# GENERALIZED EINSTEIN-KAUFMAN THEORY AND THE ELECTROMAGNETIC TENSOR

BY L. J. GREGORY AND A. H. KLOTZ

Department of Applied Mathematics, University of Sydney*

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Although the Russell-Klotz electromagnetic tensor produced both a Lorentz force term in the equations of motion and a modified Coulomb law between two charges, the two results were at variance. Another tensor is introduced which provides satisfactory results and suggests the introduction of a non-Maxwellian electromagnetic theory. A third order weak field expansion is developed to examine the Russell-Klotz tensor.

## 1. Introduction

In a series of publications (Refs [1-4]), G. K. Russell and one of the present authors considered the general structure of the non-symmetric unified field theory and the equations of motion resulting from the integration of the field equations. It was shown (Ref. [2]) that if the electromagnetic field tensor is defined as

$$f_{\mu\nu} = {}^*g^{\alpha\beta} g_{\underset{+-}{\mu\nu};\alpha\beta} \tag{1}$$

(Ref. [1]) in Einstein's notation (Ref. [5]), the objection (e.g. Ref. [6]) that the equations of motion do not contain a Lorentz force in the expected approximation is not valid. Indeed, Russell and Klotz succeeded in obtaining such a force using the technique of Infeld (Ref. [6]).

The unsatisfactory nature of some the previous results compels us to once more take up the question of the electromagnetic field identification. In particular, the Russell-Klotz identification (1) leads to a Coulomb force between two spherically symmetric charges (at rest relative to each other) of the form

$$\frac{1}{r^2}\left(1-\frac{2m}{r}\right), \tag{2}$$

* Address: Department of Applied Mathematics, University of Sydney, N. S. W. 2006 Sydney, Australia.

where $m$ is a constant (presumably, the inertial mass of the particles in question) and $r$ is the distance between their centres. This is difficult to reconcile with the Lorentz force between orbiting charges,

$$\frac{1}{r^2}+k(r-r_0) \tag{3}$$

obtained from the field equations (Ref. [2] and also Ref. [7]). Moreover, the tensor $f_{\mu\nu}$ defined by (1) is not simply-skew-symmetric. Although Russell disagrees (Ref. [8]), we regard this as a serious defect in any attempt to identify $f_{\mu\nu}$ with the electromagnetic tensor. In spite of Einstein's remark that only "something like a Maxwell field" (Ref. [5]) might be expected from the generalized theory, the problems raised by trying to interpret any symmetric part of $f_{\mu\nu}$ are too formidable to consider at the present stage of the unified field theory. On the other hand, we do not wish to upset the equations of motion result since it restores credibility to the whole of Einstein's theory. Of course,

$$*g^{\alpha\beta}g_{\underset{+-}{\mu\check{\nu}};\alpha\beta} = -*g^{\alpha\beta}g_{\underset{-+}{\nu\check{\mu}};\alpha\beta}, \tag{4}$$

but this is not good enough.

It will be shown below that this difficulty can be very easily overcome. We shall see also that the dichotomy between the Coulomb and Lorentz forces can be resolved by the simple expedient of requiring a fixed relation between the compounds of $f_{\mu\nu}$ and the electric (***E*** or ***D***) and magnetic (***B*** or ***H***) vectors (it is well known that the classical theory allows two possibilities). Finally, we shall consider whether the (modified) Russell–Klotz field tensor can be expressed as the curl of a field vector.

## 2. *A note on the field equations*

It is shown in Ref. [4] that the most general field equations satisfying the principle of Transposition Invariance (or Hermitian symmetry, with respect to the field tensor $g_{\mu\nu}$ and pseudo-connection $U^{\lambda}_{\mu\nu}$, linearly related to the affine connection $\Gamma^{\lambda}_{\mu\nu}$) can be written in the form

$$g_{\underset{+-}{\mu\nu};\lambda}(\Gamma)-\tfrac{2}{3}\Gamma_{\lambda}g_{\mu\nu}-\tfrac{2}{3}\Gamma_{\nu}g_{\mu\lambda} = 0, \tag{5}$$

$$R_{\mu\nu} = 0, \tag{6}$$

and

$$\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0, \tag{7}$$

where

$$g_{\underset{+-}{\mu\nu};\lambda}(\Gamma) = g_{\mu\nu,\lambda}-\Gamma^{\sigma}_{\mu\lambda}g_{\sigma\nu}-\Gamma^{\sigma}_{\lambda\nu}g_{\mu\sigma}, \tag{8}$$

is Einstein's (covariant) derivative, Hermitian in $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$.

Equation (5) can be considered as an algebraic system in the 64 unknowns $\Gamma^{\lambda}_{\mu\nu}$. It is shown elsewhere (Ref. [9]) that this system has rank 60 and a solution

$$\Gamma^{\lambda}_{\mu\nu} = \Delta^{\lambda}_{\mu\nu} - \tfrac{2}{3}\,\delta^{\lambda}_{\mu}\Gamma_{\nu}, \tag{9}$$

where $\Delta^{\lambda}_{\mu\nu}$ is uniquely defined by the equations

$$g_{\underset{+-}{\mu\nu};\lambda}(\Delta) = 0, \qquad \Delta_{\mu} = \Delta^{\sigma}_{\underset{\vee}{\mu\sigma}} = 0, \tag{10}$$

and $\Gamma_{\nu} = \Gamma^{\sigma}_{\nu\sigma}$ is indeterminate. On substituting $\Gamma^{\lambda}_{\mu\nu}$, the remaining field equations (6) and (7) become 20 differential equations in the 20 unknowns $g_{\mu\nu}$ and $\Gamma_{\nu}$. (The system is formally self-consistent, having been derived from a variational principle (Ref. [4]).) The quantities $g_{\mu\nu}$ and $\Gamma_{\nu}$ (which cannot be found by algebraic means) will be considered to have a physical significance and it is this important respect that Einstein's unified field differs from Schrödinger's Purely Affine Theory with which it is frequently but wrongly linked.

It is easily shown that

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(\Delta) - \tfrac{2}{3}\,(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}), \tag{11}$$

so that (as pointed out in Ref. [4]) the equations (5), (6) and (7) are equivalent to the weak field equations of Einstein and Straus (Ref. [10]) with the connection $\Delta^{\lambda}_{\mu\nu}$, and

$$R_{\underset{\vee}{\mu\nu}}(\Delta) = \tfrac{2}{3}\,(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}). \tag{12}$$

We shall consider later the possibility (contemplated by Einstein soon after his introduction of General Relativity and also relevant in Weyl's unified field theory) of regarding $R_{\underset{\vee}{\mu\nu}}(\Delta)$ as the electromagnetic field tensor. The vector $\tfrac{2}{3}\Gamma_{\mu}$ would then appear as the likely electromagnetic (four) vector potential (at least up to a gauge transformation).

### 3. *The electromagnetic tensor*

In Relativity there are two distinct formulations of Maxwell's equations, Firstly, if $f_{\mu\nu}$ is defined by

$$f_{ij} = B_k, \quad f_{k4} = E_k, \quad (i, j, k) \text{ a cyclic permutation of } (1, 2, 3),$$

then

$$f^{\mu\nu}{}_{;\nu} = J^{\mu}, \quad f_{\underset{\cdots}{\mu\nu,\lambda}} = 0. \tag{13}$$

On the other hand, if

$$f_{ij} = E_k \quad \text{and} \quad f_{4k} = B_k,$$

then

$$f^{\mu\nu}{}_{;\nu} = 0, \quad f_{\underset{\cdots}{\mu\nu,\lambda}} = \varepsilon_{\mu\nu\lambda\alpha}\mathfrak{J}^{\alpha}. \tag{14}$$

Let us now consider a weak field expansion of $g_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} + \varepsilon^2 q_{\mu\nu} + \varepsilon^3 \alpha_{\mu\nu} + \ldots, \qquad \varepsilon^2 \ll \varepsilon. \tag{15}$$

Let also

$$g^{*\mu\nu} = \eta^{\mu\nu} + \varepsilon H^{\mu\nu} + \varepsilon^2 Q^{\mu\nu} + \varepsilon^3 A^{\mu\nu} + \ldots . \tag{16}$$

Here $\eta_{\mu\nu}$ is the Minkowski tensor which we will use to raise and lower tensor indices so that, for example,

$$h^{\mu}{}_{\nu} \overset{\mathrm{Df}}{=} \eta^{\mu\sigma} h_{\sigma\nu}, \qquad \alpha^{\mu\nu} = \eta^{\mu\varrho}\eta^{\nu\sigma}\alpha_{\varrho\sigma}. \tag{17}$$

The relation

$$g^{*\mu\sigma} g_{\mu\varrho} = g^{*\sigma\mu} g_{\varrho\mu} = \delta^{\sigma}_{\varrho}, \tag{18}$$

then gives

$$H^{\varrho\sigma} = -h^{\sigma\varrho}, \tag{19}$$

$$Q^{\varrho\sigma} = -q^{\sigma\varrho} + h^{\sigma\alpha} h_{\alpha}{}^{\varrho}, \tag{20}$$

$$A^{\varrho\sigma} = -\alpha^{\sigma\varrho} - \eta^{\varrho\alpha} h_{\beta\alpha} h^{\sigma\gamma} h_{\gamma}{}^{\beta} + h_{\beta\alpha}(\eta^{\sigma\alpha} q^{\beta\varrho} + \eta^{\varrho\alpha} q^{\sigma\beta}), \tag{21}$$

and so on. It is clearly convenient, and indeed common, to assume that

$$h_{\mu\nu} = h_{\nu\mu}, \tag{22}$$

when also

$$Q^{\underset{\vee}{\mu\nu}} = q^{\underset{\vee}{\mu\nu}}. \tag{23}$$

Russell's conclusions concerning the Lorentz force are valid for any skew tensor $f_{\mu\nu}$, satisfying the conditions

($\alpha$) that, in the order $\varepsilon^2$

$$\underset{2}{f_{\mu\nu}} = \Box q_{\underset{\vee}{\mu\nu}} \tag{24}$$

where $\Box$ is the D'Alembertian operator, and

($\beta$) $f_{\mu\nu}$ involves second derivatives of $g_{\underset{\vee}{\mu\nu}}$.

It is easily seen that both these conditions are satisfied if we take as the electromagnetic field tensor

$$w_{\mu\nu} = -w_{\nu\mu} = {}^{*}g^{\alpha\beta} g_{\underset{\vee}{\mu\nu};\underset{\circ\circ}{\alpha\beta}}, \tag{25}$$

the subscripts "o" denoting, with Einstein, covariant derivatives with respect to the affine connection $\Delta^{\lambda}_{\mu\nu}$.

Properties ($\alpha$) and ($\beta$) are also satisfied by the tensor $R_{\underset{\vee}{\mu\nu}}(\Delta)$ (given by relation (12)) since in view of the $\varepsilon$-expansion (15),

$$\underset{2}{R_{\underset{\vee}{\mu\nu}}} = \tfrac{1}{2}\Box q_{\underset{\vee}{\mu\nu}}, \tag{26}$$

and

$$\underset{\vee}{R}_{\mu\nu}(\Delta) = -\underset{\vee}{\Delta}_{\mu\nu;\sigma} + \ldots$$

$$= -A^{\sigma\lambda} \underset{\vee}{g}_{\mu\nu,\lambda\sigma} + \ldots \tag{27}$$

where $A^{\sigma\lambda}$ is a symmetric tensor whose exact form (Ref. [10]) is immaterial to our calculations. The tensor $\underset{\vee}{R}_{\mu\nu}$ is less complicated than $w_{\mu\nu}$. We shall investigate below the extent to which these two tensors differ with respect to the $\varepsilon$-expansion of $g_{\mu\nu}$. At the moment, however, we must digress to consider some known exact solutions of the field equations.

## 4. *The electromagnetic tensor and the exact solutions*

(*i*) Static, cylindrical symmetric field.

Using the isothermal form

$$\underline{g}_{\mu\nu} = \text{diag}\,(-\alpha, -\alpha, -\beta, \gamma), \tag{28}$$

Russell and Klotz have shown (Ref. [11]) that the non-vanishing components of $\underset{\vee}{R}_{\mu\nu}$ are

$$\underset{\vee}{R}_{23} = a_1, \qquad \underset{\vee}{R}_{34} = a_2, \tag{29}$$

where $a_1$ and $a_2$ are arbitrary constants of integration. In either Maxwell's theory or the Born–Infeld nonlinear electrodynamic (Ref. [12]), this solution can represent the fields

$$\boldsymbol{D} = \frac{2q}{r}\,\hat{r}, \qquad \boldsymbol{H} = \frac{qi}{r}\,\hat{\theta}, \tag{30}$$

where $q$ is the charge per unit length along the source at $r = 0$, and $i$ is the steady current flowing along it. The weak field equations for the isothermal form have not, as yet, been solved so that the form of $\underline{g}_{\mu\nu}$ is unknown.

(*ii*) Static, spherically symmetric field.

A static, spherically symmetric, skew tensor, satisfying the equation

$$\underset{\vee}{R}_{\underset{\cdot\cdot}{\mu}\underset{\cdot}{\nu},\underset{\cdot}{\lambda}} = 0, \tag{31}$$

has the general form

$$\underset{\vee}{R}_{23} = W \sin\theta, \qquad \underset{\vee}{R}_{14} = f(r), \tag{32}$$

where $W$ is a constant, and $f(r)$ an arbitrary function of $r$. Similarly, the components $g_{\mu\nu}$ of the fundamental tensor are given by

$$\underline{g}_{\mu\nu} = \text{diag}\,(-\alpha, -\beta, -\beta\sin^2\theta, \gamma), \qquad \underset{\vee}{g}_{23} = p\sin\theta, \qquad \underset{\vee}{g}_{14} = w, \tag{32'}$$

with $\alpha$, $\beta$, $\gamma$, $p$ and $w$ being arbitrary functions of $r$. The vanishing of $w$ is a sufficient, but not necessary, condition for the vanishing of $f$ and the same is true of $p$ and $W$ (Refs [13, 14]). The general solution for

$$pw \neq 0$$

(Ref. [15]) is unfortunately almost unworkable because of its own complexity.

Papapetrou's solution when $p(r) = 0$ and $w(r) \neq 0$ is (Ref. [13])

$$\alpha^{-1} = \left(1 - \frac{2m}{r}\right), \quad \alpha\gamma = \left(1 + \frac{l^2}{r^4}\right), \quad w = \pm \frac{l^2}{r^2},$$

$$\underset{\vee}{R}_{14} = -\frac{2}{r}\left(\frac{w}{\alpha}\right)' - \frac{2}{r^2}\left(\frac{w}{\alpha}\right), \tag{33}$$

(dash denoting differentiation with respect to $r$, $m$ and $l^2$ being constants). According to (33), both $\underset{\vee}{R}_{\mu\nu}$ and $w_{\mu\nu}$ describe fields of the order $r^{-4}$ and, as Papapetrou himself observed, cannot represent a simple point charge. This difficulty can be overcome by identifying

$$f_{23}$$

as the electric field. In general,

$$\underset{\vee}{R}_{23} = \frac{W}{r^2}\,\hat{r}, \tag{34}$$

so that, if $W$ is the charge, we obtain the classical Coulomb field. However, $\underset{\vee}{R}_{\mu\nu}$ appears as the curl of a vector $\Gamma_\mu$ (equation (12)) and this indicates the form (13) rather than (14) of Maxwell's electrodynamics. To avoid contradiction, therefore, we are driven to the conclusion that the electrodynamics arising from the nonsymmetric unified field theory is that of Born and Infeld for which the field equations for the induction and the intensity tensors $p^{\mu\nu}$ and $s^{\mu\nu}$ respectively are

$$(\sqrt{-h}\; p^{\mu\nu})_{,\nu} = 0 = \underset{\cdots}{s_{\mu\nu,\lambda}}, \tag{35}$$

$h$ = determinant of $\underline{g_{\mu\nu}}$.

## 5. *Some exact relations*

Consider the affine connection $\Gamma^{\lambda}_{\mu\nu}$ defined by the equation (5), $\Delta^{\lambda}_{\mu\nu}$ given by the equation (9) and the Ricci tensor $R_{\mu\nu}$:

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\mu\varrho}\Gamma^{\varrho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\varrho}_{\sigma\varrho}. \tag{36}$$

We have

$$\underline{\Delta^{\sigma}_{\lambda\sigma}} = \frac{\partial}{\partial x^{\lambda}} \ln \sqrt{-g} = \underline{\Gamma^{\sigma}_{\lambda\sigma}} + \tfrac{5}{3}\,\Gamma_{\lambda},$$

whence

$$\Delta^{\sigma}_{\underline{\lambda\sigma},\kappa} = \Delta^{\sigma}_{\underline{\kappa\sigma},\lambda}$$

and

$$\Gamma^{\sigma}_{\underline{\lambda\sigma},\kappa} - \Gamma^{\sigma}_{\underline{\kappa\sigma},\lambda} = \tfrac{5}{3}(\Gamma_{\kappa,\lambda} - \Gamma_{\lambda,\kappa}) \tag{37}$$

is thus a tensor. We have already noted the relation (11) between $R_{\underset{\vee}{\mu\nu}}(\Gamma)$ and $R_{\underset{\vee}{\mu\nu}}(\Delta)$.

In the sequel we shall denote by a semicolon the covariant derivative with respect to the connection $\Delta^{\lambda}_{\underline{\mu\nu}}$, thus dropping the "zero-indicator" from the equation (25) which now reads

$$w_{\mu\nu} = {}^{*}g^{\alpha\beta} g_{\underset{\vee}{\mu\nu};\alpha\beta}. \tag{25}$$

When necessary, we shall use a stroke to denote the covariant derivative with the connection $\Gamma^{\lambda}_{\underline{\mu\nu}}$.

It follows immediately from the definition (36) and the equation (37) that

$$R_{\underset{\vee}{\mu\nu}}(\Gamma) = -\Gamma^{\sigma}_{\underset{\vee}{\mu\nu}|\sigma} - \Gamma^{\sigma}_{\underset{\vee}{\mu\nu}}\Gamma_{\sigma} + \tfrac{1}{3}(\Gamma_{\nu,\mu} - \Gamma_{\mu,\nu}). \tag{38}$$

Because $\Delta_{\lambda}$ vanishes identically, we also have

$$R_{\underset{\vee}{\mu\nu}}(\Delta) = -\Delta^{\sigma}_{\underset{\vee}{\mu\nu};\sigma}. \tag{39}$$

Since

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\nu}} = \Delta^{\lambda}_{\underset{\vee}{\mu\nu}} - \tfrac{1}{3}(\Gamma_{\nu}\sigma^{\lambda}_{\mu} - \Gamma_{\mu}\sigma^{\lambda}_{\nu}),$$

and

$$\Gamma^{\lambda}_{\underline{\mu\nu}} = \Delta^{\lambda}_{\underline{\mu\nu}} - \tfrac{1}{3}(\Gamma_{\nu}\sigma^{\lambda}_{\mu} + \Gamma_{\mu}\sigma^{\lambda}_{\nu}),$$

we can readily verify that

$$\Delta^{\sigma}_{\underset{\vee}{\mu\nu}|\sigma} = \Delta^{\sigma}_{\underset{\vee}{\mu\nu};\sigma} - \Gamma_{\sigma}\Delta^{\sigma}_{\mu\nu},$$

and

$$\Gamma_{\sigma}\Delta^{\sigma}_{\underset{\vee}{\mu\nu}} = \Gamma_{\sigma}\Gamma^{\mu}_{\underset{\vee}{\mu\nu}}.$$

Also (if $R_{\mu\nu}(\Gamma) = 0$) it follows that

$$\Gamma^{\sigma}_{\underset{\vee}{\mu\nu}|\sigma} = \tfrac{1}{3}(\Gamma_{\nu,\mu} - \Gamma_{\mu,\nu}) - \Gamma^{\sigma}_{\underset{\vee}{\mu\nu}}\Gamma_{\sigma},$$

$$= \Delta^{\sigma}_{\underset{\vee}{\mu\nu};\sigma} - \tfrac{1}{3}(\Gamma_{\nu,\mu} - \Gamma_{\mu,\nu}).$$

Hence we get the exact formula (analogous to the equation (12))

$$\Delta^{\sigma}_{\underset{\vee}{\mu\nu};\sigma} = \tfrac{2}{3}(\Gamma_{\nu,\mu} - \Gamma_{\mu,\nu}). \tag{40}$$

In the rest of this work we shall seek to compare the tensor

$$\phi_{\mu\nu} = -\tfrac{3}{2} R_{\mu\underset{\vee}{\nu}}(\Delta) = \Gamma_{\nu,\mu} - \Gamma_{\mu,\nu},$$

which satisfies, of course, equations

$$\phi_{\underset{\cdots}{\mu\nu,\lambda}} = 0, \tag{41}$$

identical in form with one set of Maxwell's (or Born-Infeld) equations of electrodynamics with the field tensor $w_{\mu\nu}$ defined by the equation (25) and identified by Russell and Klotz with the electromagnetic tensor. The aim is to calculate the successive terms in the power series expansion in $\varepsilon$ of $w_{\mu\nu}$ when the fundamental tensor $g_{\mu\nu}$ is expanded according to the equations (14) and (22). We shall use as one of the field equations

$$\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = (\sqrt{-g}\,{}^{*}g^{\underset{\vee}{\mu\nu}})_{,\nu} = 0, \tag{42}$$

valid both in the generalized unified field theory and in the Einstein-Straus theory.

Since

$$g_{\mu\nu,\alpha} - \Delta^{\sigma}_{\mu\alpha} g_{\sigma\nu} - \Delta^{\sigma}_{\alpha\nu} g_{\mu\sigma} = 0, \tag{43}$$

we get, after a straightforward calculation,

$$\begin{aligned} w_{\mu\nu} = {}^{*}g^{\alpha\beta} g_{\underset{\vee}{\mu\nu};\alpha\beta} = {}^{*}g^{\alpha\beta}[\Delta^{\sigma}_{\underset{\vee}{\mu}\underline{\alpha;\beta}} g_{\sigma\nu} + \Delta^{\sigma}_{\alpha\underset{\vee}{\nu};\beta} g_{\mu\sigma} + \Delta^{\sigma}_{\underset{\vee}{\mu}\alpha} \Delta^{\varrho}_{\underset{\vee}{\sigma}\beta} g_{\underset{\vee}{\varrho}\nu} + \Delta^{\sigma}_{\alpha\underset{\vee}{\nu}} \Delta^{\varrho}_{\beta\underset{\vee}{\sigma}} g_{\mu\underset{\vee}{\varrho}}] \\ + {}^{*}g^{\underline{\alpha\beta}}(\Delta^{\sigma}_{\underset{\vee}{\mu}\alpha} \Delta^{\varrho}_{\beta\underset{\vee}{\nu}} - \Delta^{\sigma}_{\alpha\underset{\vee}{\nu}} \Delta^{\varrho}_{\underset{\vee}{\mu}\beta}) g_{\sigma\underset{\vee}{\varrho}}. \end{aligned} \tag{44}$$

We shall find this formula useful in carrying out the approximations.

## 6. *Auxiliary calculations*

Let us assume the $\varepsilon$-expansion (15) of $g_{\mu\nu}$, the symmetry condition (22), and that it induces the following expansion of the components $\Delta^{\lambda}_{\mu\nu}$ of the Einstein-Straus affine connection

$$\Delta^{\lambda}_{\mu\nu} = \underset{0}{\Delta^{\lambda}_{\mu\nu}} + \varepsilon \underset{1}{\Delta^{\lambda}_{\mu\nu}} + \varepsilon^2 \underset{2}{\Delta^{\lambda}_{\mu\nu}} + \varepsilon^3 \underset{3}{\Delta^{\lambda}_{\mu\nu}} + \ldots . \tag{45}$$

Equating to zero coefficients of the successive powers of $\varepsilon$ from the equation (43), we get

$$\underset{0}{\Delta^{\sigma}_{\mu\lambda}} \eta_{\sigma\nu} + \underset{0}{\Delta^{\sigma}_{\lambda\nu}} \eta_{\mu\sigma} = 0, \tag{46}$$

$$h_{\mu\nu,\lambda} - \underset{1}{\Delta^{\sigma}_{\mu\lambda}} \eta_{\sigma\nu} - \underset{1}{\Delta^{\sigma}_{\lambda\nu}} \eta_{\mu\sigma} = 0, \tag{47}$$

$$q_{\mu\nu,\lambda} - \underset{2}{\Delta^{\sigma}_{\mu\sigma}} \eta_{\sigma\nu} - \underset{1}{\Delta^{\sigma}_{\mu\lambda}} h_{\sigma\nu} - \underset{2}{\Delta^{\sigma}_{\lambda\nu}} \eta_{\mu\sigma} - \underset{1}{\Delta^{\sigma}_{\lambda\nu}} h_{\mu\sigma} = 0, \tag{48}$$

$$\alpha_{\mu\nu,\lambda} - \underset{1}{\Delta^{\sigma}_{\mu\lambda}} q_{\sigma\nu} - \underset{2}{\Delta^{\sigma}_{\mu\lambda}} h_{\sigma\nu} - \underset{3}{\Delta^{\sigma}_{\mu\lambda}} \eta_{\sigma\nu} - \underset{1}{\Delta^{\sigma}_{\lambda\nu}} q_{\mu\sigma} - \underset{2}{\Delta^{\sigma}_{\lambda\nu}} h_{\mu\sigma} - \underset{3}{\Delta^{\sigma}_{\lambda\nu}} \eta_{\mu\sigma} = 0, \tag{49}$$

and so on, although equation (49) is as far as we shall go. The usual method of permuting the indices cyclically and adding and subtracting the resulting equations gives immediately

$$\underset{0}{\Delta}{}^{\sigma}_{\mu\nu} = 0,$$

$$\underset{1}{\Delta}{}^{\sigma}_{\nu\mu} = \tfrac{1}{2}\,\eta^{\sigma\lambda}(h_{\lambda\mu,\nu}+h_{\nu\lambda,\mu}-h_{\mu\nu,\lambda}) = \underset{1}{\Delta}{}^{\sigma}_{\mu\nu},$$

$$\underset{2-}{\Delta}{}^{\sigma}_{\mu\nu} = \tfrac{1}{2}\,\eta^{\sigma\lambda}(m_{\lambda\mu,\nu}+m_{\nu\lambda,\mu}-m_{\mu\nu,\lambda}), \qquad m_{\mu\nu} = \underline{q_{\mu\nu}}-h_{\mu\nu},$$

$$\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\nu} = \tfrac{1}{2}\,\eta^{\sigma\lambda}(q_{\mu\nu,\lambda}+q_{\lambda\nu,\mu}+q_{\mu\lambda,\nu}),$$

$$\underset{3-}{\Delta}{}^{\sigma}_{\mu\nu} = \tfrac{1}{2}\,\eta^{\sigma\lambda}(n_{\mu\nu\lambda}+n_{\lambda\nu\mu}-n_{\nu\mu\lambda}),$$

$$n_{\lambda\mu\nu} = \alpha_{\lambda\mu,\nu}-\underset{1}{\Delta}{}^{\sigma}_{\lambda\nu}q_{\sigma\mu}-\underset{1}{\Delta}{}^{\sigma}_{\nu\mu}q_{\lambda\sigma}-\underset{2}{\Delta}{}^{\sigma}_{\lambda\nu}h_{\sigma\mu}-\underset{2}{\Delta}{}^{\sigma}_{\nu\mu}h_{\lambda\sigma},$$

and

$$\underset{3\vee}{\Delta}{}^{\sigma}_{\mu\nu} = \eta^{\sigma\lambda}\big[\tfrac{1}{2}\,(\underset{\vee}{\alpha}_{\mu\lambda,\nu}+\underset{\vee}{\alpha}_{\lambda\nu,\mu}+\underset{\vee}{\alpha}_{\mu\nu,\lambda})-\underset{1}{\Delta}{}^{\sigma}_{\mu\lambda}\underset{\vee}{q}_{\sigma\nu}-\underset{1}{\Delta}{}^{\sigma}_{\lambda\nu}\underset{\vee}{q}_{\mu\sigma}-\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\nu}h_{\mu\lambda}\big].$$

Hence,

$$\underset{-}{\Delta}{}^{\lambda}_{\mu\nu} = \varepsilon\underset{1-}{\Delta}{}^{\lambda}_{\mu\nu}+\varepsilon^{2}\underset{2-}{\Delta}{}^{\lambda}_{\mu\nu}+\varepsilon^{3}\underset{3-}{\Delta}{}^{\lambda}_{\mu\nu}+\ \dots, \tag{50}$$

$$\underset{\vee}{\Delta}{}^{\lambda}_{\mu\nu} = \varepsilon^{2}\underset{2\vee}{\Delta}{}^{\lambda}_{\mu\nu}+\varepsilon^{3}\underset{3\vee}{\Delta}{}^{\lambda}_{\mu\nu}+\ \dots, \tag{51}$$

is the resulting expansion of the affine connection. Also, taking into account the equation (16),

$$\begin{aligned}
w_{\mu\nu} = {} & \{[\varepsilon^{2}\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\alpha,\beta}+\varepsilon^{3}(\underset{3\vee}{\Delta}{}^{\sigma}_{\mu\alpha,\beta}+\underset{1}{\Delta}{}^{\sigma}_{\varrho\beta}\underset{2\vee}{\Delta}{}^{\varrho}_{\mu\alpha}-\underset{1}{\Delta}{}^{\varrho}_{\mu\beta}\underset{2\vee}{\Delta}{}^{\sigma}_{\varrho\alpha}-\underset{1}{\Delta}{}^{\varrho}_{\alpha\beta}\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\varrho})+O(\varepsilon^{6})]\\
& \times[\eta_{\sigma\nu}+\varepsilon h_{\sigma\nu}+O(\varepsilon^{2})]-[\text{as above with } \mu \text{ and } \nu \text{ interchanged}]\}\\
& \times\{\eta^{\alpha\beta}-\varepsilon h^{\alpha\beta}+\varepsilon^{2}Q^{\alpha\beta}+O(\varepsilon^{3})\}\\
= {} & \varepsilon^{2}(\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\alpha,\beta}\eta_{\sigma\nu}+\underset{2\vee}{\Delta}{}^{\sigma}_{\alpha\nu,\beta}\eta_{\mu\sigma})\eta^{\alpha\beta}+\varepsilon^{3}\{[(\underset{3\vee}{\Delta}{}^{\sigma}_{\mu\alpha,\beta}+\underset{1}{\Delta}{}^{\sigma}_{\varrho\beta}\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\alpha}\\
& -\underset{1}{\Delta}{}^{\varrho}_{\mu\beta}\underset{2\vee}{\Delta}{}^{\sigma}_{\varrho\alpha}-\underset{1}{\Delta}{}^{\varrho}_{\alpha\beta}\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\varrho})\eta_{\sigma\nu}+\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\alpha,\beta}h_{\sigma\nu}]\eta^{\alpha\beta}+[(\underset{3\vee}{\Delta}{}^{\sigma}_{\alpha\nu,\beta}\\
& +\underset{1}{\Delta}{}^{\sigma}_{\varrho\beta}\underset{2\vee}{\Delta}{}^{\varrho}_{\alpha\nu}-\underset{1}{\Delta}{}^{\varrho}_{\alpha\beta}\underset{2\vee}{\Delta}{}^{\sigma}_{\varrho\nu}-\underset{1}{\Delta}{}^{\varrho}_{\nu\beta}\underset{2\vee}{\Delta}{}^{\sigma}_{\alpha\varrho})\eta_{\mu\sigma}+\underset{2\vee}{\Delta}{}^{\sigma}_{\alpha\nu,\beta}h_{\mu\sigma}]\eta^{\alpha\beta}-(\underset{2\vee}{\Delta}{}^{\sigma}_{\alpha\nu,\beta}\eta_{\mu\sigma}+\underset{2\vee}{\Delta}{}^{\sigma}_{\mu\alpha,\beta}\eta_{\sigma\nu})h^{\alpha\beta}\}+O(\varepsilon^{4}).
\end{aligned} \tag{52}$$

The equation (9) is insufficient to determine the Einstein-Kaufman connection $\Gamma^{\lambda}_{\mu\nu}$ if $\Delta^{\lambda}_{\mu\nu}$ is regarded as given. If, however, we assume an expansion in the $\varepsilon$-power series of the former of the same pattern as that for $\Delta^{\lambda}_{\mu\nu}$ (equation (45)), then from the equations (9), (40) and (51) we get

$$\underset{0}{\Gamma}_{\lambda}+\tfrac{3}{5}\underset{0}{\Gamma}^{\sigma}_{\underline{\lambda\sigma}} = 0,$$

and

$$\underset{1}{\Gamma}_{\nu,\mu}-\underset{1}{\Gamma}_{\mu,\nu} = 0.$$

It follows from the latter that there exists a scalar function (of position) $\phi$ such that

$$\underset{1}{\Gamma}_{\mu} = \phi_{,\mu},$$

and is seems natural to assume that

$$\underset{0}{\Gamma}^{\lambda}_{\mu\nu} = 0.$$

## 7. *The equation* $\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0$

We must consider now the effect of the fast (as distinguished from the expansion in a small, time-parameter used by Infeld) expansion (15), (50) and (51) on the equation (42) which, as observed before, is one of the field equations of the generalised non-symmetric unified field theory. Using equation (16) we have

$$\tfrac{1}{2}\,{}^{*}g^{\alpha\beta}g_{\alpha\beta,\nu}(\varepsilon^{2}Q^{\underset{\vee}{\mu\nu}}+\varepsilon^{3}A^{\underset{\vee}{\mu\nu}}+\ \ldots)+\varepsilon^{2}Q^{\underset{\vee}{\mu\nu}}{}_{,\nu}+\varepsilon^{3}A^{\underset{\vee}{\mu\nu}}{}_{,\nu}+\ \ldots\ = 0, \tag{53}$$

and

$$\tfrac{1}{2}\,{}^{*}g^{\alpha\beta}g_{\alpha\beta,\nu} = \frac{\partial}{\partial x^{\nu}}\ln\sqrt{-g} = \Delta^{\sigma}_{\underline{\nu\sigma}}.$$

Also, from the equations (20) and (21),

$$Q^{\underset{\vee}{\mu\nu}} = q^{\underset{\vee}{\mu\nu}}, \tag{54}$$

and

$$A^{\underset{\vee}{\mu\nu}} = \alpha^{\underset{\vee}{\mu\nu}}+\tfrac{1}{2}\,(h^{\nu}{}_{\sigma}Q^{\sigma\mu}-h^{\mu}{}_{\sigma}Q^{\sigma\nu})+\eta^{\mu\varrho}h^{\nu\sigma}q_{\underset{\vee}{\varrho\sigma}}. \tag{55}$$

Equation (50) enables us to rewrite (53) as

$$\varepsilon^{2}Q^{\underset{\vee}{\mu\nu}}{}_{,\nu}+\varepsilon^{3}(A^{\underset{\vee}{\mu\nu}}{}_{,\nu}+\underset{1}{\Delta}^{\sigma}_{\nu\sigma}Q^{\underset{\vee}{\mu\nu}})+\ \ldots\ = 0. \tag{56}$$

Because of (54) we can easily verify that

$$Q^{\underset{\vee}{\mu\nu}}{}_{,\nu} = -\eta^{\nu\varrho}\eta^{\mu\sigma}q_{\underset{\vee}{\varrho\sigma},\nu} = -\eta^{\nu\varrho}\eta^{\mu\sigma}(\underset{2}{\Delta}^{\alpha}_{\underset{\vee}{\varrho\nu}}\eta_{\alpha\sigma}+\underset{2}{\Delta}^{\alpha}_{\underset{\vee}{\nu\sigma}}\eta_{\varrho\alpha}) = -\eta^{\mu\sigma}\underset{2}{\Delta}^{\alpha}_{\underset{\vee}{\alpha\sigma}} = 0, \tag{57}$$

since $\Delta_\mu$ vanishes identically. Because of (57), the result of equating to zero the coefficient of $\varepsilon^3$ in (56), can be written in the form

$$A^{\underset{\vee}{\mu\nu}}{}_{,\nu} + h_{,\nu} Q^{\underset{\vee}{\mu\nu}} + h Q^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0,$$

or

$$(A^{\underset{\vee}{\mu\nu}} + hQ^{\underset{\vee}{\mu\nu}})_{,\nu} = 0, \tag{58}$$

where

$$h = \tfrac{1}{2}\eta^{\alpha\beta} h_{\alpha\beta}.$$

After some calculation, we conclude from (58) that

$$\hat{\alpha}^{\sigma}_{\lambda,\sigma} + q\eta^{\sigma\varrho}\underset{1}{\Delta}{}^{\tau}_{\lambda\sigma} q_{\underset{\vee}{\varrho\tau}} + \eta^{\sigma\tau}\underset{1}{\Delta}{}^{\varrho}_{\varrho\tau} q_{\underset{\vee}{\lambda\sigma}} = 0. \tag{59}$$

Here

$$\hat{\alpha}^{\sigma}_{\lambda} = \eta^{\sigma\varrho}\alpha_{\underset{\vee}{\lambda\varrho}}.$$

Another result which is useful in the sequel is as follows. If we define

$$D^{\sigma}_{\mu\nu} = \eta^{\sigma\beta}\eta_{\alpha\nu}\underset{1}{\Delta}{}^{\alpha}_{\beta\mu} - \underset{1}{\Delta}{}^{\sigma}_{\mu\nu},$$

and

$$D_{\lambda\mu\nu} = \eta_{\lambda\sigma} D^{\sigma}_{\mu\nu},$$

then, substituting the expression for $\underset{1}{\Delta}{}^{\alpha}_{\beta\gamma}$ in terms of the first derivatives of $h_{\mu\nu}$,

$$D_{\lambda\mu\nu} = h_{\mu\nu,\lambda} - h_{\lambda\mu,\nu}.$$

Hence

$$(\eta^{\nu\tau}\eta^{\varrho\lambda} + \eta^{\nu\varrho}\eta^{\tau\lambda}) D_{\lambda\mu\nu} \equiv 0.$$

## *8. The four-vector potential*

Our aim is to extract from the approximate form (52) of the tensor $w_{\mu\nu}$, those terms which appear as the curl of a vector. The latter can then be considered as the electromagnetic four-vector potential if the appropriate identification of $w_{\mu\nu}$ is made.

If we write

$$w_{\mu\nu} = \varepsilon^2 \underset{2}{w}_{\mu\nu} + \varepsilon^3 \underset{3}{w}_{\mu\nu} + \ldots,$$

then we have from equation (52)

$$\underset{2}{w}_{\mu\nu} = (\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\alpha},\beta}\eta_{\sigma\nu} + \underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\alpha\nu},\beta}\eta_{\mu\sigma})\eta^{\alpha\beta},$$

and because of the equations (54) and (57) and the known solution of the equation (48)

$$\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\alpha},\nu}\eta_{\sigma\nu}\eta^{\alpha\beta} = \tfrac{1}{2}\,\eta^{\alpha\beta}\underset{\vee}{q}_{\mu\nu,\alpha\beta} = \underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\nu},\sigma}.$$

Thus

$$\underset{2}{w}_{\mu\nu} = \eta^{\alpha\beta}\underset{\vee}{q}_{\mu\nu,\alpha\beta} = 2\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\nu},\sigma}.$$

But, to the order $\varepsilon^2$,

$$\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\alpha},\beta} = \underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\alpha};\beta},$$

because of (50) and (51). Hence, from equation (40) which does not involve approximations,

$$\underset{2}{w}_{\mu\nu} = \tfrac{4}{3}\,(\underset{2}{\Gamma}_{\nu,\mu} - \underset{2}{\Gamma}_{\mu,\nu}). \tag{60}$$

The expression for $\underset{3}{w}_{\mu\nu}$ from the equation (52) can be written in the form

$$\begin{aligned}&[\underset{3}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\nu};\sigma} - \underset{1}{\Delta}{}^{\sigma}_{\varrho\sigma}\underset{2}{\Delta}{}^{\varrho}_{\underset{\vee}{\mu\nu}} + \underset{1}{\Delta}{}^{\varrho}_{\mu\sigma}\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\varrho\nu}} + \underset{1}{\Delta}{}^{\varrho}_{\nu\sigma}\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\varrho}} + \eta_{\sigma\nu}\eta^{\alpha\beta}(\underset{1}{\Delta}{}^{\sigma}_{\varrho\beta}\underset{2}{\Delta}{}^{\varrho}_{\underset{\vee}{\mu\alpha}} - \underset{1}{\Delta}{}^{\varrho}_{\mu\beta}\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\varrho\alpha}} - \underset{1}{\Delta}{}^{\varrho}_{\alpha\beta}\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\varrho}})\\ &- \eta^{\alpha\beta}\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\alpha}}h_{\sigma\nu,\beta} - \underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\alpha},\beta}\eta_{\sigma\nu}h^{\alpha\beta} + \eta^{\alpha\beta}\underset{\vee}{\alpha}_{\nu\alpha,\beta\mu} + \eta^{\alpha\beta}(\underset{1}{\Delta}{}^{\sigma}_{\alpha\mu}\underset{\vee}{q}_{\sigma\nu} + \underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\nu}}h_{\sigma\alpha})_{,\beta}] - [\nu \leftrightarrow \mu].\end{aligned} \tag{61}$$

And, from the equations (59) and (57)

$$\hat{\alpha}_{\nu}{}^{\sigma}{}_{,\sigma\mu} + 2\eta^{\varrho\alpha}\underset{1}{\Delta}{}^{\sigma}_{\nu\alpha,\mu}\underset{\vee}{q}_{\varrho\sigma} + \eta^{\sigma\beta}\underset{1}{\Delta}{}^{\alpha}_{\alpha\beta,\mu}\underset{\vee}{q}_{\nu\sigma} = 0. \tag{62}$$

Using this equation and the results of Section 7, we can show, after some straightforward calculations, that

$$\begin{aligned}\underset{3}{w}_{\mu\nu} = {}& 2\underset{2}{\Delta}{}^{\sigma}_{\underset{\vee}{\mu\nu};\sigma} + (h^{\alpha\beta}\underset{\vee}{q}_{\alpha\nu,\beta})_{,\mu} - (h^{\alpha\beta}\underset{\vee}{q}_{\alpha\mu,\beta})_{,\nu} + (p^{\sigma}\underset{\vee}{q}_{\sigma\mu})_{,\nu} - (p^{\sigma}\underset{\vee}{q}_{\sigma\nu})_{,\mu}\\ &+ \eta^{\alpha\beta}[3\underset{\vee}{q}_{\alpha\sigma}\underset{1}{\Delta}{}^{\sigma}_{\mu\beta,\nu} - 3\underset{\vee}{q}_{\alpha\sigma}\underset{1}{\Delta}{}^{\sigma}_{\nu\beta,\mu} + \underset{\vee}{q}_{\sigma\nu}\underset{1}{\Delta}{}^{\sigma}_{\mu\nu,\beta} + \underset{\vee}{q}_{\alpha\nu}\underset{1}{\Delta}{}^{\sigma}_{\sigma\beta,\mu}\\ &+ \underset{\vee}{q}_{\sigma\nu}\underset{1}{\Delta}{}^{\sigma}_{\alpha\beta,\mu} - \underset{\vee}{q}_{\sigma\mu}\underset{1}{\Delta}{}^{\sigma}_{\nu\alpha,\beta} - \underset{\vee}{q}_{\alpha\mu}\underset{1}{\Delta}{}^{\sigma}_{\sigma\beta,\nu} - \underset{\vee}{q}_{\sigma\mu}\underset{1}{\Delta}{}^{\sigma}_{\alpha\beta,\nu}] = \underset{3}{\phi}_{\nu,\mu} - \underset{3}{\phi}_{\mu,\nu} + \Omega_{\mu\nu},\end{aligned} \tag{63}$$

where

$$p^{\sigma} = \eta^{\alpha\beta}\underset{1}{\Delta}{}^{\sigma}_{\alpha\beta},$$

$$\underset{3}{\phi}_{\mu} = \tfrac{4}{3}\,\underset{3}{\Gamma}_{\mu} + h^{\alpha\beta}\underset{\vee}{q}_{\alpha\mu,\beta} - p^{\sigma}\underset{\vee}{q}_{\sigma\mu},$$

and

$$\underset{\vee}{\Omega}_{\mu\nu} = \eta^{\alpha\beta}[\ldots],$$

is the quantity by which $\underset{3}{w}_{\mu\nu}$ differs from the curl of a (pseudo) vector $\underset{3}{\phi}_{\mu}$.

Thus, although

$$\underset{2}{w}_{\mu\nu} = \tfrac{4}{3}(\underset{2}{\Gamma}_{\nu,\mu} - \underset{2}{\Gamma}_{\mu,\nu}) = 2\underset{2}{R}_{\mu\nu}(\check{\Delta}),$$

$\underset{3}{w}_{\mu\nu}$ is not, in general, the curl of a vector.

## *9. Discussion*

The lack of skew-symmetry and the disparity between the Lorentz force and the Coulomb law casts doubts on the aptness of identifying the Russell-Klotz tensor (1) as the electromagnetic tensor. The modified tensor, $w_{\mu\nu}$ (25), is skew-symmetric but the Lorentz force and the Coulomb law are still in discord.

It is possible to overcome these difficulties by using $R_{\mu\nu}(\Delta)$ as the electromagnetic tensor whereby the Lorentz force is present and an unmodified Coulomb law is predicted. However, it is necessary to consider non-Maxwellian theories (for example, the non-linear theory of Born and Infeld) to ensure suitable representations of the magnetic and electric fields.

In a future paper, we will use the third order approximations to continue our investigations of the equations of motion and of the form of the current vector in the generalised theory.

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