Vol. B8 (1977) ACTA PHYSICA POLONICA No 9

COMPARISON OF DIFFERENT KINDS OF REGULARIZATION OF PERTURBATION CALCULATIONS IN QUANTUM FIELD THEORY

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(*Received February 24, 1977*)

Different methods of regularization in quantum field theory are compared. It is argued that a regularization is correct if it gives the amplitude with analytical properties predicted by the Cutkosky lemma.

1. *Introduction*

The difficulties with ultraviolet divergences in perturbation calculations of QFT are widely known. They force us to use regularization in order to avoid the divergences and to obtain a finite result comparable with experiment.

Treating the problem as generally as possible, one may say, that in every regularization either a free parameter is introduced (auxiliary masses M_i in the Pauli–Villars regularization (1, 2]) or a constant parameter present in the unrenormalized theory is treated as a variable. This parameter is for example the power of the denominator in the analytical regularization [3] (before regularization equal to 1) or the dimension of momentum space in the dimensional regularization [4, 5]. Then we make use of the fact, that the whole integral treated as a function of the chosen parameter is regular for some values of this parameter and has a pole of some order for the usual value of this parameter. In order to regularize the integral, the pole part must be dropped out from the Laurent expansion with respect to the parameter mentioned above.

One may try to treat the Pauli-Villars regularization as the analytical one. For example the regularization of the self-energy diagram (Fig. 1)

$$\sum (p^2) = \int \frac{d^4k}{(m^2-k^2-i\varepsilon)\,(m^2-(p-k)^2-i\varepsilon)} \tag{1}$$

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is realized by replacing the expression $\sum(p^2)$ by

$$\sum_{\text{reg}}(M, p^2) = \int\left(\frac{1}{m^2-k^2-i\varepsilon} - \frac{1}{M^2+m^2-k^2-i\varepsilon}\right)\frac{d^4k}{m^2-(p-k)^2-i\varepsilon}$$

$$= M^2\int\frac{d^4k}{(m^2-k^2-i\varepsilon)(M^2+m^2-k^2-i\varepsilon)(m^2-(p-k)^2-i\varepsilon)}. \tag{2}$$

We have formally

$$\lim_{M\to\infty} \sum_{\text{reg}}(M, p^2) = \sum(p^2). \tag{3}$$

Furthermore M could be regarded as a free parameter. Integration is possible for $M \neq \infty$ and the integrated formula is singular for $M = \infty$ like $\ln(M^2)$. This singularity, however, is not a simple pole at infinity and cannot be dropped out as in the case of analytical and dimensional regularization.

The purpose of this paper is to look through some known regularization methods in order to find the reason for the uniqueness of these methods.

2. *Different methods of regularization*

2.1. Regularization with the application of dispersion relations containing subtractions [6,7]

The reason we start with this method will be clear later. We shall try to show that this method of regularization is a fundamental one. It is based on the so called Cutkosky lemma [8]. The lemma allows in some cases to define the analytical properties of the amplitude connected with a certain divergent Feynman diagram without calculation of the divergent integrals.

For example, we can calculate the imaginary part of the amplitude connected with the diagram of Fig. 1. (see Appendix A). The result is

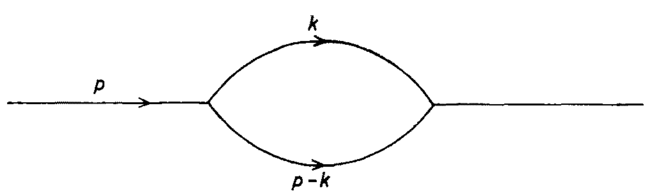

Fig. 1

$$\text{Im}\sum(p^2) = \theta(p^2-4m^2)\left[\frac{p^2-4m^2}{p^2}\right]^{1/2}. \tag{4}$$

Now a dispersion relation with one subtraction can be written down, for the amplitude treated as an analytical function of p^2 and after calculations we have a formula for $\sum(p^2)$ (see Appendix B)

$$\sum(p^2) = \frac{e^2}{2^4\pi}\left[\frac{p^2-4m^2}{p^2}\right]^{1/2} \ln\frac{4m^2-p^2+[p^2(p^2-4m^2)]^{1/2}}{4m^2-p^2-[p^2(p^2-4m^2)]^{1/2}} + C = \frac{e^2}{2^4\pi}\varphi(p^2)+C,$$

$$C = C^*, \tag{5}$$

where the real constant C is arbitrary. The condition $C = C^*$ comes from (4).

In this approach the problem of regularization is equivalent to looking for an analytical function with analytical properties predicted by the Cutkosky lemma. The problem of finding this analytical function can be solved with various methods (the dispersion relation method is the simplest). These various methods are equivalent to various regularizations and any regularization has to be one of these methods.

In what follows we shall prove it for the Pauli-Villars and the so called analytical regularization method.

2.2. The Pauli-Villars regularization [1, 2]

This method is very closely connected with the method of dispersion relations, but to see this we have to change the usual interpretation of this method.

The following steps are usualy taken:

Instead of $\sum(p^2)$ the expression $\sum_{\text{reg}}(M, p^2)$ is calculated. Then the limit (3) is taken and the finite part of the regularized expression is separated from the divergent one. This step is not uniquely defined but the correct realization of it is guaranteed by the following approach. Let us remind ourselves that

$$\sum\nolimits_{\text{reg}}(M, p^2) = \sum(p^2, m_1^2, m_2^2) - \sum(p^2, m_1^2+M^2, m_2^2)$$

$$= \int\frac{d^4k}{(m_1^2-k^2-i\varepsilon)(m_2^2-(p-k)^2-i\varepsilon)} - \int\frac{d^4k}{(m_1^2+M^2-k^2-i\varepsilon)(m_2^2-(p-k)^2-i\varepsilon)}, \tag{6}$$

where the masses m_1 and m_2 are numerated to draw a distinction between them. The calculation of the right hand side of (6) gives a difference between two identical functions with different "mass arguments"

$$\sum(p^2, m_1^2, m_2^2) - \sum(p^2, (m_1^2+M^2), m_2^2) = \varphi(p^2, m_1^2, m_2^2) - \varphi(p^2, m_1^2+M^2, m_2^2),$$

where

$$\varphi(p^2, m_1^2, m_2^2) = \left[\frac{p^2-(m_1+m_2)^2}{p^2}\right]^{1/2} \ln\frac{(m_1+m_2)^2-p^2+[p^2(p^2-(m_1+m_2)^2)]^{1/2}}{(m_1+m_2)^2-p^2+[p^2(p^2-(m_1+m_2)^2)]^{1/2}}. \tag{7}$$

Formally it is not possible to conclude here that

$$\sum(p^2, m_1^2, m_2^2) = \varphi(p^2, m_1, m_2) \tag{7'}$$

as was done in Appendix B because the function $\sum' = \varphi + g(p^2)$ is also good for every $g(p^2)$. However, we take $\varphi(p^2, m)$ as a correct solution of the problem, because the analytical properties of $\varphi(p^2, m)$ are in agreement with the Cutkosky lemma. The question which we should answer now is: why it is so?

This question would be trivial if we had realized the correct subtraction (Appendix B) instead of the Pauli-Villars regularization. In this case we would have

$$\sum (p^2, m_1^2, m_2^2) - \sum (p^2, m_1^2, m_2^2)$$

$$= \int_{(m_1+m_2)^2}^{\infty} \frac{dz}{z-p^2-i\varepsilon}\left[\frac{z-(m_1+m_2)^2}{z}\right]^{1/2} - \int_{(m_1+m_2)^2}^{\infty} \frac{dz}{z-p_0^2-i\varepsilon}\left[\frac{z-(m_1+m_2)^2}{z}\right]^{1/2}. \quad (8)$$

After a substitution

$$\left[\frac{z}{(m_1+m_2)^2}-1\right]^{1/2} = -\left[\frac{z}{(m_1+m_2)^2}\right]^{1/2} + [t]^{1/2},$$

we have for both integrals

$$\int_{(m_1+m_2)^2}^{\infty} \frac{dz}{z-p^2-i\varepsilon}\left[\frac{z-(m_1+m_2)^2}{z}\right]^{1/2} = \ln t|_1^{\infty} + \left[\frac{p^2-(m_1+m_2)^2}{p^2}\right]^{1/2}$$

$$\times \ln \frac{t(m_1+m_2)^2-2p^2+(m_1+m_2)^2-2[p^2(p^2-(m_1+m_2)^2)]^{1/2}}{t(m_1+m_2)^2-2p^2+(m_1+m_2)^2+2[p^2(p^2-(m_1+m_2)^2)]^{1/2}}\Bigg|_1^{\infty}$$

$$= \ln t|_1^{\infty} + \varphi(p^2, m_1, m_2). \quad (9)$$

We take the difference of these integrals, the terms

$$\ln t|_1^{\infty}$$

cancel and the difference of two logarithm functions $\varphi(p^2)$ remains.

In this case the conclusion $\sum(p^2) = \varphi(p^2) + C$ was correct, because the equation

$$\sum (p^2) - \sum (p_0^2) = \varphi(p^2) - \varphi(p_0^2) \quad (10)$$

is satisfied for every p^2, p_0^2.

Now we come back to the Pauli-Villars regularization. The calculations are quite similar, but in this case the difference between two $\sum$-s, is not in the arguments p^2, but in the masses: instead of m_1^2 we have in the second sigma $(M^2+m_1^2)$. A glimpse at (9) makes clear, why the $\varphi(p^2)$ is correct in (7).

2.3. The analytical and dimensional regularization [3, 4, 5]

As was mentioned above, methods of this kind are based on the following. At first a parameter must be found, which is constant in the nonregularized calculations. Then we treat this parameter as a variable and sometimes we can finish all integrations keeping this parameter free. The integral becomes a function of the chosen parameter and for its "ordinary" value the function has a pole.

There are infinitely many ways of introducing this parameter of course; for a function $f(p^2)$ many functions $F(k, p^2)$ exist, which satisfy the condition $F(k_0, p^2) = f(p^2)$. Now a question arises: are the different ways of regularization of this kind equivalent or not?

Let us consider a function $f(k, l)$ analytical in both variables: $f(k_0, l)$ is analytical in l except for some pole in l_0 and the same holds for the variable k. In that case we have

$$f(k, l_0) = \frac{a_{-1}(l_0)}{k-k_0} + a_0(l_0) + a_1(l_0)\,(k-k_0) + \ldots, \tag{11}$$

$$f(k_0, l) = \frac{a'_{-1}(k_0)}{l-l_0} + a'_0(k_0) + a'_1(k_0)\,(l-l_0) + \ldots. \tag{12}$$

The variables k and l play the role of two different parameters mentioned above; the external invariants are omitted for simplicity.

The regularization based on k gives

$$f_{\text{reg}}(k_0, l_0) = a_0(l_0) \tag{13}$$

and when l is used, then

$$f_{\text{reg}}(k_0, l_0) = a'_0(k_0). \tag{13'}$$

These two methods of regularization are equivalent if

$$a_0(l_0) = a'_0(k_0),$$

or at least if the difference between $a_0(p^2, l_0)$ and $a'_0(p^2, k_0)$ is trivial. (In our example it would be a real constant with respect to p^2). In general the answer is negative.

To prove this let us assume that our hypothesis is true for some particular function $f(k, l)$, i. e. for this function we have $a_0(l_0) = a'_0(k_0)$. Now we can construct a new function

$$F(k, l) = [\tau(k)]^{k-k_0}[\varrho(l)]^{l-l_0} f(k, l), \tag{14}$$

which is as good, as the $f(k, l)$ was, because

$$[\tau(k)]^{k-k_0}[\varrho(l)]^{l-l_0}\Big|_{\substack{k=k_0\\ l=l_0}} = 1. \tag{15}$$

However, if we expand $F(k, l)$ as before

$$\begin{aligned} F(k, l_0) &= \left\{1 + \tau^{k-k_0} \frac{d\tau}{dk} \ln \tau|_{k=k_0}(k-k_0) + O[(k-k_0)^2]\right\} \\ &\quad\times \left\{\frac{a_{-1}(l_0)}{k-k_0} + a_0(l_0) + a_1(l_0)\,(k-k_0) + O[(k-k_0)^2]\right\} \\ &= \frac{a_{-1}(l_0)}{k-k_0} + \left[a_0(l_0) + a_{-1}(l_0)\frac{d\tau}{dk} \ln \tau|_{k=k_0}\right] + O(k-k_0), \end{aligned} \tag{16}$$

and

$$F(k_0, l) = \frac{a'_{-1}(k_0)}{l-l_0} + \left[a'_0(k_0)+a'_{-1}(k_0)\frac{d\varrho}{dl}\ln\varrho(l)|_{l=l_0}\right] + O(l-l_0). \tag{16'}$$

Then we see that the contents of the brackets are not equal and in general, they can differ in a complicated way, because we are completely free to choose the functions $\tau(k)$ and $\varrho(l)$.

Let us compare now the analytical regularization with the regularization based on dispersion relations. We have

$$\sum(\alpha, p^2) = \sigma\int\frac{d^4k}{(m^2-k^2-i\varepsilon)(m^2-(p-k)^2-i\varepsilon)^\alpha}. \tag{17}$$

This integral can be calculated (Appendix C)

$$\sum(\alpha, p^2) = \sigma\pi^2 i\left[\frac{1}{\alpha-1}+1-\ln 4m^2+\varphi(p^2)+O(\alpha-1)\right]. \tag{18}$$

The result is correct (σ is imaginary see Appendix C). However, we can start with

$$\overline{\sum}(\alpha, p^2) = \sigma\int\frac{d^4k}{(k^2-m^2+i\varepsilon)((p-k)^2-m^2+i\varepsilon)^\alpha} = (-1)^{\alpha-1}\sum(\alpha, p^2) \tag{19}$$

instead of (17). For $\alpha = 1$ this change should be unessential, because $(-1)^2 = (-1)^0 = 1$. Now, if we continue the calculations we obtain a formula

$$\overline{\sum}(\alpha, p^2) = \frac{\sigma\pi^2 i}{\alpha-1}(-1)^{\alpha-1}\int_0^1 dz\frac{(1-z)}{[z(z-1)p^2+m^2-i\varepsilon]^{\alpha-1}} \tag{20}$$

instead of formula (3.8) (Appendix C) and the expansion with respect to α about $\alpha = 1$ gives

$$\overline{\sum}(\alpha, p^2) = \frac{\sigma\pi^2 i}{\alpha-1}[1+\ln(-1)(\alpha-1)+O((\alpha-1)^2)]$$
$$\times\{1+[1-\ln 4m^2+\varphi(p^2)](\alpha-1)+O((\alpha-1)^2)\}$$
$$= \sigma\pi^2 i\left[\frac{1}{\alpha-1}+\ln(-1)+1-\ln 4m^2+\varphi(p^2)+O(\alpha-1)\right]. \tag{21}$$

The behaviour of the imaginary part is correct, but the constant is wrong (it should be real). It is because $(-1)^{\alpha-1}$ was introduced in (19) which is a particular case of the freedom discussed at the beginning of this section (16), (16').

$$\overline{\sum}(\alpha, p^2) = (-1)^{\alpha-1}\sum(\alpha, p^2) = [1+\ln(-1)(\alpha-1)+O((\alpha-1)^2)]$$
$$\times\left[\frac{a_{-1}(p^2)}{\alpha-1}+a_0(p^2)+a_1(p^2)(\alpha-1)+O((\alpha-1)^2)\right]. \tag{22}$$

The product of the pole term in $\sum(\alpha, p^2)$ and the linear term of the expansion of $(-1)^{\alpha-1}$ gives an expression $\ln(-1)a_{-1}(p^2)$, which does not vanish with $\alpha \to 1$ and the behaviour of the whole imaginary part $\lim_{\alpha\to 1} \operatorname{Im} \sum (\alpha, p^2)$ is not correct. This is another example where the finite part is separated from the divergent one and we see once more that this step is nonuniquely defined[1].

The last example shows how much freedom exists in the analytical methods of regularization. In order to obtain the correct result one has to choose the method of calculation which leads to the result predicted by the Cutkosky lemma.

APPENDIX A

The calculation of the discontinuity of the amplitude (Based on Cutkosky lemma)

We consider the amplitude $\sum(p^2)$ connected with the self-energy diagram (Fig. 1). For this diagram the Cutkosky lemma gives

$$\operatorname{Im} \sum (p^2) = \int \delta_+(p^2-m^2)\delta_+[(p-k)^2-m^2]d^4k. \tag{A.1}$$

We pass to hyperbolic coordinates

$$\begin{aligned} k_0 &= k \cosh \vartheta, & k &\in [0, \infty), \\ k_1 &= k \sinh \vartheta \sin \theta \cos \varphi, & \cosh \vartheta &\in [1, \infty), \\ k_2 &= k \sinh \vartheta \sin \theta \sin \varphi, \\ k_3 &= k \sinh \vartheta \cos \theta, \end{aligned} \tag{A.2}$$

where $k = +\sqrt{k_0^2-\vec{k}^2}$.

The element of the momentum space is

$$d^4k = \tfrac{1}{2}\, k^2dk^2[\cosh^2 \vartheta-1]^{1/2}d \cosh \vartheta d\Omega. \tag{A.3}$$

In the new coordinates we have

$$\begin{aligned} \operatorname{Im} \sum (p^2) &= \tfrac{1}{2} \int \delta(k^2-m^2)\delta(p^2-2pk \cosh \vartheta+k^2-m^2) \\ &\quad \times k^2dk^2d \cosh \vartheta[\cosh^2 \vartheta-1]^{1/2}d\Omega \\ &= 2\pi \int_1^\infty \delta(p^2-2pm \cosh \vartheta)m^2d \cosh \vartheta[\cosh^2 \vartheta-1]^{1/2} \\ &= \left|\cosh \vartheta = \frac{t}{2pm}\right| = 2\pi m^2 \int_{1/2pm}^{\infty} \delta(p^2-t)\, \frac{dt}{2pm}\left[\frac{t}{4p^2m^2}-1\right]^{1/2} \\ &= \frac{\pi}{2}\left[\frac{p^2-4m^2}{p^2}\right]^{1/2} \theta(p^2-4m^2). \end{aligned} \tag{A.4}$$

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[1] Let us mention here, that this step is absent in the regularization with dispersion relations.

In the case, when the masses connected with propagators are different, we have

$$\operatorname{Im}\sum(p^2) = \frac{\pi}{2}\left[\frac{p^2-(m_1+m_2)^2}{p^2}\right]^{1/2}\theta(p^2-(m_1+m_2)^2). \tag{A.5}$$

The Cutkosky lemma gives the imaginary part of the amplitude exact to a multiplicative constant containing the combinatorial factor and the power of the coupling constant and (2π). For this reason the factor $\pi/2$ is omitted in (4).

The exact formula for $\operatorname{Im}\sum(p^2)$ is given by the exact calculation of the amplitude $\sum(p^2)$ ((2.1) and [9]). It is

$$\operatorname{Im}\sum(p^2) = \frac{e^2}{16}\left[\frac{p^2-4m^2}{p^2}\right]^{1/2}\theta(p^2-4m^2). \tag{A.6}$$

The relation between the Cutkosky lemma and the Kadyshevsky formalism was derived in [10].

APPENDIX B

Regularization of the amplitude $\sum(p^2)$ with the help of dispersion relations

Using the Kadyshevsky formalism [9] we have for the amplitude $\sum(p^2)$

$$\sum(p^2) = \gamma\int_{4m^2}^{\infty}\frac{dz}{z-p^2-i\varepsilon}\left[\frac{z-4m^2}{z}\right]^{1/2} \tag{B.1}$$

with

$$\gamma = \frac{e^2}{16\pi}.$$

The principal value of this integral is logarithmically divergent, while the discontinuity of the imaginary part (the residue) is in agreement with formulas (A.4) and (A.6) from Appendix A.

The formula (B.1) looks like an unproperly written dispersion relation (the subtraction is missing). Now we can formulate the problem as follows:

We look for the most general function $\sum(p^2)$, regular at infinity (with respect to p^2) and with analytical properties predicted by the Cutkosky lemma (and by the pole part of the integral (B.1)). Of course one subtraction is needed

$$\sum(p^2)-\sum(p_0^2) = \frac{1}{2\pi i}\left[\oint_C\frac{\sum(z)}{z-p^2}dz-\oint_C\frac{\sum(z)}{z-p^2}dz\right]$$

$$= \frac{p^2-p_0^2}{2\pi i}\oint_C\frac{\sum(z)dz}{(z-p^2)(z-p_0^2)} = \frac{p^2-p_0^2}{\pi}\int_{4m^2}^{\infty}\frac{\operatorname{Im}\sum(z)dz}{(z-p^2)(z-p_0^2)}$$

$$= \gamma(p^2-p_0^2)\int_{4m^2}^{\infty}\frac{dz}{(z-p^2)(z-p_0^2)}\left[\frac{z-4m^2}{z}\right]^{1/2}, \tag{B.2}$$

where p_0^2 is some point on the complex plane of p^2; the contour C is shown in the figure below.

The integral (B.2) is convergent. Passing to the new variables $\zeta = z/4m^2$ and then $(\zeta-1)^{1/2} = -\zeta^{1/2}+t^{1/2}$ with

$$\zeta = \frac{(t+1)^2}{4t}, \quad d\zeta = \frac{t^2-1}{4t^2}\,dt, \quad \left[\frac{\zeta-1}{\zeta}\right]^{1/2} = \frac{t-1}{t+1} \tag{B.3}$$

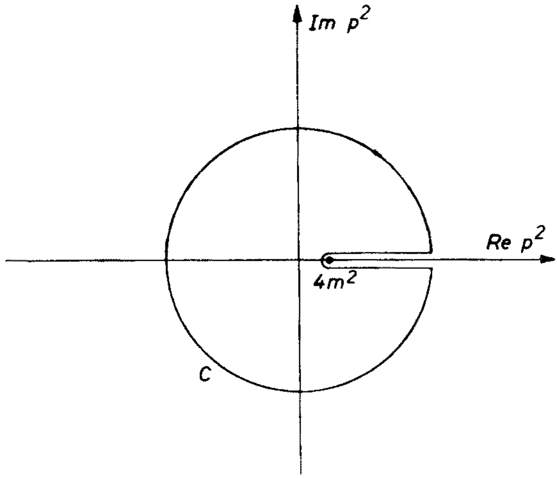

Fig. 2

we have

$$\sum(p^2)-\sum(p_0^2) = \gamma(P^2-P_0^2)\int_1^\infty \frac{dt(t-1)^2}{[(t+1)^2-tP^2]\,[(t+1)^2-tP_0^2]}, \tag{B.4}$$

where

$$P^2 = \frac{p^2}{m^2}, \quad P_0^2 = \frac{p_0^2}{m^2}.$$

This can be integrated by elementary methods

$$\sum(p^2)-\sum(p_0^2)$$

$$= \gamma\left[\frac{P^2-4}{P^2}\right]^{1/2}\int_1^\infty\left(\frac{1}{t-t_1}-\frac{1}{t-t_2}\right)dt-\gamma\left[\frac{P_0^2-4}{P_0^2}\right]^{1/2}\int_1^\infty\left(\frac{1}{t-t_3}-\frac{1}{t-t_4}\right)dt, \tag{B.5}$$

where

$$t_{1,2} = \frac{P^2-2\pm[P^2(P^2-4)]^{1/2}}{2}, \quad t_{3,4} = \frac{P_0^2-2\pm[P_0^2(P_0^2-4)]^{1/2}}{2}.$$

After integration we obtain

$$\sum(p^2)-\sum(p_0^2) = \gamma\left[\frac{p^2-4m^2}{p^2}\right]^{1/2} \ln\frac{4m^2-p^2+[p^2(p^2-4m^2)]^{1/2}}{4m^2-p^2-[p^2(p^2-4m^2)]^{1/2}}$$

$$-\gamma\left[\frac{p_0^2-4m^2}{p_0^2}\right]^{1/2} \ln\frac{4m^2-p_0^2+[p_0^2(p_0^2-4m^2)]^{1/2}}{4m^2-p_0^2-[p_0^2(p_0^2-4m^2)]^{1/2}}. \tag{B.6}$$

This is a difference equation

$$\sum(p^2)-\sum(p_0^2) = \varphi(p^2)-\varphi(p_0^2), \quad \forall p^2, \forall p_0^2, \tag{B.7}$$

which determines $\sum(p^2)$ up to one arbitrary real constant C

$$\sum(p^2) = \varphi(p^2)+C = \gamma\left[\frac{p^2-4m^2}{p^2}\right]^{1/2} \ln\frac{4m^2-p^2+[p^2(p^2-4m^2)]^{1/2}}{4m^2-p^2-[p^2(p^2-4m^2)]}+C. \tag{B.8}$$

It is easy to prove that the function $\varphi(p^2)+C$ has the correct behaviour of the imaginary part; for $0 < p^2 < 4m^2$ we have

$$\mathrm{Im}\sum(p^2) = \mathrm{Im}\left\{\gamma i\left[\frac{4m^2-p^2}{p^2}\right]^{1/2}\left[\ln\left|\frac{4m^2-p^2+i[p^2(4m^2-p^2)]^{1/2}}{4m^2-p^2-i[p^2(4m^2-p^2)]^{1/2}}\right|\right.\right.$$

$$\left.\left.+2i\,\mathrm{arctg}\left[\frac{p^2}{4m^2-p^2}\right]^{1/2}\right]\right\} = 0 \tag{B.9}$$

because the absolute value of the expression in the logarithm is 1. For $p^2 > 4m^2$

$$\mathrm{Im}\sum(p^2) = \mathrm{Im}\left\{\gamma\left[\frac{p^2-4m^2}{p^2}\right]^{1/2}\left[\ln\left|\frac{p^2-4m^2-[p^2(p^2-4m^2)]^{1/2}}{p^2-4m^2+[p^2(p^2-4m^2)]^{1/2}}\right|+i\pi\right]\right\}$$

$$= \frac{e^2}{16}\left[\frac{p^2-4m^2}{p^2}\right]^{1/2}. \tag{B.10}$$

In the case $p^2 < 0$ the arguments of the square root before the logarithm and of the logarithm (Eq. B.8) are both positive, so $\mathrm{Im}\sum(p^2) = 0$.

APPENDIX C

The analytical regularization of the self-energy diagram

We have before regularization

$$\sum(p^2) = \frac{-ie^2}{2(2\pi)^3}\int\frac{d^4k}{(m^2-k^2+i\varepsilon)(m^2-(p-k)^2+i\varepsilon)}. \tag{C.1}$$

Now a parameter α is introduced

$$\sum(\alpha, p^2) = \sigma \int \frac{d^4k}{(m^2-k^2+i\varepsilon)\,(m^2-(p-k)^2+i\varepsilon)^\alpha}\,. \tag{C.2}$$

In what follows

$$\sigma = \frac{-ie^2}{2(2\pi)^3}\,.$$

Using the Feynman formula

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dz\, \frac{z^{\alpha-1}(1-z)^{\beta-1}}{[az+b(1-z)]^{\alpha+\beta}}\,, \tag{C.3}$$

we can write

$$\sum(\alpha, p^2) = \sigma \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} \int d^4k \int_0^1 dz\, \frac{(z-1)^{\alpha-1}}{[(k^2-m^2)z+(1-z)\,((p-k)^2-m^2+i\varepsilon)]^{\alpha+1}}\,. \tag{C.4}$$

A substitution $k+p(1-z) = k'$ transforms this integral to the form

$$\sum(\alpha, p^2) = \sigma \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^1 dz(z-1)^{\alpha-1} \int \frac{d^4k}{(k^2-C)^{\alpha+1}}\,, \tag{C.5}$$

where

$$C = z(z-1)p^2+m^2-i\varepsilon.$$

We can rotate the contour of integration over k^0 about 90° (in order to integrate over the imaginary axis) and then we pass to four-dimensional spherical coordinates with Jacobian $2\pi^2k^3$ [11]:

$$\sum(\alpha, p^2) = \sigma 2\pi^3 i \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^1 dz(1-z)^{\alpha-1} \int_0^\infty \frac{k^3dk}{(k^2+C)^{\alpha+1}}$$

$$= \sigma\pi^2 i \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^1 dz(1-z)^{\alpha-1} \int_0^\infty \frac{xdx}{(x+C)^{\alpha+1}}\,. \tag{C.6}$$

Making use of the standard integral

$$\int_0^\infty \frac{x^{p-1}}{(x+q^{-1})^{p+r}}\,dx = q^r \frac{\Gamma(p)\Gamma(r)}{\Gamma(p+r)}\,, \tag{C.7}$$

we obtain

$$\sum(\alpha, p^2) = \frac{\sigma\pi^2 i}{\alpha-1}\int\limits_0^1 dz \frac{(1-z)^{\alpha-1}}{(z(z-1)p^2+m^2-i\varepsilon)^{\alpha-1}}\,. \tag{C.8}$$

This formula is regular for $\alpha \neq 1$ and has a pole of first order for $\alpha = 1$.

$$\sum(\alpha, p^2) = \frac{a_{-1}(p^2)}{\alpha-1} + a_0(p^2) + a_1(p^2)(\alpha-1) + \ldots\,. \tag{C.9}$$

Here $a_{-1}(p^2)$ is a constant equal to $\sigma\pi^2 i$, which can be easily calculated as a residue for $\alpha = 1$.

We accept the regular part of formula (C.8) with $\alpha = 1$ for the regularized amplitude

$$\sum_{\text{reg}}(p^2) = a_0(p^2) = \frac{\partial}{\partial\alpha}\left[(\alpha-1)\sum(\alpha, p^2)\right]\Bigg|_{\alpha=1}$$

$$= \sigma\pi^2 i \int\limits_0^1 dz \ln \frac{1-z}{z(z-1)p^2+m^2-i\varepsilon}\,. \tag{C.10}$$

The integration is elementary. Finally

$$\sum_{\text{reg}}(p^2) = \frac{e^2}{2^4\pi}\,\varphi(p^2) + C, \tag{C.11}$$

where

$$\varphi(p^2) = \left[\frac{p^2-4m^2}{p^2}\right]^{1/2} \ln \frac{4m^2-p^2+[p^2(p^2-4m^2)]^{1/2}}{4m^2-p^2-[p^2(p^2-4m^2)]^{1/2}}\,,$$

$$C = C^* = \frac{e^2}{2^4\pi}(1-\ln 4m^2). \tag{C.12}$$

One could have some reservations about the dimension of the argument $4m^2$ of the logarithm in (C.12). This is because we have spoiled the dimensions introducing α in (C.2). An additional term $(\mu^2)^{\alpha-1}$) in the numerator in (C.2) would restore the proper dimensions (μ is a constant with the dimension of mass). In this case we would have μ^2 in the numerator of (C.10) and the constant C would take a form:

$$C = C^* = \frac{e^2}{2^4\pi}\left(1-\ln\frac{4m^2}{\mu^2}\right). \tag{C.12'}$$

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