Vol. B8 (1977) ACTA PHYSICA POLONICA No 7

ON THE QUANTIZATION OF YANG-MILLS AND GRAVITATIONAL FIELDS. II

By Cao Chi

Institute of Physics, Hanoi*

(Received February 23, 1977)

It is shown how to obtain the Ward identities and the equation for Green functions in our approach based on the external source method combined with the quantum dynamical equation. The results obtained previously and in this paper make transparent the connection between Mandelstam's geometric approach and the functional approach to the quantization problem of gauge fields.

1. Introduction

The Mandelstam approach [1, 2] to the quantization problem of Yang-Mills and gravitational fields may be characterized as a geometric one. In fact the path dependent quantities, the construction of which is the most important step in Mandelstam's formalism, are shown by us [3] to be just the quantities taken on a horizontal path of a fibre space. Furthermore, our approach to the quantization problem of gauge fields, based on the external source method combined with the quantum dynamical equation [4], enables us to etablish a connection between the geometric approach used by Mandelstam and the functional approach used by other authors [5–8]. We would like to note here that such a connection is regrettably absent in the book by Konopleva and Popov [9] who treat the problem of gauge fields under both aspects: geometric (first part) and functional (second part).

The present paper is the continuation of [4] and is aimed at showing how to obtain the generating equation for Green functions and the Ward identities in our approach and at making more transparent the connection, which is mentioned above.

We shall use indices from the beginning of the Greek alphabet to denote components in isotopic space and indices from the middle of the Greek alphabet to denote components in the space-time.

* Address: Viên Vâtly, Nghia đô, Hanôi, ViêtNam.

We have obtained the following functional differential equation for the vacuum-to--vacuum amplitude Z in the case of Yang-Mills fields (Eq. (2.13) in [4]):

$$\left(\frac{\delta S}{\delta A^{\alpha}_{\mu}(x)}+iJ^{\alpha}_{\mu}(x)-ifc^{\alpha\beta\gamma}\partial_{\mu}G^{\beta\gamma}(x, y)|_{x=y}\right)_{A\to\frac{\delta}{\delta J}}Z = 0, \tag{1.1}$$

where $G^{\alpha\beta}(x, y)$ is the Green function defined by (Eq. (2.12) in [4])

$$\nabla^{\alpha\gamma}_{\mu}(x)\partial_{\mu}G^{\gamma\delta}(x, y) = \delta^{\alpha\delta}\delta(x-y), \tag{1.2}$$

with

$$\nabla^{\alpha\gamma}_{\mu} = \delta^{\alpha\gamma}\partial_{\mu}+fc^{\alpha\varepsilon\gamma}A^{\varepsilon}_{\mu}. \tag{1.3}$$

An important identity necessary for the deduction of the Ward identities was also proved (Eq. (2.10) in [4])

$$\nabla^{\beta\alpha}_{\mu}J^{\alpha}_{\mu} = -\nabla^{\beta\alpha}_{\mu} : \partial_{\mu}\int[\nabla^{\gamma\delta}_{\nu}(y)J^{\delta}_{\nu}(y)]G^{\alpha\gamma}(x, y)dy : -\nabla^{\beta\alpha}_{\mu}fc^{\alpha\delta\gamma}[\partial_{\mu}G^{\delta\gamma}(x, y)]_{x=y}. \tag{1.4}$$

In (1.4) the symbol :...: indicates that in the expression between colons the external source J^{α}_{μ} and the field A^{α}_{μ} are to be ordered so that J^{α}_{μ} appears before A^{α}_{μ}.

In the gravitational case the functional differential equation for Z is of the form (Eq. (3.13) in [4])

$$\left(\frac{\delta S}{\delta \mathfrak{g}^{\mu\nu}(x)}+iJ_{\mu\nu}(x)-i\partial_{\mu}\partial_{\nu}G(x, y)|_{x=y}\right)_{\mathfrak{g}\to\frac{\delta}{\delta J}}Z = 0, \tag{1.5}$$

where $\mathfrak{g}^{\mu\nu}(x) \equiv \sqrt{-g}g^{\mu\nu}(x)$ and the Green function $G(x, y)$ is defined by (Eq. (3.12) in [4])

$$\nabla^{\mu\nu}_{\lambda}(x)\partial_{\mu}G(x, y) = \delta^{\nu}_{\lambda}\delta(x-y) \tag{1.6}$$

with

$$\nabla^{\mu\nu}_{\lambda} = \mathfrak{g}^{\mu\nu}\partial_{\lambda}-\partial_{\varrho}\mathfrak{g}^{\varrho\nu}\delta^{\mu}_{\lambda}-\partial_{\varrho}\mathfrak{g}^{\varrho\mu}\delta^{\nu}_{\lambda}-\mathfrak{g}^{\nu\varrho}\delta^{\mu}_{\lambda}\partial_{\varrho}-\mathfrak{g}^{\mu\varrho}\delta^{\nu}_{\lambda}\partial_{\varrho}. \tag{1.7}$$

The identity analogous to (1.4) in the gravitational case is the following (Eq. (3.10) in [4])

$$\nabla^{\mu\nu}_{\lambda}J_{\mu\nu} = -\nabla^{\mu\nu}_{\lambda}\partial_{\nu} : \int G(x, y)\nabla^{\varrho\sigma}_{\mu}(y)J_{\varrho\sigma}(y)dy : -\nabla^{\mu\nu}_{\lambda}\partial_{\mu}\partial_{\nu}G(x, y)|_{y=x}. \tag{1.8}$$

2. *Yang-Mills fields*

A. Equation for Green functions

Let us consider Eq. (1.1). We have

$$\frac{\delta S}{\delta A^{\alpha}_{\mu}(x)} = \partial_{\nu}(\partial_{\nu}A^{\alpha}_{\mu}(x)-\partial_{\mu}A^{\alpha}_{\nu}(x))+fj^{\alpha}_{\mu}(x),$$

where

$$j^{\alpha}_{\mu} = c^{\alpha\gamma\beta}(-A^{\gamma}_{\mu}\partial_{\nu}A^{\beta}_{\nu}+2A^{\gamma}_{\nu}\partial_{\nu}A^{\beta}_{\mu}-A^{\gamma}_{\nu}\partial_{\mu}A^{\beta}_{\nu})+fc^{\alpha\gamma\beta}c^{\beta\varepsilon\delta}A^{\gamma}_{\nu}A^{\varepsilon}_{\nu}A^{\delta}_{\mu}. \tag{2.1}$$

In (1.1) the expression enclosed in parentheses is of the form

$$\partial_\nu(\partial_\nu A^\alpha_\mu - \partial_\mu A^\alpha_\nu) = \mathscr{J}^\alpha_\mu. \tag{2.2}$$

If we add to the LHS of the preceding equation a divergence $\frac{1}{\alpha}\partial_\mu\theta^\alpha(\theta^\alpha \equiv \partial_\nu A^\alpha_\nu)$ we obtain the equation

$$\Box A^\alpha_\mu - \left(1-\frac{1}{\alpha}\right)\partial_\mu\theta^\alpha = \mathscr{J}^\alpha_\mu, \tag{2.3}$$

with solution

$$A^\alpha_\mu(x) = \int\left[\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}(1-\alpha)\right]\langle x|\Box^{-1}|u\rangle\mathscr{J}^\alpha_\nu(u)du, \tag{2.4}$$

where $\langle x|\Box^{-1}|y\rangle \equiv \Delta^0_F(x, y)$.

Thus the adding of a divergence leads only to the change of the gauge of the Green function

$$D_{\mu\nu}(x, y) = \left[\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}(1-\alpha)\right]\langle x|\Box^{-1}|y\rangle.$$

Integrating (1.1) in the gauge α we have

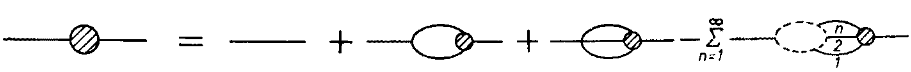
$$\begin{aligned}&(A^\alpha_\mu(x) + f\textstyle\int D_{\mu\nu}(x, u)j^\alpha_\nu(u)du + \int D_{\mu\nu}(x, u)J^\alpha_\nu(u)du \\ &- \textstyle\int D_{\mu\nu}(x, u)fc^{\alpha\beta\gamma}\partial_\nu G^{\beta\gamma}(u, y)|_{y=u}du)_{A\to\frac{\delta}{\delta J}}Z = 0.\end{aligned} \tag{2.5}$$

If we operate from the left with $P\left[\frac{\delta}{\delta J}\right]$ and set $J \to 0$ after commuting P with the source, we obtain the following equation for Green functions from (2.5)

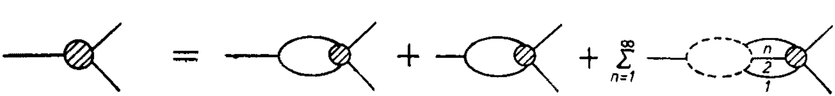
$$\begin{aligned}&\langle 0|T\left\{A^\alpha_\mu(x)P[A] + f\int D_{\mu\nu}(x, u)j^\alpha_\nu(u)duP[A] + \int D_{\mu\nu}(x, u)\frac{\delta}{\delta A^\alpha_\mu(u)}P[A]du\right. \\ &\left. - \int D_{\mu\nu}(x, u)fc^{\alpha\beta\gamma}\partial_\nu G^{\beta\gamma}(u, y)|_{y=u}duP[A]\right\}|0\rangle = 0.\end{aligned} \tag{2.6}$$

(In (2.6) $P[A]$ is an arbitrary functional of $A^\alpha_\mu(x)$.

(i) Inserting $P[A] = A^\beta_\nu(y)$ into (2.6) we get



(ii) If we set $P[A] = A^\beta_\nu(y)\, A^\gamma_\lambda(z)$ in (2.6) we have



In the preceding diagram equation the continuous lines describe the gauge field, the dotted — the Faddeev-Popov ghost field.

Similarly we can obtain equation for other Green functions from the generating equation (2.6) by setting $P[A]$ equal to various polynomials of A_μ^α and θ^β.

B. Ward identities

After finding Z from (1.1) in the general gauge α and setting $\delta Z = 0$, where δZ is the variation of under the transformations $A_\mu^\alpha \to A_\mu^\alpha - f^{-1} \nabla_\mu^{\alpha\beta} u^\beta$ (u^β — arbitrary function of x) we get the following equation

$$\frac{1}{\alpha} \nabla_\mu^{\beta\alpha} \partial_\mu \theta^\alpha + \nabla_\mu^{\beta\alpha} f c^{\alpha\delta\gamma} [\partial_\mu G^{\delta\gamma}(x, y)]_{x=y} + \nabla_\mu^{\beta\alpha}(y) J_\mu^\alpha(y) = 0. \tag{2.7}$$

By using the important identity (1.4) we can convert (2.7) into

$$\frac{1}{\alpha} \theta^\alpha = : \int G^{\alpha\gamma}(x, y) \nabla_\nu^{\gamma\delta}(y) J_\nu^\delta(y) dy :. \tag{2.8}$$

From (2.8) we get the Ward identities

$$\frac{1}{\alpha} \langle 0 | T\theta^\alpha(x) P[A] | 0 \rangle = \langle 0 | T \int du G^{\alpha\gamma}(x, u) \nabla_\nu^{\gamma\delta}(u) \frac{\delta}{\delta A_\nu^\delta(u)} P[A] | 0 \rangle. \tag{2.9}$$

If we set $P[A] = \theta^\beta$ and then $P[A] = \theta^\beta(y) A_\nu^\gamma(z)$ the Ward identities (2.9) yield respectively $p_\mu D_{\mu\nu}^{\alpha\beta} = \alpha \delta^{\alpha\beta} p_\nu p^{-2}$ and the Slavnov identity.

3. *Gravitational field*

A. Equation for Green functions

Let us consider Eq. (1.5). Here

$$\frac{\delta S}{\delta \mathfrak{g}^{\mu\nu}(x)} = R_{\mu\nu}(x). \tag{3.1}$$

Now rewritte $\mathfrak{g}^{\mu\nu}$ as

$$\mathfrak{g}^{\mu\nu} = \delta^{\mu\nu} + \lambda h^{\mu\nu},$$

where $\delta^{\mu\nu}$ is the Minkowski metric. Separating the terms which are proportional to λ from $R_{\mu\nu}(x)$ we have

$$R_{\mu\nu} = -\lambda(\tfrac{1}{4} \Box h \delta_{\mu\nu} + \tfrac{1}{2} \partial_\nu \partial_\lambda h_\mu^\lambda + \tfrac{1}{2} \partial_\mu \partial_\lambda h_\nu^\lambda - \tfrac{1}{2} \Box h_{\mu\nu}) + \mathscr{R}_{\mu\nu}(\lambda^2). \tag{3.2}$$

In (3.2), $\mathscr{R}_{\mu\nu}(\lambda^2)$ is that part of $R_{\mu\nu}$ which contains second, third and higher powers of λ. If in (1.5) we add the divergences $\frac{1}{2\alpha} \delta_{\{\nu} \theta_{\mu\}}$ ($\theta_\mu \equiv \partial_\sigma \mathfrak{g}_\mu^\sigma$, $\{\ldots\}$ denotes symmetrization of indices represented by three points) to the expression in parentheses we get the following equation

$$\lambda\left(\tfrac{1}{4}\,\Box h\delta_{\mu\nu}+\tfrac{1}{2}\,\partial_\nu\partial_\lambda h^\lambda_\mu+\tfrac{1}{2}\,\partial_\mu\partial_\lambda h^\lambda_\nu-\tfrac{1}{4}\,\Box h_{\mu\nu}-\frac{1}{2\alpha}\,\partial_\nu\partial_\lambda h^\lambda_\mu-\frac{1}{2\alpha}\,\partial_\mu\partial_\lambda h^\lambda_\nu\right)$$
$$=\mathscr{R}_{\mu\nu}(\lambda^2)+J_{\mu\nu}-\partial_\mu\partial_\nu G(x,y)|_{x=y} \tag{3.3}$$

with a solution

$$\lambda h^{\lambda\sigma}(x)=\int D^{\lambda\sigma\mu\nu}(x,u)\mathscr{I}_{\mu\nu}(u)du \tag{3.4}$$

where $\mathscr{I}_{\mu\nu}$ is the LHS of (3.3) and

$$D^{\lambda\sigma\mu\nu}(x,y)=\Big[(2-\alpha)\delta^{\lambda\sigma}\delta^{\mu\nu}-\delta^{\lambda\mu}\delta^{\sigma\nu}-\delta^{\lambda\nu}\delta^{\sigma\mu}+\frac{2(\alpha-1)}{\Box}(\delta^{\lambda\sigma}\partial^\mu\partial^\nu+\delta^{\mu\nu}\partial^\lambda\partial^\sigma)$$
$$+(1-\alpha)(\delta^{\lambda\mu}\partial^\sigma\partial^\nu+\delta^{\lambda\nu}\partial^\sigma\partial^\mu+\delta^{\sigma\mu}\partial^\lambda\partial^\nu+\delta^{\sigma\nu}\partial^\lambda\partial^\mu)\Big]\langle x|\Box^{-1}|y\rangle.$$

Integrating Eq. (1.5) in the gauge α we obtain

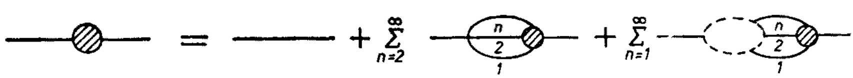
$$\{\lambda h^{\mu\nu}(x)-\int D^{\mu\nu\lambda\sigma}(x,u)\,[\mathscr{R}_{\lambda\sigma}(u)+J_{\lambda\sigma}(u)$$
$$-\partial_\lambda\partial_\sigma G(u,y)|_{y=u}]du\}_{\mathfrak{g}\to\frac{\delta}{\delta J}}Z=0. \tag{3.5}$$

By the same procedure used for obtaining (2.6) from (2.5) we convert (3.4) into the following generating equation for Green functions

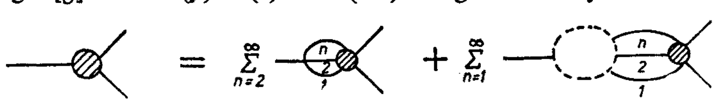
$$\langle 0|T\Big\{\lambda h^{\mu\nu}(x)P[\mathfrak{g}]-\int D^{\mu\nu\lambda\sigma}(x,u)\Big[\mathscr{R}_{\lambda\sigma}(u)P[\mathfrak{g}]$$
$$+\frac{\delta}{\delta\mathfrak{g}^{\lambda\sigma}(u)}P[\mathfrak{g}]-\partial_\lambda\partial_\sigma G(u,y)|_{y=u}P[\mathfrak{g}]\Big]du\Big\}|0\rangle=0 \tag{3.6}$$

where $P[\mathfrak{g}]$ is an arbitrary functional of $\mathfrak{g}^{\mu\nu}$.

(*i*) Substituting $P[\mathfrak{g}]=\mathfrak{g}^{\varrho\sigma}(y)-\delta^{\varrho\sigma}=\lambda h^{\varrho\sigma}(y)$ into (3.5) we get



(*ii*) Substituting $P[\mathfrak{g}]=\lambda^2 h^{\varrho\sigma}(y)h^{\tau\omega}(z)$ into (3.5) we get similarly



The equation for higher Green functions may be obtained by setting $P(\mathfrak{g}]$ equal to various polynomials of $\mathfrak{g}^{\mu\nu}$ and θ^λ.

B. Ward identities

In order to obtain Ward identities we must find Z from (1.5) in the general gauge α and then set $\delta Z=0$ (δZ is now the variation of Z under the transformations $\mathfrak{g}^{\mu\nu}\to\mathfrak{g}^{\mu\nu}$

$-u^{\varrho}\partial_{\varrho}\mathfrak{g}^{\mu\nu}+\mathfrak{g}^{\mu\varrho}\partial_{\varrho}u^{\nu}+\mathfrak{g}^{\nu\varrho}\partial_{\varrho}u^{\mu}-\mathfrak{g}^{\mu\nu}\partial_{\varrho}u^{\varrho}$, here u^{ν} is an arbitrary function of x). As result, we obtain

$$\frac{1}{\alpha}\nabla_{\lambda}^{\mu\nu}\partial_{\nu}\theta^{\mu}+\nabla_{\lambda}^{\mu\nu}\partial_{\mu}\partial_{\nu}G(x,y)|_{y=x}+\nabla_{\lambda}^{\mu\nu}J_{\mu\nu} = 0. \tag{3.7}$$

If we make use of the identity (1.8) Eq. (3.6) can be expressed as follows

$$\frac{1}{\alpha}\theta^{\mu}(x) = :\int G(x,u)\nabla_{\mu}^{\varrho\sigma}(u)J_{\varrho\sigma}(u)du: \tag{3.8}$$

Eq. (3.8) yields us the following Ward identities

$$\frac{1}{\alpha}\langle 0|T\theta^{\nu}(x)P[\mathfrak{g}]\,|0\rangle = \langle 0|T\int G(x,u)\nabla_{\nu}^{\varrho\sigma}(u)\frac{\delta}{\delta\mathfrak{g}^{\varrho\sigma}(u)}P[\mathfrak{g}]\,|0\rangle. \tag{3.9}$$

If we set $P[\mathfrak{g}] = \theta^{\varphi}(y)$ and then $P[\mathfrak{g}] = \theta^{\varrho}(y)\,\mathfrak{g}^{\omega\tau}(z)$, the Ward identities (3.9) yield respectively $p_{\mu}p_{\lambda}D^{\mu\nu\lambda\varphi} = \alpha\delta^{\nu\varphi}$ and the Slavnov identity for gravitational field obtained in [8].

It was noted in [4] that the important identities (1.4) and (1.8), which enable us to derive all results, obtained in [4] and in the present paper, are analogous to the equations (4.43) in [1] and (6.15) in [2] for path independent Green functions of Mandelstam's formalism. Thus our approach furnishes a way for establishing a transparent connection between the functional approach and the geometric approach used in Mandelstam's papers. The geometric meaning of Mandelstam's approach was shown by us in [3] by using the theory of fibre bundle.

REFERENCES

[1] S. Mandelstam, *Phys. Rev.* **175**, 1580 (1968).
[2] S. Mandelstam, *Phys. Rev.* **175**, 1604 (1968).
[3] Cao Chi, Doan Nhuong, *Acta Phys. Pol.* **B7**, 11 (1976).
[4] Cao Chi, *Acta Phys. Pol.* **B7**, 3 (1976).
[5] B. S. DeWitt, *Dynamical Theory of Groups and Fields*, Gordon and Breach 1965.
[6] L. D. Faddeev, V. N. Popov, *Phys. Lett.* **25B**, 30 (1967).
[7] E. S. Fradkin, I. V. Tyutin, *Phys. Rev.* **D2**, 2841 (1970).
[8] E. S. Fradkin, G. A. Vilkovisky, *Phys. Rev.* **D8**, 4241 (1973).
[9] N. P. Konopleva, V. N. Popov, *Kalibrovochnye polya*, Atomizdat 1972.