THERMODYNAMICS OF A GAS OF MIT BAGS

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The thermodynamics of the grand canonical ensemble of MIT bags is studied; the finite extension of the bags is taken into account as covolume à la van der Waals. Due to this finite extension, proportional to the bag mass, the gas has no ultimate temperature though the level density is similar to that of the statistical bootstrap model. The bag gas has a first order phase transition into a quark continuum. Connections to the hydrodynamical model of Landau are outlined.

1. Introduction

The MIT bag model [1] has found considerable success on the one hand by reproducing fairly well the static properties of baryons and mesons [2] and certain features of the Pomeron [3] and on the other hand by unifying several features which are at present believed to characterize the hadronic world as [1, 4] quark and colour confinement, an exponentially rising level density, linearly rising Regge trajectories and – at least in one space dimension – some features of dual models. The bag has a thermodynamical analogue, a bubble filled with a free, to the lowest order of sophistication massless quark gas [1]. Since we think that the treatment of multiparticle production processes within the bag model will have to rely on thermodynamical and possibly hydrodynamical approaches we have studied the thermodynamic properties of a gas of bags and its transition to a quark continuum phase. An earlier attempt to relate the bag model to the hydrodynamical model can be found in Ref. [5]; it is, in our opinion, incomplete.

The basic properties of the bag – bubble analogy are given in Ref. [1] and recalled in Chapter 2. The bubble reproduces the exponentially rising level density characteristic for the statistical bootstrap model [6], in addition its volume is specified to be proportional to its mass. We take this into account as excluded volume à la van der Waals in our treatment of the bag gas. This represents certainly a crude approximation. We simply argue that two bags which overlap form just a new bag of larger volume and mass, since the boundaries are by no means rigid. This new bag, formed in a scattering process, needs not respect the volume-mass proportionality which is derived from a virial theorem

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implying time averaging. Treating it, as we do, in the same way as a static bag is therefore analogous to a narrow resonance approximation to a scattering amplitude, which may not be so bad at low energies.

Our bag gas is depicted in Fig. 1a) and its partition function is evaluated in Chapter 3. It has, due to the finite extension of the bags, no ultimate temperature. It has a phase transition to a phase depicted in Fig. 1b), a big bag rattling inside the volume V. This

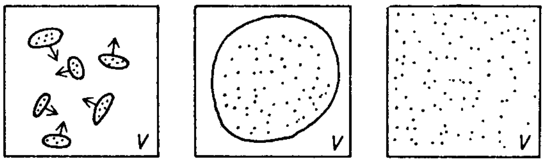

Fig. 1. The three phases considered in this paper: a) gas of bags, b) single big bag, c) quark continuum

phase is however statistically irrelevant against another system we can make up out of free quarks, shown in Fig. 1c). It is a configuration where the whole volume is filled with the quark gas, not respecting the bag boundary conditions. The partition function of this system is evaluated in Chapter 4. This homogeneous phase wins the statistical competition at a transition pemperature $T_{tr} = 1.2\, T_0$, where T_0 is the "Hagedorn temperature" characterising the level density. The transition is of first order.

The behaviour of the whole system is very reminiscent of a system considered in the hydrodynamical model of Landau [7]. There it is assumed, that in a high energy collision of hadrons a kind of prematter is formed which behaves as a free relativistic massless gas; this system expands according to relativistic hydrodynamics. If the temperature (dpending on space and time) has decreased to $T \approx m_\pi$, free hadrons can emerge, which form the final state of the scattering process. The value m_π of the transition temperature comes from the condition that a gas of hadrons of typical extension m_π^{-3} ceases to interact. This picture is quite analogous to the thermodynamic analogue to the bag model presented here, in the bag model we have the additional bonus, that both phases are made up out of the same material.

Before we go to the technical developments, we should mention that we are using Boltzmann statistics throughout, with the only excuse that at the degree of sophistication of using correct Fermi and Bose statistics more corrections like finite quark masses, the triality zero condition etc. should be included, going beyond the scope of this more qualitative presentation.

2. *Partition function for a single bag*

The thermodynamical properties of a single MIT - bag have been discussed already in Ref. [1]

The basic ingredients are:

(*i*) the contents of the bag are free massless quark fields, to be translated to a free massless relativistic gas, (*ii*) from (*i*) and the boundary conditions one derives a virial theorem:

$$m = 4B\langle V\rangle$$

where m is the energy (mass) of the bag and B is the bag constant. For an equilibrium configuration $\langle V\rangle = V$.

(*iii*) The energy of the empty bag in BV, so $3BV = 3/4\, m$ is the energy of the fields or, in this context, the gas.

We obtain then for the grand microcanonical partition function

$$z(m) = \sum \frac{1}{n!}\left(\frac{m}{4B}\right)^{n-1} (2\pi)^3 d^n \Omega_n(\tfrac{3}{4}\, m) \tag{2.2}$$

where d is the degeneracy and

$$\Omega_n(E) = \int \prod_{i=1}^{n} \frac{d^3k_i}{(2\pi)^3}\, \delta\left(E - \sum |\vec{k}_i|\right) \delta^3\left(\sum \vec{k}_i\right) \tag{2.3}$$

is the noncovariant phase space. We have included the momentum constraint since the interesting region will be at low m and therefore also small n. For Ω_n we obtain, using the method of Lurçat and Mazur [8] the approximation

$$\Omega_n(\tfrac{3}{4}\, m) = \frac{81\pi}{4\sqrt{3}}\, \frac{(3/4m)^{3n-4} n^2}{(3n)^{3n} e^{-3n} \pi^{2n}}\,. \tag{2.4}$$

Inserting into (2.2) yields after some rearrangement of factorials using Stirlings formula

$$z(m) = \frac{2^7\pi}{\sqrt{3}}\, 4BT_0^{-4} m^{-1} \sum_{n=2}^{\infty} \frac{(m/T_0)^{4n-4} n^2}{(4n)!}\,. \tag{2.5}$$

Asymptotically this behaves as

$$z(m) \sim m^{-2} e^{m/T_0} \tag{2.6}$$

where T_0, the analogue to the Hagedorn temperature, is

$$T_0 = \sqrt[4]{\frac{4B\pi^2}{4d}}\,. \tag{2.7}$$

3. *Partition function for a gas of bags*

We consider now a gas of bags in the grand canonical ensemble and in the thermodynamic limit. The chemical potential is zero, since the number of bags is not fixed, bags can coalesce and separate freely. We take into account the volume of the bags in a van der Waals type of way by using

$$V - \sum V_i = V - \sum \frac{m_i}{4B} \tag{3.1}$$

as the phase volume. Lorentz contraction has been taken into account in the actual calculation in an iterative way, since the use of

$$V-\sum \frac{m_i}{E_i}\,\frac{m_i}{4B} \tag{3.2}$$

instead od (3.1) prohibits any of the calculations given below. The grand canonical partition function reads then (using (3.1))

$$Z(T,V)=\sum \frac{1}{n!}\int \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3}\, dm_i z(m_i) e^{-\sqrt{p_i^2+m_i^2/T}} \left(V-\sum \frac{m_i}{4B}\right)^n$$

$$=\sum \frac{1}{n!}\int \prod_{i=1}^{n} dm_i \left(z(m_i)\,\frac{m_i T^2}{2\pi^2}\,(m_i/T) K_2\left(\frac{m_i}{T}\right)\right)\left(V-\sum \frac{m_i}{4B}\right)^n. \tag{3.3}$$

Using the abbreviation

$$\varphi(m_i, T)=\ln\left(z(m_i)\,\frac{m_i T^2}{2\pi^2}\,\frac{m_i}{T}\,k_2\left(\frac{m_i}{T}\right)\right), \tag{3.4}$$

this can be rewritten as

$$Z(T,V)\approx \sum_n \int \prod dm_i \exp\left\{\sum_{i=1}^{n} \varphi(m_i, T)+n\ln\left(V-\sum \frac{m_i}{4B}\right)+n-n\ln n\right\}. \tag{3.5}$$

The maximum of the integrand with respect to the m_i is determined by

$$\frac{\partial \varphi}{\partial m_i}=\frac{n}{V-\sum_{i=1}^{n} m_i/4B}\,\frac{1}{4B}. \tag{3.6}$$

So — apart from the possibility of having different solutions of $d\varphi/dm = \text{const}$ — the maximum occurs at equal masses $m_i = \bar{m}$ (the alternative of two solutions is discussed in the appendix). We do the m_i integration in (3.5) by expanding the exponent around the maximum up to second order in $(m_i-\bar{m})$, and treating the exponential as a gaussian.

The matrix of second derivatives of the exponent is

$$M_{ij}=\delta_{ij}\left.\frac{\partial^2 \varphi(m,T)}{\partial m^2}\right|_{m=\bar{m}}-\frac{n}{\left(V-n\dfrac{\bar{m}}{4B}\right)^2}\left(\frac{1}{4B}\right)^2 \tag{3.7}$$

and the gaussian integration yields a factor

$$\frac{\sqrt{2\pi}^n}{\sqrt{\det(-M_{ij})}}. \tag{3.8}$$

The determinant can be evaluated to give

$$\det(-M_{ij}) = \left(-\frac{\partial^2\varphi(\overline{m}, T)}{\partial m^2}\right)^{n-1}\left(-\frac{\partial^2\varphi(\overline{m}, T)}{\partial m^2} + \frac{1}{\left(\frac{4BV}{n} - \overline{m}\right)^2}\right) \tag{3.9}$$

which has the correct sign if

$$\frac{\partial^2\varphi(\overline{m}, T)}{\partial m^2} < 0 \tag{3.10}$$

as a condition for a maximum.

In the thermodynamic limit

$$n \to \infty, \qquad V \to \infty, \qquad \varrho = \frac{n}{V} \text{ finite} \tag{3.11}$$

we obtain then from (3.5), (3.8) and (3.9)

$$\lim_{V\to\infty}\frac{1}{V}\ln Z(T, V) = \max_{\varrho}\, \varrho\left\{\varphi(\overline{m}, T) - \ln\left(\varrho^{-1} - \frac{\overline{m}}{4B}\right) + 1 + \tfrac{1}{2}\ln\frac{2\pi}{\varphi''(\overline{m}, T)}\right\}. \tag{3.12}$$

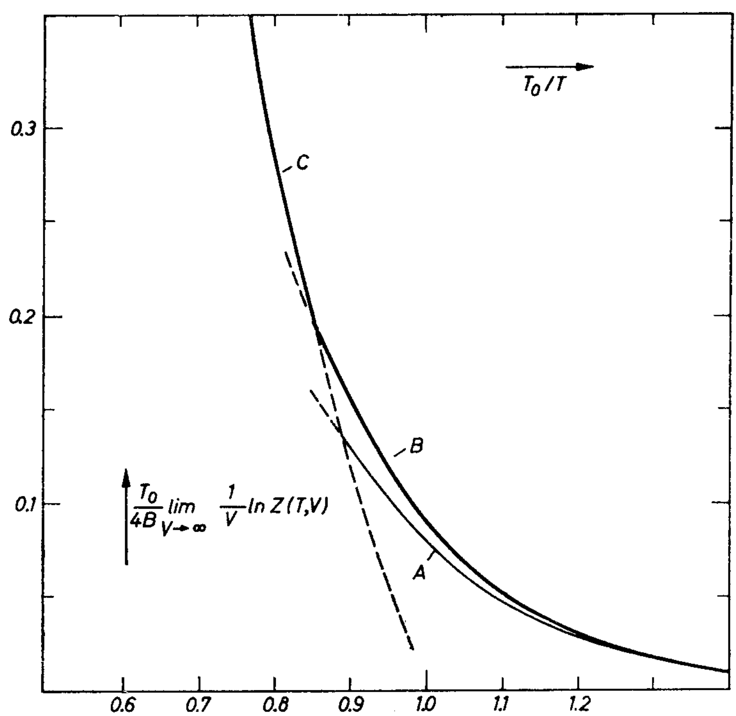

Fig 2. Lim $\frac{1}{V}\ln Z(T, V)$ as a function of T_0/T; curve A: for bag gas without Lorentz contraction, curve B: for bag gas with Lorentz contraction, curve C: for the quark continuum

It is convenient to get all the arguments of logarithms dimensionless by separating a term $\ln \frac{4B}{T_0^2}$ from $\varphi(m, t)$:

$$\varphi(m, T) = \ln \frac{4B}{T_0^2} + \hat{\varphi}(m, T) \tag{3.13}$$

so that $\hat{\varphi}(m, T)$ is a function of m/T_0 and T/T_0 alone and to transfer $\ln \frac{4B}{T_0}$ into the second term and $\ln 1/T_0$ into the last term in the exponent. Then

$$\lim_{V\to\infty} \frac{1}{V} \ln Z(T, V) = \frac{4B}{T_0} \max_{\hat{\varrho}} \hat{\varrho} \left\{ \hat{\varphi}(\overline{m}, T) - \ln\left(\hat{\varrho}^{-1} - \frac{\overline{m}}{T_0}\right) + 1 + \tfrac{1}{2} \ln\left(\frac{2\pi}{T_0^2 \hat{\varphi}''(\overline{m}, T)}\right) \right\} \tag{3.14}$$

with $\hat{\varrho} = \varrho T_0/4B$.

So using T_0 as the scale for $\overline{m}$ and T and $\frac{4B}{T_0}$ as the scale for $\frac{1}{V} \ln Z$ the results are independent of the degeneracy d. In the numerical calculation we have maximised the first to terms in the exponential in (3.14) with respect to $\overline{m}/T_0$ at fixed $\hat{\varrho}$ and then maximised with respect to $\hat{\varrho}$. This procedure leads to curve A in Fig. 2. The mean energy per particle may be calculated via

$$\overline{E} = \frac{\partial}{\partial 1/T} \ln Z/(\varrho V) = \frac{\partial}{\partial 1/T} \left(\frac{\ln Z}{4BV/T_0}\right) \Big/ \hat{\varrho} \tag{3.15}$$

and the Lorentz contraction factor $\overline{m}/\overline{E}$ is then inserted into (3.11) in analogy to (3.2). This is repeated until the procedure gets stable. This way we obtain curve B in Fig. 2.

4. Partition function for the quark continuum and the phase transition

Let us consider just one bag at rest in the volume V. This is, below T_0, a configuration which is statistically irrelevant. However if we raise its energy it will expand until, at $E = 4BV$, its extension coincides with V and its surface with the walls; it is no longer a free surface against the vacuum. Then the bag boundary conditions and therefore the virial theorem no longer apply, we have a new phase, where the internal temperature exceeds T_0. The total energy is however still the sum of the energy of the gas and the empty bag energy BV. So we have

$$Z(T, V) = \int_{4BV}^{\infty} dE e^{-E/T} \sum_{n=1}^{\infty} \frac{1}{n!} V^n d^n \Omega_n(E - BV) \tag{4.1}$$

with, neglecting the 3-momentum constraint,

$$\Omega_n(E) = \int \prod_{i=1}^{n} \frac{d^3 k_i}{(2\pi)^3} \delta\left(\sum |\vec{k}_i| - E\right) = \frac{1}{\pi^{2n}} \frac{E^{3n-1}}{(3n-1)!}. \tag{4.2}$$

From this we obtain for $T > T_0$

$$\lim_{V\to\infty} \frac{1}{V} \ln Z = \frac{4B}{T_0} \tfrac{1}{4} \left(\left(\frac{T}{T_0} \right)^3 - \frac{T_0}{T} \right). \tag{4.3}$$

For $T < T_0$, ln Z becomes negative, so this phase does not exist. The result (4.3) is plotted in Fig. 2, curve C. At $T = T_0$ this phase is not yet statistically competitive with the bag gas phase, we find a transition temperature $T_t = 1.18\, T_0$ (intersection of curve B and C). The transition is a first order phase transition, implying a discontinuity in the energy.

5. *Remarks and conclusion*

Since the qualitative features of the thermodynamical system have already been presented in the introduction, we should like to restrict ourselves to some additional remarks here. At the transition temperature we find an average hadron mass of about 4.5 T_0. With a bag constant of $B^{1/4} \approx 120$ MeV and a corresponding T_0 of 114 MeV (if a degeneracy $d = 12$ is used) this average mass is still below the ground state mass calculated in the bag model with the same B and massless quarks which is about 800 MeV. This is of course due to the use of a continuous level density $z(m)$ (Eq. 2.5) in this low mass region. On the other hand the low mass spectrum presents also problems in the bag model even in its refined version [2] and specially the pion, which is the most prominent object in multiparticle production, is still in an unclear situation within the bag model. So in order to close the picture also on a quantitative level, a more sophisticated version of our analysis on the one hand and a better understanding of the meson spectrum, especially of the pion, on the other hand are necessary.

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APPENDIX

A configuration with two different bag masses

As mentioned in Chapter 3, the equation (3.6) for the average bag mass may have more than one solution. This occurs actually for $T > T_0$ where $d\varphi/dm$ is no longer biunique. However the second derivative of φ is positive at the high mass solution m_+. The analysis of the matrix of second derivatives of the exponent shows that a configuration with several bags of high mass corresponds to a saddle point and not to a maximum. The only possibility is one big bag of mass m_+ and an ensemble of small bags of mass m_-. The one big bag is however statistically irrelevant in the thermodynamic limit unless its mass is proportional to the volume V, tending to infinity. With the definition $\mu = m_+/V$ the maximum condition (3.6) becomes

$$\frac{\partial\varphi(m_-, T)}{\partial m} = \frac{\varrho/4B}{1 - \dfrac{\mu}{4B} - \dfrac{\varrho}{4B} m_-} = \frac{\partial\varphi(m_+, T)}{\partial m} = \frac{1}{T_0} - \frac{1}{T} + O\left(\frac{1}{V}\right) \tag{A.1}$$

where ϱ is the density of the small bags. The last equation in (A.1) gives the asymptotic behaviour of $d\varphi/dm$ as $m \to \infty$. The small bag mass is therefore determined by the temperature and does not depend on μ and ϱ. The equality of the second and the last expression in (A.1) gives then a relation between ϱ and μ if T and therefore m_- are fixed. μ can then be eliminated and one obtains as the analogue to Eq. (3.12)

$$\lim \frac{1}{V} \ln Z = 4B\left(\frac{1}{T_0} - \frac{1}{T}\right) + \max_{\varrho} \varrho \left\{ \varphi(m_-) - m_- \left(\frac{1}{T_0} - \frac{1}{T}\right) - \ln 4B\left(\frac{1}{T_0} - \frac{1}{T}\right) - \tfrac{1}{2} \ln (\varphi''(m_-)/2\pi) \right\}. \tag{A.2}$$

Now the expression in the parenthesis is independent of ϱ, so that, depending on the sign of that expression the maximum is obtained either for $\varrho = 0$, i. e. a single big bag, or for $\varrho = \varrho_{\max} = 4B\Big/\left(m_- + \dfrac{TT_0}{T-T_0}\right)$; this implies $\mu = 0$, i. e. no big bag at all. For the single big bag configuration one obtains

$$\lim_{V\to\infty} \frac{1}{V} \ln Z(T, V) = 4B\left(\frac{1}{T_0} - \frac{1}{T}\right) \tag{A.3}$$

which is always below the corresponding expression for the quark continuum, Eq. (4.3). For the phase without big bag the values for ϱ and m_- give certainly a smaller value for $\dfrac{1}{V} \ln Z$ than ϱ and $\overline{m}$ obtained by unconstrained maximization in Chapter 3. So the configuration discussed in this appendix is statistically irrelevant, as was te be shown. This "phase" seems to be a reminiscence to a similar phase in the statistical bootstrap model (see the second of Refs [6]) which there is responsible for the ultimate temperature.

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