recurrence are also arranged along a straight line in the Chew–Frautschi diagrams. These experimental data serve as a justification for a large number of models based on approximately linear Regge trajectories, and in some of these models the linearity of trajectories is raised to the level of a strict regularity in strong interactions dynamics.

On the other hand, it is reasonable to assert that Regge trajectories are nonlinear, e.g. on account of resonance instability. And the question is, to what extent is the neglect of nonlinear terms of Regge trajectories justified. Leaving the arguments for the nonlinearity of trajectories alone, we want to emphasize a simple, but rather interesting effect arising when the nonlinearity of trajectories is taken into account in the framework of reggeon calculus.

2. *Statement of the problem*

Let us represent the contribution of a (simple) Regge pole to the amplitude of a high energy hadron interaction in the form

$$M_1(s, t) = \eta(\alpha(t))g^2(t)\,(s/s_0)^{\alpha(t)-1}, \tag{1}$$

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where $\eta(\alpha)$ is the signature factor, $g(t)$ is a vertex function and $\alpha(t)$ is a Regge pole trajectory.

The contribution from the n-multiple exchange of the reggeons considered is then defined by the expression [1, 2]

$$M_n(s, t) = \left(\frac{i}{\pi}\right)^{n-1} \int d^2\vec{\kappa}_1 \dots d^2\vec{\kappa}_{n-1} N^2(\vec{\kappa}_1, \dots, \vec{\kappa}_n)$$
$$\times \eta(\alpha(-\kappa_1^2)) \dots \eta(\alpha(-\kappa_n^2)) (s/s_0)^{\beta_n(\kappa_1, \dots, \kappa_n)-n}, \tag{2}$$

where $N(\vec{\kappa}_1, \dots \vec{\kappa}_n)$ is some vertex function (we do not specify its form since, in this paper, we are interested in the behaviour of the trajectories of n-reggeon singularities), $\vec{\kappa}_i$ is the transversal component of the i-th reggeon momentum, $\kappa_i = \sqrt{-t_i}$, while

$$\kappa_1 + \dots + \kappa_n = \kappa = \sqrt{-t}\,. \tag{3}$$

The function β_n is

$$\beta_n(\kappa_1, \dots, \kappa_n) = \alpha(-\kappa_1^2) + \dots + \alpha(-\kappa_n^2). \tag{4}$$

To find the trajectory of the n-reggeon branch point, we must extract the leading asymptotic term of expression (2). From (2) we see that the main contribution to the asymptotics is given by a maximum of the function $\beta_n(\kappa_1, \dots, \kappa_n)$. i.e. at $t \leqslant 0$

$$\alpha_n(t) = \max \beta_n(\kappa_1, \dots, \kappa_n) - n + 1, \tag{5}$$

when κ_i belong to the ray $[0, \infty)$ and are connected by condition (3).

If all reggeons are the same and their trajectories are linear, then the transferred momentum is distributed uniformly between all reggeons,

$$\kappa_1 = \dots = \kappa_n = \frac{\kappa}{n} = \sqrt{-t/n^2}\,, \tag{6}$$

and [1–5]

$$\alpha_n(t) = n\alpha(t/n^2) - n + 1. \tag{7}$$

However, an entirely different picture may arise in the case of nonlinear reggeon exchanges.

3. Exchange of two nonlinear reggeons

In this case (after integration over angular variables)

$$M_2(s, t) = i\sqrt{\pi} \int d\kappa_1 \kappa_1 \frac{N^2(\kappa, \kappa - \kappa_1)}{\sqrt{\alpha'(-(\kappa-\kappa_1)^2)\kappa\kappa_1\xi}} \exp\{\xi[\beta_2(\kappa_1, \kappa-\kappa_1) - 2]\}, \tag{8}$$

where $\xi = \ln(s/s_0)$ and

$$\beta_2(\kappa_1, \kappa - \kappa_1) = \alpha(-\kappa_1^2) + \alpha(-(\kappa-\kappa_1)^2). \tag{9}$$

The extremum condition for the function $\beta_2(\kappa_1, \kappa-\kappa_1)$ in the variable κ_1 is

$$\kappa_1\alpha'(-\kappa_1^2) = (\kappa-\kappa_1)\alpha'(-(\kappa-\kappa_1)^2). \tag{10}$$

The value $\kappa_1 = \kappa/2$ always satisfies condition (10), i.e. this point is an extremum of the function $\beta_2(\kappa_1, \kappa-\kappa_1)$. However, for the point $\kappa_1 = \kappa/2$ to be a maximum, it is necessary to implement the additional condition

$$\frac{\partial^2}{\partial\kappa_1^2}\beta_2(\kappa_1, \kappa-\kappa_1)|_{\kappa_1=\kappa/2} < 0, \tag{11}$$

i.e.

$$2\alpha'(t/4) > (-t)\alpha''(t/4). \tag{12}$$

We now study the conditions (10), (12) in more detail. Let a Regge pole trajectory satisfy the dispersion relation with one subtraction

$$\alpha(t) = \alpha(0) + \frac{t}{\pi}\int_{t_0}^{\infty} d\vartheta\, \frac{\operatorname{Im}\alpha(\vartheta)}{\vartheta(\vartheta-t)}. \tag{13}$$

Equation (10) is then reduced to

$$(\kappa_1-\kappa/2)\int_{t_0}^{\infty} d\vartheta\, \operatorname{Im}\alpha(\vartheta)K(\vartheta, \kappa, \kappa_1) = 0, \tag{14}$$

where

$$K(\vartheta, \kappa, \kappa_1) = \frac{\vartheta^2-\kappa_1(\kappa-\kappa_1)\,[2\vartheta+\kappa_1^2+\kappa(\kappa-\kappa_1)]}{[\vartheta+(\kappa-\kappa_1)^2]^2[\vartheta+\kappa_1^2]^2}. \tag{15}$$

From (14) we see that the point $\kappa_1 = \kappa/2$ is an extremum for the function $\beta_2(\kappa_1, \kappa-\kappa_1)$. But since the function $K(\vartheta, \kappa, \kappa_1)$ is not constant in sign in the integration region, condition (14) can also be satisfied for other relations between κ_1 and κ.

Condition (11), (12) now has the form

$$\int_{t_0}^{\infty} d\vartheta\, \operatorname{Im}\alpha(\vartheta)\,\frac{4\vartheta+3t}{(4\vartheta-t)^3} > 0. \tag{16}$$

If we suppose (as it is usually done) that $\operatorname{Im}\alpha(\vartheta)$ is positive in the entire physical region, then (16) is satisfied for all $t \geqslant -4t_0/3$ (irrespective of the trajectory form). Consequently, in this kinematic region the point $\kappa_1 = \kappa/2$ is a maximum of the function $\beta_2(\kappa_1, \kappa-\kappa_1)$ and the leading branch point will move along the trajectory

$$\alpha_2(t) = 2\alpha(t/4)-1. \tag{17}$$

However, when $t < -4t_0/3$, the integrand in (16) is of a different sign in the region of integration and therefore the validity of the inequality (16) depends on the behaviour of $\operatorname{Im}\alpha(t)$. It is easy to see that condition (16) can be violated if $\operatorname{Im}\alpha(t)$ is great enough near the threshold and increases not too rapidly with t.

Let us consider the situation when condition (16) is violated, and the point $\kappa_1 = \kappa/2$ is not longer a maximum of the function $\beta_2(\kappa_1, \kappa-\kappa_1)$. In the vicinity of the point $\kappa_1 = \kappa/2$

$$\beta_2(\kappa_1, \kappa-\kappa_1)-\beta_2(\kappa/2, \kappa/2) \simeq \tfrac{1}{2}\, b_2(\kappa)\Delta^2+\frac{1}{4!}\, b_4(\kappa)\Delta^4, \tag{18}$$

where

$$\Delta = \kappa_1-\kappa/2, \qquad b_n(\kappa) = \frac{\partial^n}{\partial\kappa_1^n}\, \beta_2(\kappa_1, \kappa-\kappa_1)|_{\kappa_1=\kappa/2}.$$

Suppose now that κ increases and at some $\kappa = \kappa_K$ condition (16) is violated. Then the coefficient functions in the expansion (18) exhibit the following behaviour in some vicinity of this point

$$\begin{aligned} \operatorname{sign} b_2(\kappa) &= \operatorname{sign}(\kappa-\kappa_K), \\ b_2(\kappa_K) &= 0, \\ b_4(\kappa) &< 0. \end{aligned} \tag{19}$$

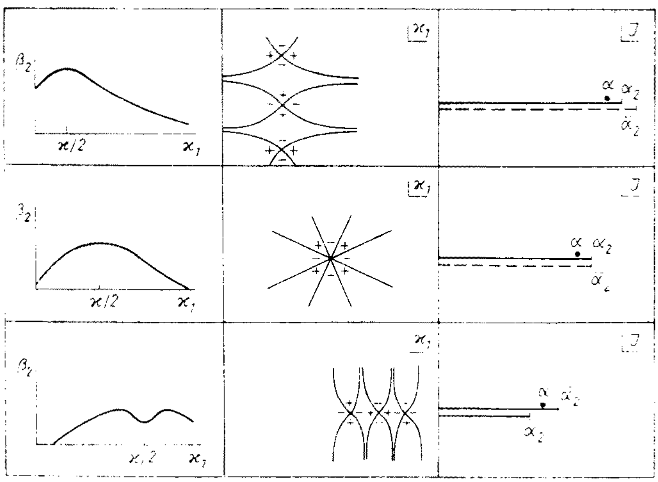


Fig. 1. The left column demonstrates the behaviour of the function $\beta_2(\varkappa_1, \varkappa-\varkappa_1)$, defined by (9), at different values of $t = -\varkappa^2$. The middle column shows the saddle point position of this function in the $\varkappa_1$ variable plane (signs $+(-)$ indicate the directions of rises (falls) on the relief of the considered function). The right column represents the position of the branch points $\alpha_2(t)$ and $\bar{\alpha}_2(t)$ at various t (dashed lines denote the cuts situated on the second sheet)

From (18) we see that at $\kappa > \kappa_K$ two maxima in κ_1 emerge in $\beta_2(\kappa_1, \kappa-\kappa_1)$, both maxima being symmetric relative to the point $\kappa/2$. The latter point will now be a minimum (see Fig. 1). Thus at the point $\kappa = \kappa_K$ there occurs a reconstruction of the relief of the function $\beta_2(\kappa_1, \kappa-\kappa_1)$ (see Fig. 1) and at $\kappa > \kappa_K$ the asymptotics of the amplitude $M_2(s, t)$ are defined by the saddle point at $\kappa_1 = \kappa_\pm \equiv \kappa/2\pm\zeta(t)$ (the function $2\zeta(t)$ gives the distance between κ_+ and κ_- for real $\kappa > \kappa_K$). Consequently, in this kinematic region the leading contribution is given by the branch point[1] with

$$\bar{\alpha}_2(t) = \alpha\left(-\left(\frac{\sqrt{-t}}{2}+\zeta(t)\right)^2\right)+\alpha\left(-\left(\frac{\sqrt{-t}}{2}-\zeta(t)\right)^2\right)-1. \tag{20}$$

In the language of the J-plane, this picture looks as follows. When κ varies from 0 to κ_K, a branch point $\alpha_2(t)$ moves on the physical sheet of the J-plane. A branch point at $J = \bar{\alpha}_2(t)$ lies on the second sheet with respect to the branch point $J = \alpha_2(t)$ (see Figs 1, 2) and does not contribute to the asymptotics of the amplitude. When κ passes through the value κ_K, the branch point $J = \bar{\alpha}_2$ emerges from the second sheet, settles to the right

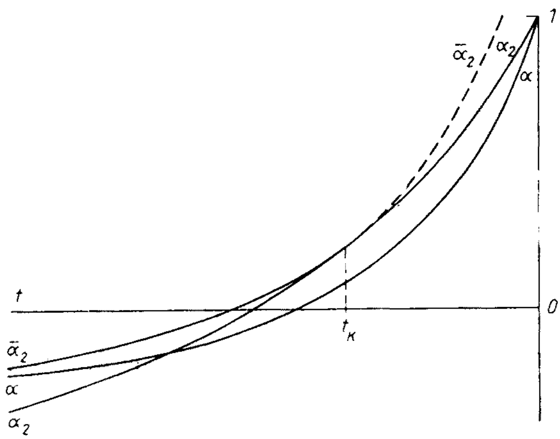


Fig. 2. The typical behaviour of the Regge pole trajectory and the branch point trajectories $\alpha_2(t)$ and $\bar{\alpha}_2(t)$. The dashed line indicates that the given singularity lies on the second sheet of the J-surface

of α_2 and gives the main contribution to the asymptotics of $M_2(s, t)$. The typical behaviour of a pole trajectory $\alpha(t)$ and branch point trajectories $\alpha_2(t)$ and $\bar{\alpha}_2(t)$ is shown in Fig. 2. In the Appendix it is shown that after such a reconstruction of the asymptotic regime, the branch point at $J = \alpha_2$ remains on the physical sheet of the J-plane and continues to contribute to the asymptotics, though its contribution will be decreasing compared to that of the branch point $J = \bar{\alpha}_2$.

From expression (18) one can find the behaviour of the function $\zeta(t)$ at t near $t_K = -\kappa_K^2$. Differentiating (18) with respect to κ_1 and setting the derivative equal to zero, we obtain the equation for the extrema of $\beta_2(\kappa_1, \kappa-\kappa_1)$ in κ_1

$$(\kappa_1-\kappa/2)\,[b_2(\kappa)+\tfrac{1}{6}b_4(\kappa)\,(\kappa_1-\kappa/2)^2] = 0. \tag{21}$$

The condition (21) is satisfied at

$$\kappa_1 = \kappa/2$$
$$\kappa_1 = \kappa_\pm = \kappa/2\pm\sqrt{-6b_2(\kappa)/b_4(\kappa)}. \tag{22}$$

[1] We call it the anomalous branch point in contrast to the normal branch point with trajectory (17).

Comparing with (20), we find that at $\kappa \simeq \kappa_K$

$$\zeta(t) \simeq \sqrt{-6b_2(\kappa)/b_4(\kappa)}. \tag{23}$$

As soon as $b_2(\kappa) \sim \kappa - \kappa_K$ (see (19)), $\zeta(t) \sim \sqrt{\kappa - \kappa_K}$, the function $\zeta(t)$ being real for real $\kappa > \kappa_K$. Such a behaviour of $\zeta(t)$ leads, in particular, to the fact that

$$\alpha_2(t_K) = \bar{\alpha}_2(t_K), \quad \alpha_2'(t_K) = \bar{\alpha}_2'(t_K), \tag{24}$$

i. e. the trajectories $\alpha_2(t)$ and $\bar{\alpha}_2(t)$ do not cross, but are only tangent at the point $t = t_K$ (see Fig. 2). In the language of the J-plane, this means that the branch point at $J = \bar{\alpha}_2$ is always placed to the right of the branch point $J = \alpha_2$.

The reconstruction of the J-plane at $t = t_K$ is also displayed in a change of the asymptotics of $M_2(s, t)$, namely

$$\begin{aligned} M_2(s, t) &\sim s^{\alpha_2(t)}/\ln s \quad \text{at } t > t_K, \\ &\sim s^{\alpha_2(t_K)}/(\ln s)^{3/4} \quad \text{at } t = t_K, \\ &\sim s^{\bar{\alpha}_2(t)}/\ln s \quad \text{at } t < t_K. \end{aligned} \tag{25}$$

Whether the anomalous branch point $\bar{\alpha}_2(t)$ remains dominant with further increase of $(-t)$, or at some values of t the reconstruction of J-plane would occur again and the branch point at $\alpha_2(t)$ would be dominant, depends on the behaviour of the Regge pole trajectory. However, it can be shown that a decisive factor here is the asymptotic behaviour of Regge trajectories. Suppose that for large $(-t)$

$$\alpha(t) \simeq -\gamma(-t)^\nu, \quad 0 < \nu \leqslant 1. \tag{26}$$

Then

$$\frac{\partial^2}{\partial \kappa_1^2} \beta_2(\kappa_1, \kappa - \kappa_1)|_{\kappa_1 = \kappa/2} = -4\gamma\nu(2\nu - 1)(\kappa/2)^{2\nu - 2}. \tag{27}$$

Since as is seen from (27) condition (11) is violated at $\nu < 1/2$ the branch point $\bar{\alpha}_2(t)$ will be dominant for all $t < t_K$. For $\nu > 1/2$ the branch point at $J = \bar{\alpha}_2$ (if it has emerged on the first sheet of J-surface at some t) will again go to the second sheet (with respect to the branch point $J = \alpha_2$). The case $\nu = 1/2$ requires taking into account next terms in the asymptotic expansion of a Regge trajectory.

We now investigate condition (10), defining extrema of β_2, at $\nu < 1/2$ and $t \to -\infty$. If $\kappa_1/\kappa \to \text{const} \neq 0$ or 1 with increasing κ, then (10) is satisfied only at the point $\kappa_1 = \kappa/2$, which is a minimum of β_2 in the case considered. If at $\kappa \to \infty$

$$\kappa_1 \to \infty, \quad \kappa_1/\kappa \to 0,$$

then the left- and right-hand sides of the equality (10) behave as $\kappa_1^{2\nu-1}$ and $\kappa^{2\nu-1}$, respectively, i. e. β_2 has no maxima in this region of κ_1.

Now let $\kappa_1 \to 0$. Then it follows from (10) that a maximum is reached at

$$\kappa_1 \simeq \frac{\kappa}{\alpha'(0)} \alpha'(-\kappa^2) \simeq \frac{\gamma\nu}{\alpha'(0)} \kappa^{2\nu-1}. \tag{28}$$

The second maximum of β_2 in κ_1 is situated at

$$\kappa-\kappa_1 \simeq \frac{\gamma\nu}{\alpha'(0)}\kappa^{2\nu-1} \to 0.$$

Thus, both the maxima in κ_1 go with κ away from the point $\kappa_1 = \kappa/2$ and approach asymptotically the points $\kappa_1 = 0$ and $\kappa_1 = \kappa$. The function $\zeta(t)$ at $t \to -\infty$ behaves as follows

$$\zeta(t) \simeq \frac{\sqrt{-t}}{2} + O((-t)^{\nu-1/2}), \tag{29}$$

and, consequently, at large $(-t)$

$$\bar{\alpha}_2(t) \simeq \alpha(t)+\alpha(0)-1. \tag{30}$$

Thus at $\nu < 1/2$ and $\alpha(0) = 1$, trajectories of a Regge pole and a two-reggeon branch point coincide asymptotically.

We also note that if the dispersion relation (13) has a linear term bt, then the condition (16) changes to

$$\int_{t_0}^{\infty} d\vartheta \, \mathrm{Im}\, \alpha(\vartheta) \frac{4\vartheta+3}{(4\vartheta-t)^3} > -\pi b/16. \tag{31}$$

It is "more difficult" to break this condition, but it can also be violated for a large enough value of Im $\alpha(t)$, so that the branch point $J = \bar{\alpha}_2$ emerges on the physical sheet. However, with increasing $(-t)$ the branch point with trajectory $\alpha_2(t)$ will be dominant.

4. The behaviour of the trajectory of two-reggeon branch point under particular parametrization of Regge pole trajectories

Suppose that only one threshold branch point at $t = 4m^2$ contributes to a Regge trajectory. If this trajectory behaves asymptotically as $(-t)^\nu$, then its simplest parametrization is

$$\alpha(t) = \lambda-\gamma(1-t)^\nu, \tag{32}$$

where we put, for simplicity, $2m = 1$.

Trajectories of the type (32) are of interest as regards their use in model amplitudes, e. g. in the dual analytic models [6].

The extremum condition for $\beta_2(\kappa_1, \kappa-\kappa_1)$

$$\kappa_1[1+(\kappa-\kappa_1)^2]^{1-\nu} = (\kappa-\kappa_1)\,[1+\kappa_1^2]^{1-\nu} \tag{33}$$

is naturally satisfied at $\kappa_1 = \kappa/2$. But the second derivative at this point

$$\frac{\partial^2}{\partial\kappa_1^2}\beta_2(\kappa_1, \kappa-\kappa_1)|_{\kappa_1=\kappa/2} = -2\nu\left(1+\frac{\kappa^2}{4}\right)[2+(\nu-\tfrac{1}{2})\kappa^2] \tag{34}$$

is negative for all κ if $\nu \geqslant 1/2$. If $\nu < 1/2$, then the second derivative becomes positive at

$$\kappa^2 > \kappa_K^2 = \frac{4}{1-2\nu}. \tag{35}$$

Consequently, the brach point $J = \bar{\alpha}_2(t)$ emerges at $t < -\kappa_K^2$. To illustrate this, let $\nu = 1/3$. Then

$$\bar{\alpha}_2(t) = 2\lambda - 1 - \gamma\left[1+\left(\frac{\sqrt{-t}}{2} - \zeta(t)\right)^2\right]^{1/3} - \gamma\left[1+\left(\frac{\sqrt{-t}}{2} + \zeta(t)\right)^2\right]^{1/3}, \tag{36}$$

where

$$\zeta(t) = \left\{-\frac{t}{4} - \tfrac{2}{3} - \left[-\frac{t}{2} - \tfrac{1}{27} + \sqrt{\frac{t^2}{4} + \tfrac{1}{27}}\right]^{1/3} - \left[-\frac{t}{2} - \tfrac{1}{27} - \sqrt{\frac{t^2}{4} + \tfrac{1}{27}}\right]^{1/3}\right\}^{1/2}. \tag{37}$$

The function $\zeta(t)$ has square root branch points at $t = 0$ and $t = -12$, as well as a third-order branch point at infinity. The trajectory $\bar{\alpha}_2(t)$ has third-order branch points on the physical sheet at $t = 0$ and $t = \infty$. When $t \to \infty$

$$\zeta(t) \simeq \frac{\sqrt{-t}}{2} - (-t)^{-1/6},$$

$$\bar{\alpha}_2(t) \simeq \alpha(t) + \alpha(0) - 1 + \frac{\gamma}{3}(-t)^{-1/3}. \tag{38}$$

In Fig. 3 we show the behaviour of $\beta_2(\kappa_1, \kappa-\kappa_1)$ when $\nu = \frac{1}{3}$, $\lambda = \gamma = 1$ in trajectory (32). The behaviour of the trajectories $\alpha(t)$, $\alpha_2(t)$ and $\bar{\alpha}_2(t)$ in this case is represented in Fig. 4.

We consider two more modifications of expression (32). Let

$$\alpha(t) = \lambda - \gamma(1-t)^\nu - \gamma_1\sqrt{t_1 - t}, \quad \nu < 1/2. \tag{39}$$

In this case the reconstruction of the J-plane will be the same as for the expression (32). If the trajectory has a linear term,

$$\alpha(t) = \lambda - \gamma(1-t)^\nu + bt, \quad \nu < 1/2, \tag{40}$$

then for a sufficiently large value of b, namely, for

$$b/\gamma \geqslant \frac{\nu(1-2\nu)^{2-\nu}}{(2-\nu)(13-2\nu)^{1-\nu}}, \tag{41}$$

only the branch point $\alpha_2(t)$ will be on the physical sheet of the J-surface. For small b, when (41) is violated, the branch point $J = \bar{\alpha}_2(t)$ will be dominant at $\kappa > \kappa_K'$. But at $\kappa > \kappa_K'' > \kappa_K'$ the branch point with trajectory $\alpha_2(t)$ will again be dominant. The values of κ_K' and κ_K'' are defined, respectively, as a smaller and a larger root of the equation

$$\gamma\nu(1-2\nu)\kappa_K^2 = 4b(1+\kappa_K^2/4)^{2-\nu} + 2. \tag{42}$$

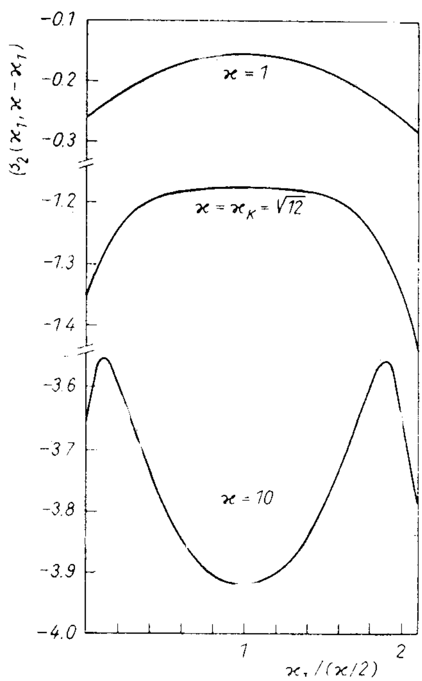


Fig. 3. The behaviour of $\beta_2(\varkappa_1, \varkappa-\varkappa_1)$ in the case when the pole trajectory is defined by the expression (32) with $\nu = 1/3$, $\lambda = \gamma = 1$

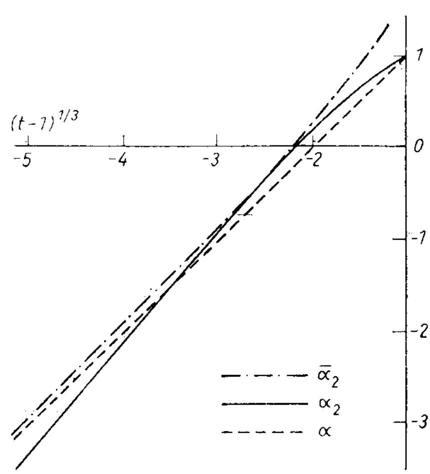
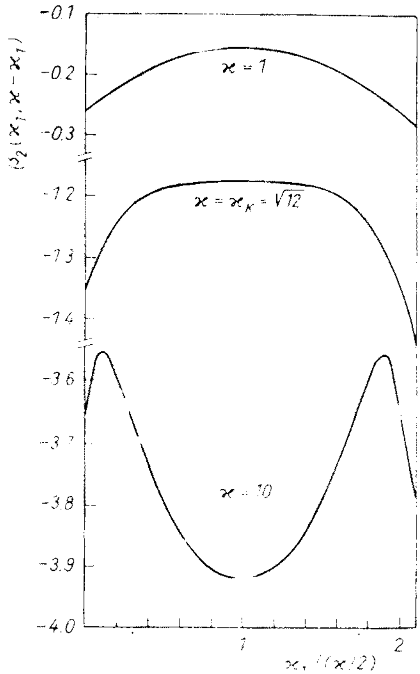


Fig. 4. The motion of the branch points $\bar{\alpha}_2(t)$ and $\alpha_2(t)$ when the pole moves according to (32) with $\nu = 1/3$, $\lambda = \gamma = 1$



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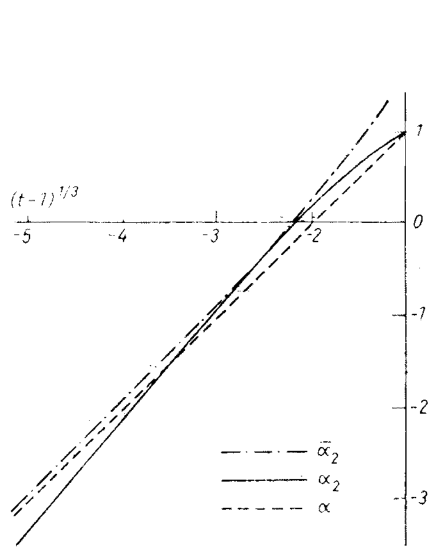

Fig. 4. The motion of the branch points $\alpha_2(t)$ and $\bar{\alpha}_2(t)$ when the pole moves according to (32) with $\nu = 1/3$, $\lambda = \gamma = 1$

5. *Multiple rescatterings*

It is not difficult to generalize the results obtained above to the case of n-reggeon exchanges. Let us represent the expression (4) for β_n in the form

$$\beta_n(\kappa_1, \ldots, \kappa_n) = \beta_2(\tau_2-\kappa_1, \kappa_1)+\beta_{n-2}(\kappa_2, \ldots, \kappa_{n-1}),$$

where

$$\tau_2 = \kappa - \sum_{i=2}^{n-1} \kappa_i.$$

For $\beta_2(\tau_2-\kappa_1, \kappa_1)$ to have a maximum at $\kappa_1 = \tau_2/2$, the conditions (11), (12), (16) must be satisfied. Then the integral over κ_1 in (2) will be asymptotically saturated by the contribution of the saddle point $\kappa_1 = \tau_2/2$, and we must find saddle points of the functions

$$\beta_n\left(\frac{\tau_2}{2}, \frac{\kappa_2}{2}, \kappa_3, \ldots, \tau_2/2\right) = \beta_3\left(\frac{\tau_3-\kappa_2}{2}, \frac{\tau_3-\kappa_2}{2}, \kappa_2\right)+\beta_{n-3}(\kappa_3, \ldots, \kappa_{n-1}),$$

where

$$\tau_3 = \kappa - \sum_{i=3}^{n-1} \kappa_i,$$

$$\beta_3\left(\frac{\tau_3-\kappa_2}{2}, \frac{\tau_3-\kappa_2}{2}, \kappa_2\right) = 2\alpha\left(-\left(\frac{\tau_3-\kappa_2}{2}\right)^2\right)+\alpha(-\kappa_2^2).$$

If the behaviour of the trajectory $\alpha(t)$ and the value of τ_3 are such that the condition

$$\alpha'(-\tau_3^2/9) > (2\tau_3^2/9)\alpha''(-\tau_3^2/9), \tag{43}$$

is satisfied, then the maximum of $\beta_3((\tau_3-\kappa_2)/2, (\tau_3-\kappa_2)/2, \kappa_2)$ in the variable κ_2, will be at $(\tau_3-\kappa_2)/2$, i. e. at $\kappa_2 = \tau_3/3$.

Applying further the same arguments, we come to the conclusion that the main contribution to the integral over κ_{n-1} is given by the maximum of the function

$$\beta_n\left(\frac{\kappa-\kappa_{n-1}}{n-1}, \ldots, \frac{\kappa-\kappa_{n-1}}{n-1}, \kappa_{n-1}\right) = (n-1)\alpha\left(-\left(\frac{\kappa-\kappa_{n-1}}{n-1}\right)^2\right)+\alpha(-\kappa_{n-1}^2).$$

This expression has a maximum at $\kappa_{n-1} = \kappa/n$, if $\kappa = \sqrt{-t}$ is such that

$$\alpha'(t/n^2) > (-2t/n^2)\,\alpha''(t/n^2). \tag{44}$$

When the dispersion relation (13) is taken into account, condition (44) transforms to

$$\int_{t_0}^{\infty} d\vartheta \,\mathrm{Im}\,\alpha(\vartheta)\,\frac{\vartheta+3t/n^2}{(\vartheta-t/n^2)^3} > 0. \tag{45}$$

The conditions (12) and (16) obtained above are particular cases of the conditions (44) and (45), respectively. We see from the latter conditions that the kinematic region of t, where the normal n-reggeon branch point with trajectory (7) always dominates, is defined by the value of a single reggeon momentum. Therefore this region increases $\sim n^2$ with increasing number of reggeons, and the branch point at $J = \alpha_n(t)$ dominates at least at $-t < n^2 t_0/3$.

If parametrization (32) is valid for a pole trajectory, it is then easy to show that for $\nu < 1/2$ a reconstruction of the J-plane occurs at

$$t = -n^2\kappa_K^2/4 = -n^2/(1-2\nu), \tag{46}$$

where κ_K defines the critical point for a two-reggeon exchange.

The trajectory $\bar{\alpha}_n(t)$ has properties similar to those of $\bar{\alpha}_2(t)$. The expression for $\bar{\alpha}_n(t)$ is rather complicated. But one can show that as $t \to -\infty$

$$\alpha_n(t) \simeq \alpha(t)+(n-1)\,(\alpha(0)-1), \tag{47}$$

which is a generalization of (30).

6. Conclusions

The analysis carried out above has shown that if Regge pole trajectories are linear enough, then we have the classical picture of multiple reggeon exchanges, where the momentum is uniformly distributed between all reggeons. But when the nonlinearity of trajectories is great, asymmetry in the momentum distribution can occur, this asymmetry increases with momentum and as $t \to -\infty$ the entire momentum passing through a single reggeon only.

As has been shown above, the properties of n-reggeon branch points depend to a large extent on the asymptotic behaviour of Regge pole trajectories. If pole trajectories rise asymptotically slower than $|t|^{1/2}$, then a new type of branch points will always be dominant at sufficiently large $(-t)$. The reason for this can be explained as follows. Let us consider the exchange of several reggeons with trajectories $\alpha(t)$ and $\alpha(0) = 1$. Then there always exists such a distribution of the transferred momentum $\sqrt{-t}$ between all reggeons that

$$\sum_{i=1}^{n} [\alpha(t_i)-1] \geqslant \alpha(t)-1,$$

with equality e. g. for

$$t_1 = t; \quad t_2, t_3, \ldots = 0.$$

Therefore the branch points can not be situated (at $t \leqslant 0$) to the left of the poles. But if $\alpha(t) \simeq \gamma(-t)^\nu$, $\nu < 1/2$, then

$$\alpha_n(t) \simeq -n\gamma(-t/n^2)^\nu \simeq n^{1-2\nu}\alpha'(t) < \alpha(t),$$

i. e. the normal branch points go to the left of the pole. Before this happens, a new branch point $\bar{\alpha}_n(t)$ emerges, therefore, on the physical sheet of the J-surface. This branch point is always placed to the right of the Regge pole and approaches it asymptotically.

In conclusion we also note that the sum of the contributions of a Regge pole and the rescattering on it behaves in different ways for various asymptotic behaviours of the Regge trajectory. If $\alpha(t) = 1+\alpha' t$, then

$$M(s, t) = \sum_{n=1}^{\infty} \frac{1}{n!} M_n(s, t)$$

$$\sim s^{\alpha' t}\left[1+\varphi_1(t)\frac{s^{-\alpha' t/2}}{\ln s}+\varphi_2(t)\frac{s^{-2\alpha' t/3}}{\ln^2 s}+\ldots\right],$$

i. e. at $t < 0$ each next term of the sum rises faster with s than the foregoing ones. But if $\alpha(t) = 1+\gamma[(4m^2)^\nu-(4m^2-t)^\nu]$, where $\nu \leqslant 1/2$, then at rather large $(-t)$ (of course, t is such that $|t/s| \ll 1$)

$$M(s, t) \sim s^{-\gamma(-t)^\nu}\left[1+\frac{\bar{\varphi}_1(t)}{\ln s}+\frac{\bar{\varphi}_2(t)}{\ln^2 s}+\ldots\right],$$

and the expansion in n-reggeon exchanges will make sense for large momentum transfer. Thus, taking the nonlinearity of Regge trajectories into account allows one to extend the range of applicability of reggeon calculus.

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APPENDIX

We show here that at $\kappa > \kappa_K$, when the branch point at $J = \bar{\alpha}_2(t)$ is dominant, the normal branch point with trajectory $\alpha_2(t)$ continues to contribute to the amplitude, i. e. it remains on the physical sheet of the J-surface. For this purpose we investigate the character of the asymptotics of $M_2(s, t)$ at κ near κ_K.

At $\kappa < \kappa_K$ the expression (8), with the expansion (18) taken into account, has the form

$$M_2(s, t) \sim \xi^{-1/2} \exp\left\{\xi\left[\beta_2\left(\frac{\kappa}{2}, \frac{\kappa}{2}\right)-2\right]\right\} I(s, t), \tag{A.1}$$

where

$$I(s, t) = \int_{-\infty}^{\infty} d\Delta \exp\left\{\xi\left[\tfrac{1}{2} b_2(\kappa)\Delta^2+\frac{1}{4!} b_4(\kappa)\Delta^4\right]\right\}. \tag{A.2}$$

Using the formula (3.323.3) of [7], we obtain

$$I(s, t) = \left(\frac{3b_2(\kappa)}{b_4(\kappa)}\right)^{1/2} e^z K_{1/4}(z), \tag{A.3}$$

where $K_{1/4}(z)$ is the Bessel function of the imaginary argument, and

$$z = 3\xi b_2^2(\kappa)/4b_4(\kappa). \tag{A.4}$$

Since at $z \to \infty$

$$K_{1/4}(z) \sim e^{-z}/\sqrt{z},$$

then at $\xi b_2^2(\kappa) \gg 1$ we have $I(s, t) \sim 1/\sqrt{-\xi b_2(\kappa)}$ and thus

$$M_2(s, t) \sim s^{\alpha_2(t)-1}/\ln s. \tag{A.5}$$

When $z \to 0$

$$K_{1/4}(z) \sim z^{1/4},\ I(s, t) \sim \xi^{-1/4}$$

Therefore, in the region of κ close to κ_K, where $\xi b_2^2(\kappa) \ll 1$

$$M_2(s, t) \sim s^{\alpha_2(t)-1}/(\ln s)^{3/4}. \tag{A.6}$$

Suppose that κ has passed through the critical point κ_K. Then, according to (19), $b_2(\kappa)$ changes sign from minus to plus, and we must substitute $z \to ze^{2i\pi}$ in (A.3). In this case as $z \to \infty$ (with the formulae (8.476.5) and (8.451.5,6) taken into account from [7])

$$K_{1/4}(ze^{2i\pi}) \simeq -i\sqrt{\frac{\pi}{z}}\, e^{z} + \sqrt{\frac{\pi}{2z}}\, e^{-z}.$$

Consequently,

$$I(s, t) \simeq e^{2z}\sqrt{4\pi/\xi b_2(\kappa)} + \sqrt{2\pi/\xi b_2(\kappa)}. \tag{A.7}$$

Substituting (A.7) into (A.1), we see that the second term of (A.7) corresponds to the contribution of the branch point at $J = \alpha_2(t)$, i. e. this branch point remains on the physical sheet after the reconstruction of the J-plane. The first term in (A.7) corresponds to an anomalous branch point with trajectory (20), the function $\xi(t)$ at $\kappa \simeq \kappa_K$ being expressed through $b_2(\kappa)$ and $b_4(\kappa)$ by (23). So at $s \to \infty$ and $\kappa > \kappa_K$

$$M_2(s, t) \sim s^{\bar{\alpha}_2(t)-1}/\ln s + f(t)s^{\alpha_2(t)-1}/\ln s, \tag{A.8}$$

where $f(t)$ is some function of t.

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