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A SELF-CONSISTENT *N/D* CALCULATION OF THE RHO MESON USING THE VENEZIANO MODEL

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A self-consistent calculation for the rho-meson mass and width is performed. The Veneziano model is used to calculate a partial-wave amplitude for pion-pion scattering with isospin $I = 1$ and angular momentum $l = 1$. The left-hand cut discontinuity is obtained for this partial wave and is used as input to the Frye-Warnock N/D equations. From the N/D equations, a new partial-wave amplitude satisfying unitarity is determined. The absorption factor, $\eta(s)$, is adjusted so that the output ϱ resonance from the new partial wave is consistent with experimental values. By varying parameters associated with the right and left-hand cuts, the relative significance of the two cuts is examined.

1. Introduction

In this study we develop a mathematical formulation of pion-pion scattering, utilizing the Veneziano model [1] and the Frye-Warnock N/D formalism [2]. The major shortcoming of the Veneziano model is its lack of unitarity. However, since unitarity is incorporated into the N/D equations, it is hoped that using the Veneziano model in conjunction with the N/D method will result in a scattering amplitude which retains the desirable features of the Veneziano model and includes unitarity.

In the calculations the $I = 1$, $l = 1$ partial-wave amplitude for pion-pion scattering is obtained from the Veneziano model, and the corresponding left-hand discontinuity is then used as input to the Frye-Warnock equations. The equations are numerically solved, and a solution for the partial-wave amplitude is sought which exhibits a resonance at the observed experimental values for the rho.

In practice, there are a number of variable parameters in these equations. For the left-hand cut, the parameter values of the rho trajectory (assumed to be linear) and the value of an overall constant factor are determined from resonance data in the physical

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region. The parameter values associated with the right-hand inelastic cut are then adjusted to give the best agreement between the output data and experimental data. It is anticipated that some physical significance can be attached to the values of the parameters which produce the best output resonance mass and with.

2. N/D method

The invariant pion-pion partial-wave amplitude with $l = 1$ is written in the form

$$A(s) = \frac{\eta(s)e^{2i\delta(s)}-1}{2i\varrho(s)} \equiv \frac{N(s)}{D(s)}$$

where

$$\varrho(s) = \left(\frac{s-4}{s}\right)^{1/2}$$

and the units are such that $m_\pi = 1$. $N(s)$ is defined to contain the left-hand cut ($s < -s_L$, where $-s_L$ marks the start of the left-hand cut) and the right-hand inelastic cut ($s > 16$). $D(s)$ contains the right-hand elastic cut ($s > 4$).

Choosing the phase of $D(s)$ to be $-\delta$, we obtain

$$N(s) = \frac{1}{2i\varrho(s)}\left[\eta(s)D^*(s)-D(s)\right].$$

Separating $N(s)$ and $D(s)$ into real and imaginary parts in the above equation gives

$$\mathrm{Im}\, D(s) = -\frac{2\varrho(s)}{1+\eta(s)}\,\mathrm{Re}\, N(s), \qquad s > 4, \tag{1}$$

$$\mathrm{Im}\, N(s) = \frac{1-\eta(s)}{2\varrho(s)}\,\mathrm{Re}\, D(s), \qquad s > 16. \tag{2}$$

A subtracted dispersion relation for $D(s)$ may then be written

$$D(s) = 1-\frac{s-s_0}{\pi}\int_4^\infty \frac{ds'}{(s'-s)(s'-s_0)}\,\frac{2\varrho(s')}{1+\eta(s')}\,\mathrm{Re}\, N(s'), \tag{3}$$

where $D(s)$ has been normalized to unity at an arbitrary subtraction point, s_0, and Eq. (1) has been substituted for Im $D(s)$ under the integral. In general, the results for the amplitude $A(s)$ do not depend on the choice of s_0 [3].

As shown by Frye and Warnock [2], an integral equation for Re$N(s)$ may now be written in the form

$$\frac{2\eta(s)}{1+\eta(s)}\,\mathrm{Re}\, N(s) = \bar{B}(s)+\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}\left[\bar{B}(s')-\frac{s-s_0}{s'-s_0}\,\bar{B}(s)\right]\frac{2\varrho(s')}{1+\eta(s')}\,\mathrm{Re}\, N(s') \tag{4}$$

with

$$\bar{B}(s) = \frac{1}{\pi} \int_{-\infty}^{-s_L} \frac{ds'}{s'-s} \text{Disc}_L(s') + \frac{P}{\pi} \int_{16}^{\infty} \frac{ds'}{s'-s} \left[\frac{1-\eta(s')}{2\varrho(s')} \right]. \tag{5}$$

These equations provide a method for generating a partial-wave amplitude from the left-hand cut discontinuity and knowledge of $\eta(s)$. The left-hand discontinuity and absorption parameter are used in calculating $\bar{B}(s)$ from Eq. (5). Once $\bar{B}(s)$ is known, Re$N(s)$ is calculated from Eq. (4). Then the real and imaginary parts of $D(s)$ are found from Eqs (1) and (3). Finally, the imaginary part of $N(s)$ is calculated from Eq. (2). The partial-wave amplitude is then completely determined.

3. Veneziano model and discontinuity across the left-hand cut

Defining a Veneziano amplitude, $V(s, t)$, to be

$$V(s, t) = -\gamma \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))},$$

we may then write for the isospin-1 pion-pion scattering amplitude

$$A^1(s, t, u) = V(s, t) - V(s, u).$$

Here $\alpha(s)$ is the $\varrho - f^0$ exchange-degenerate trajectory and is assumed to have a linear form:

$$\alpha(s) = as + b.$$

The partial-wave projection of $A^1(s, t, u)$ for physical l values can be written [4]

$$V_l(s) = \gamma \frac{\alpha(s)}{ak^2} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha(s)+1)}{\Gamma(\alpha(s)+1)n!} Q_l\left(1+\frac{n+1-b}{2ak^2}\right), \quad l > \text{Re}\,\alpha(s),$$

where k is the center of mass momentum. For $b < 1$, this expression is an analytic function of s with a left-hand cut starting at $s = -s_L = 4m_\pi^2 + (b-1)/a$.

The discontinuity across the cut for $l = 1$ is given by

$$\text{Disc}\, V_1(s) = \tfrac{1}{2}\pi\gamma \frac{\alpha(s)}{ak^2} \sum_{n=0}^{p} \frac{\Gamma(n+\alpha(s)+1)}{n!\Gamma(\alpha(s)+1)} \left(1+\frac{n+1-b}{2ak^2}\right), \quad s < -s_L, \tag{6}$$

where p is the largest integer less than or equal to $b-1-4ak^2$.

4. *Results*

The left-hand cut discontinuity for the $I = 1$, $l = 1$ partial-wave, calculated from the Veneziano model, Eq. (6), was used as input to the N/D equations. A new p-wave amplitude was then obtained numerically from these equations. The absorption factor, $\eta(s)$, was adjusted until the values of the output ϱ mass and width were roughly consistent with the input values.

We have found that when the Veneziano model is used as input to the elastic N/D equations ($\eta(s) = 1$), a self-consistent calculation is not possible. Although the output ϱ resonance can be forced to the correct mass, in order to do so, it is necessary to increase the coupling constant, γ, to approximately seven times its correct value (as determined by the procedure discussed below), and even in this case the output rho width is much too large. However by using the Veneziano model in conjunction with the inelastic N/D equations, a self-consistent calculation with correct rho mass and width has been obtained.

In the following discussion, all values of s are given in units of m_π^2, with $m_\pi^2 = 1$.

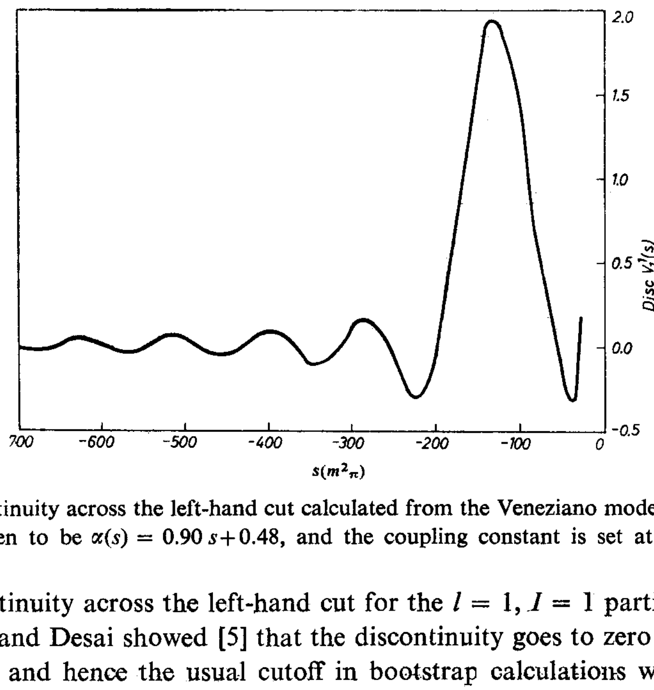

Fig. 1. The discontinuity across the left-hand cut calculated from the Veneziano model. The rho trajectory is taken to be $\alpha(s) = 0.90\,s + 0.48$, and the coupling constant is set at $\gamma = 0.5$

The discontinuity across the left-hand cut for the $l = 1$, $I = 1$ partial wave is shown in Fig. 1. Park and Desai showed [5] that the discontinuity goes to zero faster than $1/s$ as s goes to $-\infty$, and hence the usual cutoff in bootstrap calculations with rho exchange is not needed here. In calculating the discontinuity, the ϱ trajectory was assumed to be

$$\alpha(s) = 0.90s + 0.48.$$

These values of the slope and intercept were chosen to be consistent with phase-shift data at low energies [6]. The value of the coupling constant, γ, for the left-hand discontinuity was determined using the narrow resonance approximation to compare the Veneziano p-wave amplitude at the ϱ resonance to a Breit-Wigner resonance formula with $m_\varrho = 765$ Mev and $\Gamma = 100$ Mev. Using this procedure, γ was set equal to 0.5.

The right-hand cut was split into four regions: $4 < s < 16$, $16 < s < s_1$, $s_1 < s < s_2$, and $s_2 < s$, where s_1 and s_2 are variable parameters. In the region from $s = 4$ to $s = 16$, the scattering is purely elastic and $\eta(s) = 1$. In the next region, $s = 16$ to s_1, $\eta(s)$ was taken to be a power function of s [7]

$$\frac{1-\eta(s)}{2\varrho(s)} = \left(\frac{s-16}{s_1-16}\right)^{9/2} \frac{1-\eta(s_1)}{2\varrho(s_1)} .$$

This expression assures smooth behavior of $\eta(s)$ between the elastic region and the region above s_1. From s_1 to s_2, $\eta(s)$ was taken to have a constant value of η'. Finally, for large $s(> s_2)$, $\eta(s)$ was set equal to a constant value η_∞. The parameter values found to produce the best output (Fig. 2) were $s_1 = 33$, $s_2 = 1000$, $\eta' = 0.01$, and $\eta_\infty = 1.0$. The resulting

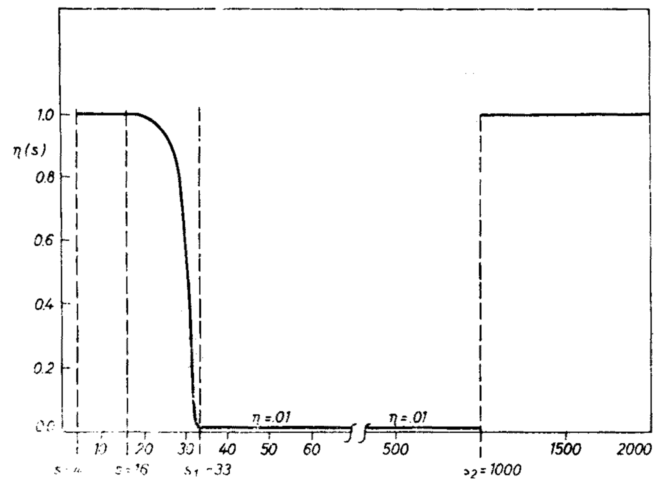

Fig. 2. The values of the absorption factor, $\eta(s)$, which produced the best output as a function of energy on the right-hand cut. The intermediate region, $\eta(s) = \eta' = 0.01$, is limited by s_1 and s_2

output resonance occurred at $s_\varrho = 29.5$, which corresponds to a mass of 765 Mev. The corresponding output width was 132 Mev.

The real and imaginary parts of the partial-wave amplitude and partial-wave cross--section are shown in Fig. 3.

After the best solution was found, the parameters for the left and right-hand cuts were varied to determine the relative significance of the two cuts. In this procedure, parameters were varied one at a time. While one parameter was being varied, all other parameters were fixed at the value which had produced the best output.

To investigate the effect of the left-hand cut on the output resonance, the slope, a, and intercept, b, of the trajectory and the coupling constant, γ, were varied. While a and b were limited to values rougly consistent with physical data, γ was extended to a wider range of values. Varying a from 0.90 Bev^{-2} to 1.0 Bev^{-2} and b from 0.40 to 0.50 produced changed of less than 1% in the position of the output resonance. The output position also changed by less than 2% with variations in γ from 0.0 to 1.0. The insensitivity of the bootstrap

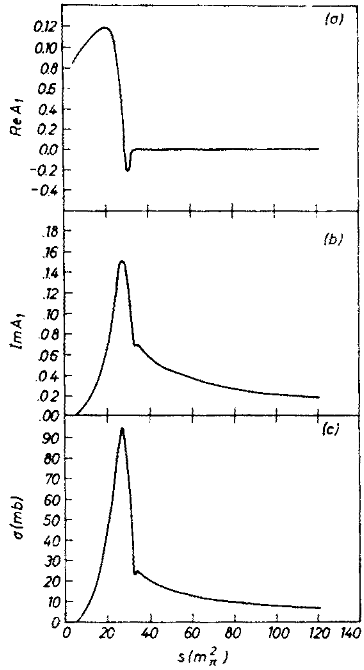

Fig. 3. The $I = 1$, $l = 1$ partial-wave amplitude and cross-section calculated from the N/D equations. (a) The real part of the partial-wave amplitude. For $s > 33\ m_\pi^2$, Re $A_1(s) < 10^{-4}$. (b) The imaginary part of the partial-wave amplitude. Im $A_1(s)$ continues to decrease until it is less than 10^{-4} for $s > 2000\ m_\pi^2$. (c) The partial-wave cross-section

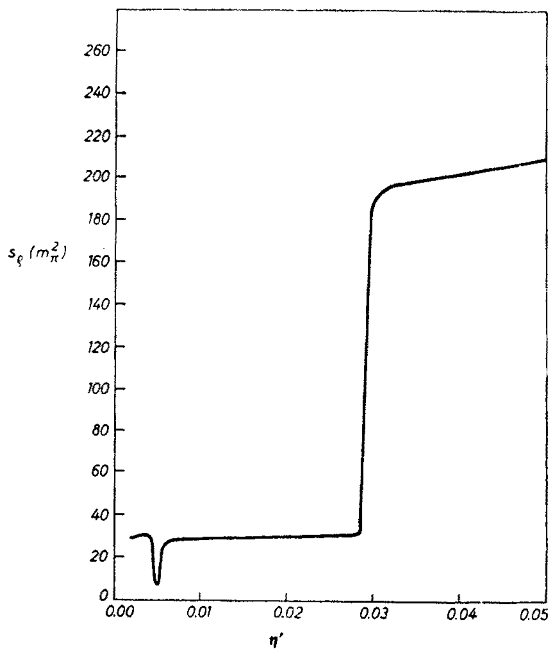

Fig. 4. The resulting output resonance position as a function of η', the value of $\eta(s)$ between s_1 and s_2

solution to variations in parameters associated with the left-hand cut as well as the fact that the correct output could be obtained with $\gamma = 0$ (no left-hand cut) apparently means that the right-hand inelastic cut is responsible for the bootstrap.

For the right-hand cut, there were three physically significant parameters: η', η_∞, and s_1. The parameter s_2 which separates the regions $\eta(s) = \eta'$ and $\eta(s) = \eta_\infty$ had negligible effect on the output for $s_2 \geqslant 1000$. The position of the output resonance was changed by less than 0.5% as s_2 was increased from 1000 to 5000. The discontinuity in the values of $\eta(s)$ at s_2 was made to permit variation in the asymptotic value of $\eta(s)$. If the effect of s_2 had been significant, the discontinuity in $\eta(s)$ could not have been tolerated and a smooth connection between the regions would have been required. The effect of varying η_∞ was fairly small. As η_∞ was varied from 0.1 to 1.0, the output resonance position changed by less than 1% and the output width varied by about 2%. It has been argued [7, 8] on the basis of a Pomeranchuk-type theorem that the discontinuities across the left and right-hand cuts asymptotically approach the same value. We have therefore chosen 1.0 as the best value for η_∞ in order to be consistent with this argument.

The most significant results came from varying η' and s_1. The effect of s_1 on the output was so large that even varying other parameters for compensation, a bootstrap could be obtained only with s_1 in the range 32 to 35. In Fig. 4, the output is shown as a function of η'. For $\eta' > 0.03$ no self-consistent calculation was possible, while for $\eta' < 0.03$ the output was fairly constant. This seems to imply that $\eta' < 0.03$ corresponds to nearly total absorption. The small effect of the asymptotic value of the absorption parameter plus the large effect of η' and s_1 implies that the scattering is dominated by inelastic effects for a broad range of energies above the mass of the rho.

In conclusion, it was found that by using the Veneziano model with the inelastic N/D equations a self-consistent calculation of the rho meson mass and width was possible. Futhermore, it was found that in this model p-wave pion-pion scattering with isospin-1 is influenced by strong absorption at energies above the rho mass. The importance of inelastic effects in the scattering comes from two general results: (1) the effects on the output due to parameters associated with the right-hand inelastic cut are much more pronounced than those due to the left-hand cut parameters, and (2) a self-consistent calculation could be obtained using the inelastic N/D equations without a left-hand cut while such a calculation was not possible with the elastic N/D equations.

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