IRREDUCIBLE MASS, UNINCREASABLE ANGULAR MOMENTUM AND ISOAREAL TRANSFORMATIONS FOR BLACK HOLE PHYSICS

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The concept of unincreasable angular momentum for a Kerr black hole is introduced and related to the isoareal transformations of the horizons. A thermodynamical interpretation is proposed for the new parameter.

In a pioneering paper Christodoulou [1] introduced the concept of reversible and irreversible transformations for Kerr black holes, by considering accretion processes.

The energy E of a particle which falls from infinity onto the black hole attains, as a function of the radial coordinate r, its minimum value:

$$E_{\min}^{(+)} = \frac{ap_\phi}{r_+^2+a^2} \tag{1}$$

on the outer horizon $r = r_+ \equiv m+(m^2-a^2)^{\frac{1}{2}}$, where m, a are the parameters of the black hole, namely mass and angular momentum per unit mass, and p_ϕ is azimuthal momentum of the particle. By integrating the infinitesimal form of (1), Christodoulou was led to the formula:

$$m^2 = m_{\rm ir}^2+\frac{L^2}{4m_{\rm ir}^2} \quad \text{with } m^2 \leqslant 2m_{\rm ir}^2, \tag{2}$$

where $L = am$ and $m_{\rm ir}$ is the mass that is left when all the rotational energy of the black hole is reversibly extracted. The quantity $m_{\rm ir}$ is called the *irreducible mass* since no classical transformation can ever decrease it. In fact, from $E \geqslant E_{\min}^{(+)}$ it follows that

$$\delta m_{\rm ir} \geqslant 0, \tag{3}$$

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the equality holding only for reversible transformations. Relation (3) is in one-to-one correspondence with the "2nd law of black hole mechanics" [2]:

$$\delta A_+ \geqslant 0, \tag{4}$$

A_+ being the proper surface area of the outer horizon, since $A_+ = 16\pi m_{ir}^2$.

Relation (1) was later extended to charged black holes [3].

The concept of reversible transformation has been recently generalized by Curir and Francaviglia [4, 5] who modified relation (1) making it valid without restrictions on m. They introduced so-called *isoareal transformations* i.e. those transformations which keep invariant either the area A_+ or the area A_- of the inner horizon $r = r_- \equiv m^2 - (m^2 - a)^{\frac{1}{2}}$. In fact, the infinitesimal form of (1) is equivalent to the condition $dA_+ = 0$.

From this point on we shall assume that the black hole is described by a Kerr metric in the outside as well as the inside of the outer horizon. Consider now the equation $dA_- = 0$. This can be shown to be equivalent to the infinitesimal form of

$$E_{\min}^{(-)} = \frac{a p_\phi}{r_-^2 + a^2}, \tag{5}$$

which gives the minimum value of E attained on the inner horizon. Both relations $dA_+ = 0$ and $dA_- = 0$ are summarized by the formula:

$$m^2 = \beta + \frac{L^2}{4\beta}, \tag{6}$$

which therefore governs all isoareal transformations, the constant β coinciding with $A_+/16\pi$ (resp. $A_-/16\pi$) when $m^2 \leqslant 2\beta$ (resp. $m^2 \geqslant 2\beta$), see [4].

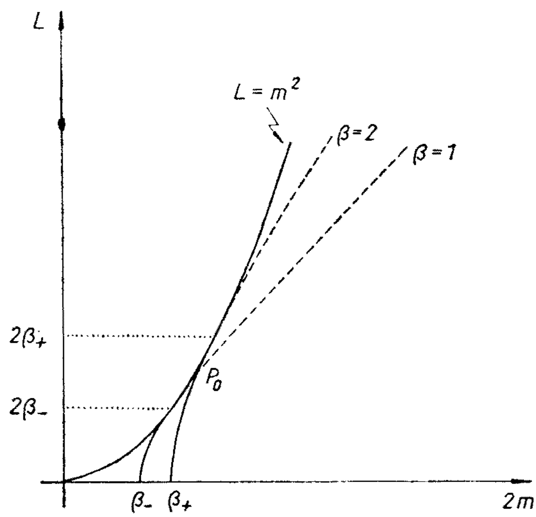

Fig. 1. The continuous (resp. dashed) part of each hyperbola represents transformations in which the area A_+ (resp. A_-) is constant. The inequalities (3) and (7) mean that only transformations which leave the black hole under the corresponding hyperbola are allowed

The integral curves (6) in the plane (m, L) form a family of hyperbolas enveloped by the parabola $L = m^2$ which separates black holes from naked singularities.

Each black hole $P_0 = (m_0, L_0)$ lies on two such hyperbolas, thus determining (see Fig. 1) two physical parameters β_+ and β_-, which, accordingly to the previous results, are respectively $A_+(P_0)/16\pi$ and $A_-(P_0)/16\pi$. It is clear that β_+ is nothing but the irreducible mass of the black hole, as introduced by Christodoulou. In this paper we investigate and clarify the physical meaning of β_-.

From the fact that $E^{(-)}_{\min}$ is a minimum of the energy E and $\beta_- = \frac{1}{2}mr_-$ it follows that:

$$\delta\beta_- = -(r_-^2+a^2)^{-\frac{1}{2}}((r_-^2+a^2)\delta m - a\delta L) \leqslant 0, \tag{7}$$

which is equivalent to $\delta A_- \leqslant 0$. Since $\frac{1}{2}\beta_-$ coincides with the angular momentum of the extreme black hole which can be reached from P_0 by means of isoareal transformations of the inner horizon, we shall call it the *unincreasable angular momentum* and we shall denote it by L_{un}. Moreover, formula (6) can be alternatively written as follows:

$$\frac{L^2}{4} = \tfrac{1}{2}L_{\text{un}}(m^2 - \tfrac{1}{2}L_{\text{un}}) \quad \text{with } L \geqslant L_{\text{un}}, \tag{8}$$

which holds for transformations on the inner horizon and it can be interpreted as a complementary relation to (6), which describes transformations on the outer horizon.

Since we believe that no classical accretion process exists capable of keeping constant the area A_+ we think that the internal isoareal transformations have relevance only with respect to quantum processes. If a black hole undergoes an internal isoareal transformation governed by (8), its irreducible mass could decrease. However, in our opinion, this is not in contradiction either with (3) or with Hawking's theorem, since they explicitly refer to a different kind of transformations and to the exterior of the black hole, completely disregarding the physical phenomena inside.

It is noteworthy that both (2) and (8) can be deduced from the 1st law of black hole dynamics:

$$dm = \tau dA + \Omega dL, \tag{9}$$

where Ω is the angular velocity of the horizon and τ its surface gravity. In fact, (9) can be written in terms of β_+ and β_- as follows:

$$dm = \frac{\hbar}{kc}\,\frac{\beta_+ - \beta_-}{4m\beta_\pm}\,d(16\pi\beta_\pm) + \frac{\sqrt{\beta_+\beta_-}}{m\beta_\pm}\,d\sqrt{(\beta_+\beta_-)}. \tag{10}$$

Keeping either β_+ or β_- constant in (10) and imposing suitable boundary conditions for the appropriate isoareal transformations, we recover (2) or (8).

Moreover, the condition $\delta(\beta_+ - \beta_-) \geqslant 0$ directly implies $\delta(m^4) \geqslant \delta(L^2)$ so that the horizons are preserved.

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