RELATIVISTIC SPHERES FILLED WITH ISENTROPIC MAGNETOFLUIDS

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(*Received August 10, 1976*; *Revised version received March 18, 1977*)

Exact solutions of the field equations for a thermodynamical perfect fluid with an infinite electrical conductivity and constant magnetic permeability are obtained under the assumptions that the space-time is spherically symmetric, the flow is isentropic and the fluid is incompressible. A class of static solutions reduces to the well known Schwarzschild solution in the absence of the magnetic field and Nordström-Jeffery type solution is recovered when the field is purely magnetic. A class of non-static and time-dependent solutions is also obtained. For numerical evaluation, the results are cast in dimensionless forms and the boundary conditions are stated.

1. *Introduction*

In the last sixty years, or so, various exact solutions of Einstein's field equations, mainly with the perfect fluid distributions in spherically symmetric space-times and with the electromagnetic fields in cylindrically symmetric space-times, have been found. General relativistic fluid spheres by Schwarzschild, Tolman, Oppenheimer and Volkoff are well-known. A series of papers dealing with new exact solutions under the assumption of the spherically symmetric distribution of a perfect fluid are due to Kuchowicz [1–4]. These solutions have been studied with special reference to the neutrino emission processes in advanced evolutionary stages of superdense stars (Kuchowicz [5, 6]). May and White [7] have given numerical analysis of cold neutron stars. Vaidya's radiating star has been studied by Lindquist et al. [8] for gravitational collapse. Solutions describing contracting as well as expanding distributions with a pressure gradient have been studied by Vaidya [9], as a non-static analog of Schwarzschild internal solution. Taub [10, 11] has found a time-dependent interior solution of the field equations of isentropic relativistic hydrodynamics.

In none of the works cited above, the magnetic field which is inevitably present in the astronomical objects like neutron stars in the intergalactic space is considered. The

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origin of the magnetic field has been a perplexing problem. The intergalactic matter is having very high electrical conductivity and strong magnetic field. Superdense or supermassive astronomical configurations like neutron stars according to our knowledge are characterized by strong magnetic field of the order 10^{12} gauss. These facts prompt one to incorporate the magnetic field effects in the models of the superdense stars. The assumption of a magnetofluid with infinite electrical conductivity and constant magnetic permeability befits theoretical considerations pertaining to cosmological models having magnetic field.

Using Lichnerowicz's [12] field equations for a thermodynamical perfect fluid with infinite electrical conductivity and constant magnetic permeability, Date [13, 14] has obtained some exact solutions and interpreted them as universes filled with irrotational and shearfree magnetofluid. Bray [15–17] has found Gödel's models and axially symmetric universes filled with a magnetofluid by solving Lichnerowicz's field equations.

This paper aims at obtaining solutions which are amenable for numerical computation. The following assumptions are made to obtain such solutions.

(a) The fluid is incompressible and isentropic with zero internal energy density.

(b) The space-time filled with a magnetofluid is spherically symmetric.

(c) The coordinate system is comoving.

(d) The velocity field and the magnetic field exhibit spherical symmetry.

2. *Field equations*

The energy-momentum tensor for a thermodynamical perfect fluid with infinite electrical conductivity and constant magnetic permeability is (Lichnerowicz, [12]).

$$T_{ab} = (\varrho_0 \nu + \mu h^2) u_a u_b - (p/c^2 + \tfrac{1}{2}\mu h^2) g_{ab} - \mu h_a h_b, \tag{2.1}$$

where ϱ_0 is the matter density, μ is the constant magnetic permeability, p is the hydrostatic pressure, c is the velocity of light. The fluid index ν is given by

$$\nu = 1 + \frac{\varepsilon}{c^2} + \frac{p}{c^2 \varrho_0}, \qquad \nu\varrho_0 = \varrho + \frac{p}{c^2}, \tag{2.2}$$

where ε is the internal energy density. The 4-velocity vector u^a and the magnetic field vector h^a satisfy the relations

$$u^a u_a = 1, \qquad h^a h_a = -h^2, \qquad h^a u_a = 0. \tag{2.3}$$

The field equations of relativistic magnetohydrodynamics (RMHD) are as follows:

Einstein field equations are

$$R_{ab} - \tfrac{1}{2} R g_{ab} = -kc^2 T_{ab}, \tag{2.4}$$

where

$$k = \frac{8\pi G}{c^2}.$$

Maxwell equations are

$$(u^a h^b - u^b h^a)_{;b} = 0, \tag{2.5}$$

where a semicolon indicates covariant differentiation and a comma denotes partial differentiation. The equation connecting rest temperature T_0 and entropy S is

$$T_0 dS = d\varepsilon + pd(1/\varrho_0). \tag{2.6}$$

Consequently the equation of continuity has the form

$$v(\varrho_0 u^a)_{;a} + \frac{1}{c^2} T_0 \varrho_0 u^a S_{,a} = 0. \tag{2.7}$$

3. *Spherically symmetric space-time and magnetofluid*

By means of the assumptions made in Section 1, we determine the thermodynamical variables ϱ, p, S, the 4-velocity u^a and the magnetic field h^a. From the assumption (a) (the isentropic motion of incompressible fluid with zero internal energy density), we have (Taub [10])

$$S = \text{constant}, \ \varrho = \text{constant}, \ \varepsilon = 0. \tag{3.1}$$

We consider the metric for spherically symmetric space-time as

$$ds^2 = e^{2A} dt^2 - \frac{1}{c^2} e^{2B} dr^2 - \frac{1}{c^2} e^{2C} d\Omega^2, \tag{3.2}$$

where A, B, C are functions of r, t and

$$d\Omega = d\Theta^2 + \sin^2 \Theta \, d\Phi^2.$$

Assumptions (c) and (d) imply

$$u^a = (0, 0, 0, e^{-A}), \tag{3.3}$$

$$p = p_0 \delta (e^{A_0 - A} - 1), \tag{3.4}$$

where

$$\delta = \frac{c^2 \varrho_0}{p_0}, \quad e^{A_0} = 1 + \frac{1}{\delta}. \tag{3.5}$$

Here p_0 is a constant having dimensions of pressure.

From the vanishing components of T_{ab} in spherically symmetric space-time, we obtain the magnetic field consistent with (2.3) and (3.3) as

$$h^a = (h^1, 0, 0, 0). \tag{3.6}$$

Maxwell equations (2.5) yield

$$h^1{}_{,t} = 0, \ h^1{}_{,r} + h^1 (B + 2C)_{,r} = 0. \tag{3.7}$$

Consequently, we get

$$h^1 = a_0 e^{-(B+2C)}, \quad h^2 = \frac{a_0^{\,2}}{c^2} e^{-4C} \tag{3.8}$$

where a_0 is an arbitrary constant.

For computation purposes, we find the Einstein equations in dimensionless forms by introducing the dimensionless variables

$$\tau = \frac{ct}{r_0}, \quad \mathrm{r} = \frac{r}{r_0}, \quad \mathrm{h} = \frac{h}{h_0}, \tag{3.9}$$

where $r_0 = (kp_0/c^2)^{-\frac{1}{2}}$ has the dimensions of length and h_0 has the dimensions of the magnetic field. In the space-time (3.2), Einstein equations become

$$e^{-2A}\{2C_{44}+3C_4^{\,2}-2C_4A_4\}-e^{-2B}\{C_1^{\,2}+2C_1A_1\} = -e^{-2C}-\frac{p}{p_0}+n^2e^{-4C}, \tag{3.10}$$

$$e^{-2A}\{B_{44}+B_4^{\,2}+C_{44}+C_4^{\,2}+B_4C_4-B_4A_4-C_4A_4\}$$

$$-e^{-2B}\{C_{11}+C_1^{\,2}+A_{11}+A_1^{\,2}+C_1A_1-A_1B_1-B_1C_1\} = -\frac{p}{p_0}-n^2e^{-4C}, \tag{3.11}$$

$$e^{-2A}\{C_4^{\,2}+2C_4B_4\}-e^{-2B}\{2C_{11}+3C_1^{\,2}-2C_1B_1\} = -e^{-2C}+\delta+n^2e^{-4C}, \tag{3.12}$$

$$C_{14}+C_1C_4-C_1B_4-C_4A_1 = 0, \tag{3.13}$$

where

$$n^2 = \mu a_0^{\,2} h_0^{\,2}/2p_0 \text{ and } C_1 = \frac{\partial C}{\partial \mathrm{r}}, \quad C_4 = \frac{\partial C}{\partial \tau} \text{ etc.}$$

The equation of continuity (2.7) yields

$$B_4+2C_4 = 0. \tag{3.14}$$

These are five equations in three unknowns A, B, C. Therefore, the solutions of any three equations either satisfy the remaining equations identically or give rise to the consistency conditions.

4. *Boundary conditions*

The equations (3.10) to (3.14) can be solved fully if the boundary conditions are known. The boundary conditions for the present problem are on the boundary (hyper-surface) separating two regions. Across the hypersurface separating space-time into interior and exterior, we have the following boundary conditions:

(*i*) The gravitational potentials are continuous, their derivatives with respect to τ are continuous, and the derivatives of A and C with respect to r are continuous. The derivatives of B with respect to r need not be continuous.

(*ii*) The total pressure from inside and from outside of the boundary must be equal.

(*iii*) The fluid velocity u^a satisfies the condition

$$[u^a\lambda_a]_{\text{inside}} = [u^a\lambda_a]_{\text{outside}},$$

where λ_a are the covariant components of the normal to the hypersurface separating interior and exterior.

(*iv*) The magnetic field h^a satisfies the condition

$$[h^a\lambda_a]_{\text{inside}} = [h^a\lambda_a]_{\text{outside}}.$$

5. *Static solution*

For static case, functions A, B, C are functions of r only and equations (3.13), (3.14) are identically satisfied. Without loss of generality, we can choose

$$C = \log \mathrm{r}$$

so that the line element (3.2) reduces to the canonical form. The equations (3.10) to (3.12) reduce to

$$e^{-2B}\left(\frac{1}{\mathrm{r}^2}+\frac{2}{\mathrm{r}}A_1\right) = \frac{1}{\mathrm{r}^2}+\frac{p}{p_0}+\frac{n^2}{\mathrm{r}^4}, \tag{5.1}$$

$$e^{-2B}\left(A_{11}+A_1{}^2+\frac{A_1}{\mathrm{r}}-A_1B_1-\frac{B_1}{\mathrm{r}}-\frac{1}{\mathrm{r}^2}\right) = \frac{p}{p_0}+\frac{n^2}{\mathrm{r}^4}, \tag{5.2}$$

$$e^{-2B}\left(\frac{1}{\mathrm{r}^2}-\frac{2B_1}{\mathrm{r}}\right) = \frac{1}{\mathrm{r}^2}-\delta-\frac{n^4}{\mathrm{r}^4}. \tag{5.3}$$

On integrating equation (5.3), we get

$$e^{-2B} = 1-\left(\frac{\delta}{3}\right)\mathrm{r}^2+\frac{n^2}{\mathrm{r}^2}-\frac{\alpha_0}{\mathrm{r}}, \tag{5.4}$$

where α_0 is a constant of integration. From equation (5.1) we have

$$e^A = \tfrac{1}{2}(1+\delta)e^{-B}\int e^{3B}\mathrm{r}d\mathrm{r}+\gamma_0 e^{-B}, \tag{5.5}$$

where γ_0 is a constant of integration. It is interesting to note that the equation (5.2) is identically satisfied by these solutions. Moreover, the effect of the magnetic field can be accounted in the universe filled with the isentropic magnetofluid. We claim that the static solution found here is a generalization of Schwarzschild interior solution as well as that of Nordström–Jeffery solution. This claim can be justified by considering the following particular cases:

Case (*i*). Magnetic field is not present

Here $\boldsymbol{h} = 0 \Leftrightarrow n = 0$ and $\delta \neq 0$, so that equation (5.4) reduces to

$$e^{-2B} = 1-\left(\frac{\delta}{3}\right)\mathrm{r}^2-\frac{\alpha_0}{\mathrm{r}}. \tag{5.6}$$

This solution has singularity at $\mathrm{r} = 0$. One can avoid it by choosing $\alpha_0 = 0$. Consequently

$$e^{-2B} = 1 - \left(\frac{\delta}{3}\right) \mathrm{r}^2. \tag{5.7}$$

Replacing $\mathrm{r} = r/r_0$, we have a static solution

$$e^{-2B} = 1 - \frac{r^2}{R^2}, \tag{5.8}$$

where

$$R^2 = \frac{3r_0{}^2}{\delta} = \frac{3c^2}{8\pi G \varrho_0}.$$

The equation (5.5) becomes

$$e^A = \tfrac{1}{2}(3\sqrt{1 - R_0{}^2/R^2} - \sqrt{1 - r^2/R^2}), \tag{5.9}$$

where R_0 is a suitably chosen constant of integration. Note that the equation (5.9) is for a perfect fluid as found by Schwarzschild (Taub, [11]).

Case (*ii*). Magnetic field in vacuo

Here $\delta = 0$ and $\boldsymbol{h} \neq 0$. Equations (5.4) and (5.5) yield

$$e^{-2B} = e^{2A} = 1 + \frac{n^2}{\mathrm{r}^2} - \frac{\alpha_0}{\mathrm{r}}, \tag{5.10}$$

when $\gamma_0 = 0$. This solution agrees in form with Nordström-Jeffery solution.

Case (*iii*). Empty space-time

Here $\boldsymbol{h} = 0$, $\delta = 0$, so that equations (5.4) and (5.5) reduce to

$$e^{-2B} = e^{2A} = 1 - \frac{\alpha_0}{\mathrm{r}}. \tag{5.11}$$

This solution is the Schwarzschild exterior solution. The constant α_0 is usually replaced by $\alpha_0 = 2GM/c^2$ so that the solution (5.11) represents the gravitational field outside a spherical mass M centred at $\mathrm{r} = 0$.

Now we find the region of validity of a static solution of the relativistic magnetohydrodynamic field equations. Let the region be restricted to

$$0 < \gamma \leqslant \gamma_0. \tag{5.12}$$

Exterior to the hypersurface $\gamma = \gamma_0$, there may be an empty space-time permeated by the magnetic field. Then the solution (5.10) can be fitted as the exterior solution. If the exterior to $\gamma = \gamma_0$ is an empty space-time, then Schwarzchild exterior solution (5.11) can be fitted. If the exterior to $\gamma = \gamma_0$ is a space-time filled with a perfect fluid, the Schwarzschild interior solution presented in case (*i*) can be fitted on the cavity filled with the magnetofluid.

6. *Non-static solution*

We obtain the non-static solutions of the field equations (3.10) to (3.14). From equation (3.14) we have

$$B(\mathrm{r}, \tau) = -2C(\mathrm{r}, \tau)+B(\mathrm{r}, 0)+2\log \mathrm{r} \tag{6.1}$$

and from equation (3.13),

$$e^{3C}C_4 = \frac{\beta_0}{\sqrt{3}}e^A, \tag{6.2}$$

where β_0 is a constant.
On using equations (3.14), (6.2), equation (3.12) becomes

$$e^{-2B}(2C_{11}+3C_1{}^2-2C_1B_1) = e^{-2C}-\beta_0{}^2e^{-6C}-\delta-n^2e^{-4C}. \tag{6.3}$$

This equation holds true for all τ if it holds true for $\tau = 0$. Hence by taking

$$C(\mathrm{r}, 0) = \log \mathrm{r},\ \tau = 0, \tag{6.4}$$

equation (6.3) reduces to

$$e^{-2B}\left(\frac{1}{\mathrm{r}^2}-\frac{2B_1}{\mathrm{r}}\right) = \frac{1}{\mathrm{r}^2}-\frac{\beta_0{}^2}{\mathrm{r}^6}-\frac{n^2}{\mathrm{r}^4}-\delta, \tag{6.5}$$

which on integration, gives rise to

$$e^{-2B} = \left(\frac{e^C}{\mathrm{r}}\right)^4\left\{1-\frac{\delta}{3}\mathrm{r}^2+\frac{\beta_0{}^4}{3\mathrm{r}^4}+\frac{n^2}{\mathrm{r}^2}+\frac{\alpha_0}{\mathrm{r}}\right\}, \tag{6.6}$$

where α_0 is a constant of integration. Again from the field equations (3.10), (3.12) we get

$$A_1e^A+B_1e^A = \tfrac{1}{2}(1+\delta)e^{2B} \tag{6.7}$$

whose solution is

$$e^A = \tfrac{1}{2}(1+\delta)e^{-B}\int e^{3B}\mathrm{r}d\mathrm{r}+\gamma e^{-B}, \tag{6.8}$$

where γ is an arbitrary function of τ. When $\beta_0 = 0$, $\gamma = \gamma_0$ this solution reduces to a static solution. In the absence of the magnetic field ($\boldsymbol{h} = 0$), the results in this section reduce to Taub's [11] results for isentropic hydrodynamics.

7. *Time dependent solution*

From equation (3.14), we get

$$B = -2C+f(\mathrm{r}), \tag{7.1}$$

where $f(\mathrm{r})$ is an arbitrary function of r. It is always possible to choose the spatial variable r by coordinate transformation (See Taub [10]) such that

$$B = -2C. \tag{7.2}$$

From equation (3.13), we have

$$(e^{3C})_4 = \sqrt{3}\,\beta(\tau)e^{A}, \tag{7.3}$$

where $\beta(\tau)$ is a function of τ and accordingly the solution is referred as *time dependent* solution. Consequently (3.12) reduces to

$$e^{4C}\{2C_{11}+7C_1{}^2\} = e^{-2C}-\delta-n^2e^{-4C}-\beta^2e^{-6C}. \tag{7.4}$$

We have on integration

$$C_1{}^2 = e^{-6C}+\frac{\beta^2}{3}e^{-10C}-\frac{\delta}{3}e^{-4C}+n^2e^{-8C}+\frac{\alpha}{3}e^{-7C}, \tag{7.5}$$

where α is a function of τ only. On setting $F = e^{3C}$, equations (7.3) and (7.5) reduce to

$$F_4 = \sqrt{3}\,\beta e^{A}, \tag{7.6}$$

$$F_1 = \sqrt{3}\,F^{-2/3}\left[\left(\beta^2+\frac{\alpha^2}{4\delta}\right)-\delta\left(F-\frac{\alpha}{2\delta}\right)^2+3F^{2/3}(n^2+F^{2/3})\right]^{\frac{1}{2}}. \tag{7.7}$$

By keeping τ constant and taking $F = F_0$ when $r = r_0$, equation (7.7) yields

$$\int_{F_0}^{F}\frac{F^{2/3}dF}{\sqrt{3}\left[\left(\beta^2+\frac{\alpha^2}{4\delta}\right)-\delta\left(F-\frac{\alpha}{2\delta}\right)+3F^{2/3}(n^2+F^{2/3})\right]^{\frac{1}{2}}} = r-r_0. \tag{7.8}$$

Subtracting equation (3.12) form equation (3.10) and using equation (7.2) we get

$$e^{-2A}\{2C_{44}+6C_4{}^2-2C_4A_4\}+e^{4C}\{6C_1{}^2+2C_{11}-2C_1A_1\} = -\frac{p}{p_0}-\delta. \tag{7.9}$$

Substituting p from equation (3.4) and $F = e^{3C}$, we have

$$1/F(e^{-A}F_4)_4+F^{1/3}e^{A}(F_{11}-F_1A_1) = -3/2(1+\delta). \tag{7.10}$$

and by equation (7.6) it reduces to

$$\frac{\sqrt{3}}{F}\beta_4-\frac{F^{1/3}F_1{}^2}{\beta\sqrt{3}}\left(\frac{F_4}{F_1}\right)_1 = -\tfrac{3}{2}(1+\delta). \tag{7.11}$$

On eliminating F from equations (7.8) and (7.11) we get a consistency condition to be satisfied by α and β as

$$\alpha_4 = \sqrt{3}\,\beta(1+\delta). \tag{7.12}$$

The line element corresponding to this solution is

$$ds^2 = \frac{r_0{}^2F_4{}^2}{3c^2\beta^2}d\tau^2-\frac{r_0{}^2}{c^2}F^{-4/3}\,dr-\frac{F^{2/3}}{c^2}d\Omega^2. \tag{7.13}$$

This can be transformed to

$$ds^2 = \frac{r_0{}^2F_4{}^2}{c^2}\,dt^2 - \frac{r_0{}^2F^{-4/3}}{c^2}\,dr^2 - \frac{F^{2/3}}{c^2}\,d\Omega^2, \tag{7.14}$$

where F is given by

$$\int\limits_{F_0}^{F} \frac{F^{2/3}dF}{\sqrt{3}\left[(1/3+\alpha^2/4\delta)-\delta\left(F-\frac{\alpha}{2\delta}\right)^2+3F^{2/3}(n^2+F^{2/3})\right]^{\frac{1}{2}}} = r-r_0, \tag{7.15}$$

and the consistency condition (7.12) reads as

$$\alpha = (t-t_0)\,(1+\delta). \tag{7.16}$$

Equation (7.10) determines the gravitational potential A and the pressure is evaluated from (3.4).

8. *Concluding remarks*

This paper presents the solutions of the Lichnerowicz's [12] magnetohydrodynamic field equations and the boundary conditions to study the self-gravitating astronomical objects which possess strong magnetic field. From static, non-static and time-dependent solutions it is apparent that the magnetic field term is present in these solutions. In absence of the magnetic field, Taub's [10, 11] results are recovered and for pure magnetic field, the static solution reduces to the Nordström-Jeffery type solution. A solution for the pure magnetic field can be interpreted as the magnetic field of a point source situated at the origin and identifies the constant n as the magnetic pole strength a point source. Despite the fact that a magnetic monopole is a physical fiction, the strong magnetic field which is present in astronomical systems does not rule out the possibility of the magnetic monopole from theoretical considerations.

The authors' thanks are due to Professor A. H. Taub for remarks on an earlier version of this paper and to Professor J. V. Narlikar for valuable discussions. We thank the referee of this paper for modifications. This work has been made possible by the University Grants Commission's grant to one of us (T. H. D.).

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