TRANSFORMATION FROM THE USUAL NUCLEAR SHELL MODEL TO THE TRANSLATIONALLY INVARIANT SHELL MODEL

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A general formula is obtained for the coefficient of transition from the wave function of the usual nuclear shell model for the dipole states in p-shell nuclei to the wave function of the translationally invariant shell model. The coefficients for the set of functions of the configurations s^3p^3 and s^4p $(2s-1d)$ are calculated and a table is given.

1. *Introduction*

It is well known that the wave function of the shell model is not translationally invariant [1, 2]. Among the excited states there exist the so-called "spurius states", the states in which the center of mass of the nucleus is excited and the intrinsic state remains unchanged. This defect in the shell model is very significant for the light nuclei. However, a complete set of pure wave functions (free of spurius states) can be constructed in the framework of the so-called "translationally invariant shell model" [3, 4], abbreviated TISM). This model also has the advantage of describing the cluster structure of the nucleus.

In TISM the wave function $|AN[f](\lambda\mu)LST\rangle$ describes the nuclear state of A nucleons characterised by N — the number of oscillator quanta, $[f]$ — Young scheme, $(\lambda\mu)$ — the irreducible representation of SU_3 group, LST — the total angular momentum, the spin and isotopic spin of the nucleus. However, the procedure for calculating the energy spectra and some other nuclear properties by the wave functions of the usual shell model is rather easier than by the TISM wave functions.

2. *Derivation of the transformation coefficients from the usual shell model to the TISM*

We consider the states of the configurations s^3p^{K+1} and s^4p^{K-1} $(2s-1d)$ which are formed when one nucleon is removed from the s-shell of the configuration s^4p^K to the p-shell or from the p-shell to the $(2s-1d)$-shell. In TISM these states are denoted by the

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wave function $|A = K+4N = K+1[f](\lambda\mu)LST\rangle$. According to Elliot and Skyrme [4] we can write the relation between the wave function of the usual shell model and the TISM wave function as follows:

$$\psi_{00}(\vec{R}_A)\,|AN[f]\,(\lambda\mu)LST\rangle = \sum_{[f_1]S_1T_1} \alpha_{[f_1]S_1T_1}|s^3p^{K+1}[f_1]LS_1T_1 : LST\rangle$$

$$+ \sum_{[f'']L''S''T'',l_3} \beta_{[f'']L''S''T'',l_3} \times |s^4p^{K-1}[f'']L'S''T'', 2l_3\, \tfrac{1}{2}\tfrac{1}{2} : LST\rangle, \quad (1)$$

where $\psi_{00}(\vec{R}_A)$ is the wave function of the center of mass of zero vibration α, β are coefficients to be found.

$$\alpha_{[f_1]S_1T_1} = \langle s^3p^{K+1}[f_1]LS_1T_1 : LST|\psi_{00}(\vec{R}_A)\,|AN[f]\,(\lambda\mu)LST\rangle,$$

$$\beta_{[f'']L''S''T'',l_3} = \langle s^4p^{K-1}[f'']L'S''T'', 2l_3 : LST|\psi_{00}(\vec{R}_A)\,|AN[f]\,(\lambda\mu)LST\rangle. \quad (2)$$

The wave function $|s^3p^{K+1}\rangle$ can be written in the form of an antisymmetrized product of the antisymmetric functions $|s^3\rangle$ and $|p^{K+1}\rangle$

$$|s^3p^{K+1}[f_1]LS_1T_1 : LST\rangle = (-1)^{A-1}\sqrt{\frac{3!(K+1)!}{A!}}\,\hat{Q}|p^{K+1}[f_1]LS_1T_1; s^30\,\tfrac{1}{2}\tfrac{1}{2} : LST\rangle, \quad (3)$$

where $\hat{Q}$ — the antisymmetrization operator.

Then,

$$\alpha_{[f_1]S_1T_1} = (-1)^{A-1}\sqrt{\frac{A!}{3!(K+1)!}}$$

$$\times \langle p^{K+1}[f_1]LS_1T_1; s^30\,\tfrac{1}{2}\tfrac{1}{2} : LST|\psi_{00}(\vec{R}_A)\,|AN[f]\,(\lambda\mu)LST\rangle. \quad (4)$$

The function on the left side of the matrix element is antisymmetric only with respect to the interchange of the p-nucleons and s-nucleons separately.

The shell function $|p^{K+1}\rangle$ (and also $|s^3\rangle$) can be written as a product of the intrinsic wave function and the wave function of the center of mass,

$$|p^{K+1}LS_1T_1\rangle = \sum_{nl,N_1\Lambda_1} \langle p^{K+1}L|\psi_{nl}; K+1N_1\Lambda_1\rangle\,|\psi_{nl}(\vec{R}_{K+1}), N_1\Lambda_1S_1T_1 : LS_1T_1\rangle,$$

$$|s^30\,\tfrac{1}{2}\tfrac{1}{2}\rangle = \langle s^30|\psi_{00}; 300\rangle\,|\psi_{00}(\vec{R}_3), 300\,\tfrac{1}{2}\tfrac{1}{2} : 0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle, \quad (5)$$

where $\psi_{nl}(\vec{R}_{K+1})$ — the wave function of the center of mass of $(K+1)$ nucleons in the p-state, $\psi_{00}(\vec{R}_3)$ — the wave function of the center of mass of three nucleons in the s-state and $n+N_1 = N$, $\vec{l}+\vec{\Lambda} = L$. The squares of the coefficients in the expressions (5) give the statistical weights of the products of $\psi_{nl}(\vec{R})$ and the wave function of the intrinsic motion in the shell wave function.

Expressing the wave function on the right side of the matrix element (4) through the coefficients of fractional parentage (c.f.p.) for separating 3 particles [5, 6], changing the coupling momentum scheme in (5) and substituting in (4) we get

$$\alpha_{[f_1]S_1T_1} = (-1)^{A-1}\sqrt{\frac{A!}{3!(K+1)!}}$$

$$\times \sum_{N_1\Lambda_1,nl} \langle ANLST|A-3N_1\Lambda_1S_1T_1;\, nl,\, 00\tfrac{1}{2}\tfrac{1}{2}\,(nl):LST\rangle \langle p^{K+1}L|\psi_{nl};\, K+1N_1\Lambda_1\rangle$$

$$\times \langle s^3|\psi_{00};\, 300\rangle \begin{pmatrix} l & \Lambda_1 & L \\ 0 & 0 & 0 \\ l & \Lambda_1 & L \end{pmatrix} \langle nl_3 00 : nl|A-3,3|00,\, nl:nl\rangle, \tag{6}$$

where nl — the principal quantum number and the orbital momentum of the wave function of the center of mass of three nucleons relative to the center of mass of $(A-3)$ nucleons. The Talmi coefficient [7] stands for the product of the wave function

$$\langle \psi_{nl}(\vec{R}_{K+1}),\, \psi_{00}(\vec{R}_3)\,|\psi_{00}(\vec{R}_A),\, \varphi_{\nu\lambda}(\vec{R}_{A-3}-\vec{R}_3)\rangle \delta_{n\nu}\delta_{l\lambda}.$$

The coefficient $\langle s^3|\psi_{00};300\rangle = 1$ and $9j$-symbol $= 1$. To find the coefficient $\langle p^{K+1}L|\psi_{nl};\, N_1\Lambda_1\rangle$ we express the function $|p^{K+1}\rangle$ through the c.f.p.,

$$|p^{K+1}LS_1T_1\rangle = \sum_{L_1'S_1'T_1'} \langle p^{K+1}LS_1T_1|p^K L_1'S_1'T_1'\rangle\, |p^K L_1'S_1'T_1',\, p\tfrac{1}{2}\tfrac{1}{2}:LS_1T_1\rangle$$

$$= \sum_{L_1'S_1'T_1',n'l',N_1\Lambda_1'} \langle p^{K+1}LS_1T_1|p^K L_1'S_1'T_1'\rangle \langle p^K L_1'|\psi_{n'l'};\, KN_1'\Lambda_1'\rangle$$

$$\times |\psi_{n'l'}(\vec{R}_K),\, KN_1'\Lambda_1'(L_1'S_1'T_1');\, p\tfrac{1}{2}\tfrac{1}{2}:LS_1T_1\rangle, \tag{7}$$

where $n'+N' = N$, $\vec{l}'+\vec{\Lambda}_1' = L_1'$.

In the wave function on the right side of (7) we change the momentum scheme $\vec{\Lambda}_1'+\vec{l}' = L_1'$ to $\vec{l}'+\vec{p} = \mathscr{L}$ and make a transformation in the wave function $|\psi_{n'l'}(\vec{R}_K),\, \varphi_p : \mathscr{L}\rangle$ to the center of mass of $(K+1)$ nucleons, using the Talmi coefficient [7],

$$|\psi_{n'l'},\, KN_1'\Lambda_1'(L_1'S_1'T_1');\, p\tfrac{1}{2}\tfrac{1}{2}:LS_1T_1\rangle$$

$$= \sum_{nl,\nu_0\lambda_0,\mathscr{L}} U(\Lambda_1'l'L1;\, L_1'\mathscr{L})\, \langle n'l',\, 11:\mathscr{L}|K,\, 1|nl,\, \nu_0\lambda_0:\mathscr{L}\rangle$$

$$\times |KN_1'\Lambda_1';\, \psi_{nl}(\vec{R}_{K+1}),\, \varphi_{\nu_0\lambda_0}(\vec{R}_K-\vec{r}_{K+1})\,\{\mathscr{L}\}:LS_1T_1\rangle,\, (-1)^{L_1'-\Lambda_1'-l'}, \tag{8}$$

where $\varphi_{\gamma_0\lambda_0}$ — the wave function of the motion of the $(K+1)$-th nucleon relative to the center of mass of K nucleons. Before the function on the right side stand Racah coefficients and Talmi coefficients. Changing the scheme $\vec{l}+\vec{\lambda}_0 = \mathscr{L}$ to $\vec{\Lambda}_1'+\vec{\lambda}_0 = \Lambda$ and expressing the wave function $|K+1N_1\Lambda_1\rangle$ through the c.f.p. we get

$$\langle p^{K+1}L|\psi_{nl};N_1\Lambda_1\rangle = \sum_{L_1'S_1'T_1',N_1'[f_1']\Lambda_1',\nu_0\lambda_0,l',\mathscr{L}} \langle p^{K+1}[f_1]LS_1T_1|p^K[f_1']L_1'S_1'T_1'\rangle$$

$$\times\langle K+1N_1[f_1]\Lambda_1S_1T_1|KN_1'[f_1']\Lambda_1'S_1'T_1',\nu_0\lambda_0\rangle$$

$$\times U(\Lambda_1'l'L1;L_1'\mathscr{L})U(l\lambda_0L\Lambda_1';\mathscr{L}\Lambda_1)\langle n'l',11:\mathscr{L}|K,1|nl,\nu_0\lambda_0:\mathscr{L}\rangle$$

$$\times\langle p^KL_1'|\psi_{n'l'};KN_1'\Lambda_1'\rangle(-1)^{L+L_1'+\Lambda_1-\Lambda_1'-l'-\lambda_0-\mathscr{L}}. \tag{9}$$

And hence, by equation (9) we can find the coefficient $\langle p^m|\psi(\vec{R}_m);N=m\rangle$ if the coefficient $\langle p^{m-1}|\psi(\vec{R}_{m-1});N=m-1\rangle$ is known.

The coefficient $\langle p^3L|\psi(\vec{R}_3);N=3\Lambda\rangle$ can be found explicitly. Using fractional parentage expanding of the function $|p^3\rangle$ and $|N\Lambda\rangle$ as in (7) and (8), we obtain the following formula

$$\langle p^3L|\psi_{nl}(\vec{R}_3);3N\Lambda\rangle = \sum_{L'S'T',N'\Lambda',\nu\lambda,n'l',\mathscr{L}} \langle p^3LST|p^2L'S'T'\rangle$$

$$\times\langle 3N\Lambda ST|2N'\Lambda'S'T',\nu\lambda\rangle U(\Lambda'l'L1;L'\mathscr{L})$$

$$\times\langle 11,11:L'|1,1|n'l',N'\Lambda':L'\rangle U(l\lambda L\Lambda';\mathscr{L}\Lambda)$$

$$\times\langle n'l',11:\mathscr{L}|2,1|nl,\nu\lambda:\mathscr{L}\rangle(-1)^{L+L'+\Lambda-\Lambda'-l'-\lambda-\mathscr{L}}. \tag{10}$$

The coefficient β in (2) can be found similarly. We write the function $|s^4p^{K-1},2l_3:LST\rangle$ in the form of the antisymmetrized product of the antisymmetric function $|s^4p^{K-1}\rangle$ and the function of the last nucleon $|2l_3\rangle$ .Then we perform in the shell function $|s^4p^{K-1}\rangle$ a transition to the function of the intrinsic motion of $(A-1)$ nucleons taking into consideration that their center of mass is in the 1s-state.

$$|s^4p^{K-1}[f'']L''S''T'',2l_3\tfrac{1}{2}\tfrac{1}{2}:LST\rangle = \sqrt{A}\,\psi_{00}(\vec{R}_{A-1})$$

$$\times|A-1N'=K-1[f'']L''S''T'';2l_3\tfrac{1}{2}\tfrac{1}{2}:LST\rangle \tag{11}$$

Using the Talmi coefficient, the function $\psi_{00}|2l_3\rangle$ is transformed to the product of the function of the center of mass of A nucleons and the function of the motion of the A-th nucleon relative to the center of mass of $(A-1)$ nucleons.

$$|s^4p^{K-1}L''S''T'',2l_3:LST\rangle = \sqrt{A}\,|A-1N'L''S''T'';\psi_{00}(\vec{R}_A),\varphi_{nl}(\vec{R}_{A-1}-\vec{r}_A):LST\rangle$$

$$\times\langle 00,2l_3:l_3|A-1,1|00,2l_3:l_3\rangle\delta_{2n}\delta_{l_3l}. \tag{12}$$

Using the c.f.p. in the function on the right side of (2) we get finally

$$\beta_{[f'']L''S''T'',l_3} = \sqrt{A}\,\langle ANLST|A-1N'[f'']L''S''T'',2l_3:LST\rangle$$

$$\times\langle 00,2l_3:l_3|A-1,1|00,2l_3:l_3\rangle. \tag{13}$$

By the formulas given above we calculated the coefficients α and β for all possible one photon-excited states of ^{6}Li nucleus. The results are given in the table shown below.

TABLE I

α and β coefficients

TISM Function			Shell function							
			$s^3p^3[f']LS'T':LST$				s^4p, 1d: LST		s^4p, 2S: LST	
		$[f']$	[3]	[21]			[2]	[11]	[2]	[11]
$[f]$	$(\lambda\mu)$	$LST \backslash S'T'$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{3}{2}$	$\frac{3}{2}$ $\frac{1}{2}$	ST	ST	ST	ST
[42]	(11)	201		$\sqrt{\frac{1}{30}}$	$\sqrt{\frac{4}{30}}$		$\sqrt{\frac{25}{30}}$			
[321]	(11)	201		$\sqrt{\frac{4}{5}}$	$-\sqrt{\frac{1}{5}}$					
[42]	(11)	210		$\sqrt{\frac{1}{30}}$		$\sqrt{\frac{4}{30}}$	$\sqrt{\frac{25}{30}}$			
[321]	(11)	210		$-\sqrt{\frac{4}{5}}$		$\sqrt{\frac{1}{5}}$				
[411]	(11)	211		$-\sqrt{\frac{1}{18}}$	$\sqrt{\frac{4}{18}}$	$\sqrt{\frac{4}{18}}$		$\sqrt{\frac{9}{18}}$		
[321]	(11)	211		$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$				
[321]	$(11)_1$	211		$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$				
[411]	$(11)_2$	200		$\sqrt{\frac{1}{2}}$				$\sqrt{\frac{1}{2}}$		
[42]	(30)	301	$-\sqrt{\frac{2}{3}}$				$\sqrt{\frac{1}{3}}$			
[42]	(30)	310	$\sqrt{\frac{2}{3}}$				$\sqrt{\frac{1}{3}}$			
[411]	(30)	311						1		
[33]	(30)	311	-1							
[411]	(30)	300						1		
[33]	(30)	300	-1							
[411]	(11)	111		$\sqrt{\frac{1}{18}}$	$-\sqrt{\frac{4}{18}}$	$-\sqrt{\frac{4}{18}}$		$\sqrt{\frac{5}{18}}$		$-\sqrt{\frac{4}{18}}$
[411]	(30)	111						$\sqrt{\frac{4}{9}}$		$\sqrt{\frac{5}{9}}$
[33]	(30)	111	-1							
[321]	$(11)_1$	111		$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$				
[321]	$(11)_2$	111		$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$				
[42]	(30)	101	$-\sqrt{\frac{18}{27}}$				$\sqrt{\frac{4}{27}}$		$\sqrt{\frac{5}{27}}$	
[42]	(11)	101		$-\sqrt{\frac{9}{270}}$	$-\sqrt{\frac{36}{270}}$		$\sqrt{\frac{125}{270}}$		$-\sqrt{\frac{100}{270}}$	
[321]	(11)	101		$-\sqrt{\frac{4}{5}}$	$\sqrt{\frac{1}{5}}$					
[42]	(30)	110	$\sqrt{\frac{18}{27}}$				$\sqrt{\frac{4}{27}}$		$\sqrt{\frac{5}{27}}$	
[42]	(11)	110		$-\sqrt{\frac{9}{270}}$		$-\sqrt{\frac{36}{270}}$	$\sqrt{\frac{125}{270}}$		$\sqrt{\frac{100}{270}}$	
[321]	(11)	110		$\sqrt{\frac{4}{5}}$		$-\sqrt{\frac{1}{5}}$				
[411]	(11)	100		$-\sqrt{\frac{9}{18}}$				$\sqrt{\frac{5}{18}}$		$-\sqrt{\frac{4}{18}}$
[411]	(30)	100						$\sqrt{\frac{4}{9}}$		$\sqrt{\frac{5}{9}}$
[33]	(30)	100	-1							
[321]	(11)	121				1				
[321]	(11)	221				-1				

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