ISOAREAL TRANSFORMATIONS OF THE KERR-NEWMAN BLACK HOLES

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(Received August 8, 1977)

We investigate a natural extension of the fundamental formula for black hole dynamics found by Christodoulou and Ruffini (1971) for Kerr-Newman black holes, by generalizing it to any value of the mass. New transformations, having irreversible character, are introduced and studied in detail. They are shown to be intimately related to the horizons of the black hole, since during any evolution governed by the improved equations either the area of the event horizon or the area of the antievent horizon are kept constant. We propose a new, more symmetrical form of the fundamental formula and we introduce two new parameters characterizing the black holes which are naturally associated with the set of new transformations. The possible physical implications are also discussed.

1. Introduction and formulation of the problem

Studying dynamical transformations of charged, rotating black-holes, Christodoulou and Ruffini (1971) deduced the well known relation

$$m^2 = (m_{\rm ir}+Q^2/4m_{\rm ir}^2)^2+L^2/4m_{\rm ir}^2, \tag{1}$$

where m, Q, and L are respectively the mass, charge and angular momentum of the black hole, and $m_{\rm ir}$ its irreducible mass. They restricted the validity of (1) to the range in which the following inequality is satisfied

$$L^2/4m_{\rm ir}^2+Q^4/16m_{\rm ir}^4 \leqslant 1. \tag{2}$$

In this range of parameters the black hole always undergoes reversible transformations.

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In a preceding paper (1977) we investigated, for the uncharged case, a generalization of (1) which is intimately related to a generalization of the concept of reversible transformation, too. By "isoareal transformations" we mean transformations which do not change the areas either of the inner or of the outer horizon. Using this notion, we shall here extend the previous result to the case of charged black holes. Furthermore, we shall suggest an interpretation of the generalized formula (1) in terms of new parameters related to the horizons, which in turn admit the physical meaning of "extreme energies" with respect to suitable processes.

We recall that the Kerr–Newman solution

$$ds^2 = \varrho^2 \Delta^{-1} dr^2 + \varrho^2 d\theta^2 + \varrho^{-2} \sin^2\theta (a dt - (r^2+a^2) d\phi)^2$$
$$- \varrho^{-2} \Delta (dt - a \sin^2\theta d\phi)^2, \tag{3}$$

where $\varrho^2 = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 - 2mr + a^2 + Q^2$ describes a rotating (if $a \neq 0$) and charged (if $Q \neq 0$) black hole.

When $a^2 + Q^2 \leqslant m^2$, the space-time described by (3) is endowed with an event horizon at $r = r_+ = m + (m - a^2 - Q^2)^{\frac{1}{2}}$ and an antievent horizon at $r = r_- = m - (m - a^2 - Q^2)^{\frac{1}{2}}$. Both $r = r_+$ and $r = r_-$ are null surfaces and the two horizons coalesce when $a^2 + Q^2 = m^2$.

The Kerr–Newman solution is also, likewise the Kerr solution, endowed with a singularity (at $r = 0$) in the equatorial plane. Consider a particle of azimuthal momentum p_ϕ, rest mass μ and charge ε at radial distance r. Its energy E, as measured from infinity, is given by the positive solution of

$$E^2(r^4 + a^2(r^2 + 2mr - Q^2)) - 2E((2mr - Q^2) a p_\phi + \varepsilon Q r (r^2 + a^2))$$
$$-(r^2 - 2mr + Q^2) p_\phi^2 + 2\varepsilon Q a r p_\phi + \varepsilon^2 Q^2 r^2 - (r^2 - 2mr + a^2 + Q^2)(\mu r^2 + X)$$
$$= ((r^2 - 2mr + a^2 + Q^2) p_r)^2, \tag{4}$$

where X is the so-called "Carter's constant of the motion" (Carter 1973). The discriminant D of equation (4) vanishes at $r = r_+$ and $r = r_-$; hence, the energy of a particle on the horizons is given by

$$E = (a p_\phi + r_\pm Q \varepsilon)/(r_\pm^2 + a^2). \tag{5}$$

Going to the infinitesimal limit of (5) (with $r = r_+$)[1] Christodoulou got the following partial differential equation:

$$dm(L, Q) = (L/m dL + r_+ Q dQ)(r_+^2 + L^2/m^2)^{-1}, \tag{6}$$

whose solution squared leads to relation (1), provided condition (2) is satisfied.

The constant of integration m_{ir} is the mass of the unique Schwarzschild black hole which is left when all the rotational and Coulomb energy of the Kerr-Newman black hole is taken away by reversible transformations: no classical process exist capable of decreasing it. m_{ir}, therefore, has been named "irreducible mass" of the black hole.

[1] Namely by taking $E = dm, p_\phi = dL, \varepsilon = dQ$.

It is useful to introduce exact 1-forms

$$f^{-1}((r_\pm Q)dQ+L/m dL-(r_\pm^2+a^2)dm) = \omega_\pm, \tag{7}$$

where $f = (m^2-a^2-Q^2)^{\frac{1}{2}}$. Integration of (7) gives $\omega_\pm = dA_\pm$, where

$$A_\pm = mr_\pm - Q^2/2 \tag{8}$$

are the areas of the event and antievent horizon respectively. From (6), (7) and (8), it is easy to see that, under the condition (2), equation (1) represents, in (m, L, Q) space, the surfaces $A_+ =$ constant, while (1), with the limitation

$$L^2/4m_{ir}^2+Q^4/16m_{ir}^4 \geqslant 1 \tag{2'}$$

represents the surface $A_- =$ constant.

2. *Isoareal transformations*

In the space (m, L, Q), the physical region $\mathfrak{F}$, i.e. the region containing true black holes, is separated from the region containing naked singularities by the surface

$$m^4 = L^2+Q^2m^2, \tag{9}$$

which is a conoidal surface having vertex at the origin of the axes.

From analogy with our previous work in the uncharged case, we rewrite (1) with m_{ir} replaced by an arbitrary parameter $\beta \in R^+$

$$m^2 = (\beta^2+Q^2/4\beta^2)^2+L^2/4\beta^2. \tag{10}$$

For each β, in the part of the corresponding surface (10) in which $m^2-Q^2/2$ is smaller than $2\beta^2$, the area $A_\pm$ of the event horizon is constant; it is precisely given by

$$A_+ = 16\pi\beta^2 = 8\pi(m(m+(m^2-a^2-Q^2)^{\frac{1}{2}})-Q^2/2) \tag{11}$$

and β coincides with m_{ir}.

Conversely, if $m^2-Q^2/2$ is greater than $2\beta^2$, the area A_- of the antievent horizon is constant, being given by

$$A_- = 4L^2/\beta^2+Q^4/\beta^2 = 8\pi(m(m-(m^2-a^2-Q^2)^{\frac{1}{2}})-Q^2/2). \tag{12}$$

Hence, the surfaces (10) shall be called "isoareal surfaces". Furthermore, the transformations represented by continuous curves lying in one of the isoareal surface (10) shall be called "isoareal transformations".

If we assign to the parameter β the values

$$\beta = \beta_+ = (A_+/16\pi)^{\frac{1}{2}} \quad \text{if} \quad L^2/4m_{ir}^2+Q^4/16m_{ir}^4 \geqslant 1, \tag{13}$$

$$\beta = \beta_- = (A_-/16\pi)^{\frac{1}{2}} \quad \text{if} \quad L^2/4m_{ir}^2+Q^4/16m_{ir}^4 \leqslant 1, \tag{14}$$

we recognize that (10) holds for every (m, L, Q) and for every fixed β. Also the identity $A = 16\pi\beta^2$ is always true on the surfaces (10), provided A is either equal to A_+ or A_-, accordingly to the validity of (2) or (2′). Furthermore, the condition $m^2 - Q^2/2 \leqslant 2\beta$ is always satisfied. For each β, the isoareal surface (10) and the conoid (9) are tangent to each other along the oblique curve

$$m^2 = L^2/m^2 + Q^2, \quad m^2 = L^2/4\beta^2 + (\beta + Q^2/4\beta)^2 \tag{15}$$

whose points represent extreme Kerr-Newman black holes having β as irreducible mass.

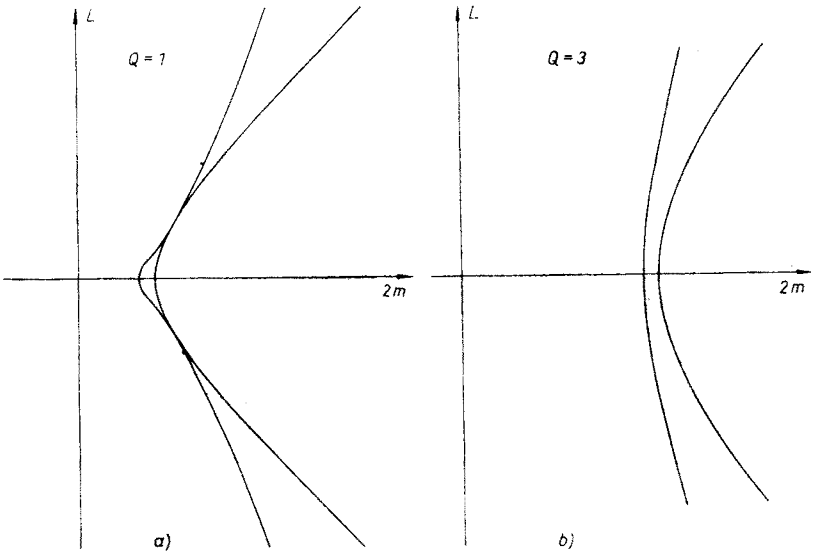

Fig. 1a, b. Isoareal transformations without exchange of charge ($\beta = 1$), $Q \neq 0$

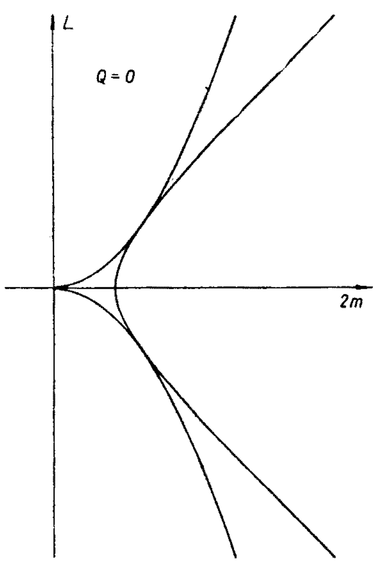

Fig. 2. The particular case $Q = 0$ (Kerr solution) ($\beta = 1$)

The curve (15) is closed and divides the corresponding isoareal surface into two parts: a compact one, which contains a unique Schwarzschild black hole and any isoareal transformation with $A_+ =$ constant lies on it, and an open part which, on the contrary,

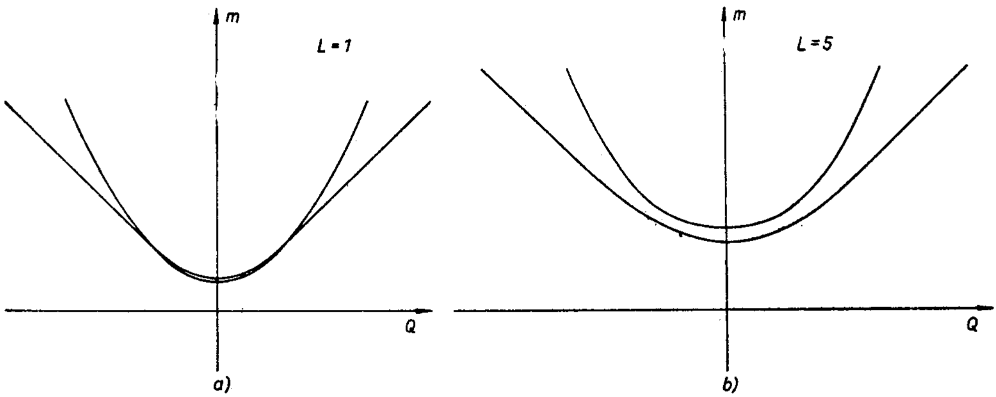

Fig. 3 a, b. Isoareal transformations without exchange of angular momentum ($\beta = 1$), $L \neq 0$

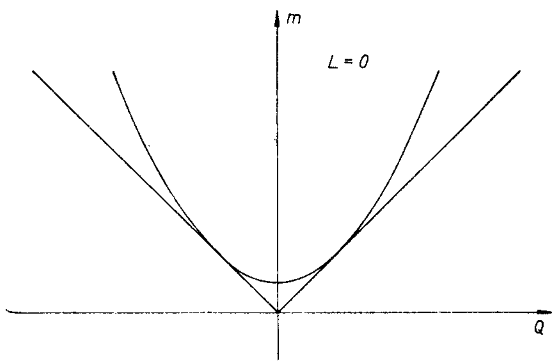

Fig. 4. The particular case $L = 0$ (Reissner-Nørdstrom solution) ($\beta = 1$)

does not contain any Schwarzschild black hole and contains transformations with $A_- =$ constant (see Fig. 5).

Now it shall be interesting to consider the behaviour of the aforementioned transformations in the particular cases: $L =$ constant or $Q =$ constant. Easy computations show that they are summarized by Figs 1, 2, 3 and 4.

Let us now consider a point $P_0(m_0, L_0, Q_0)$ which represents a black hole whose horizons, by (11) and (12), have areas: $A_{0+} = 16\pi\beta_{0+}^2$, $A_{0-} = 16\pi\beta_{0-}^2$.

Through P_0 we draw the two isoareal surfaces Σ_{0+} and Σ_{0-} which correspond to the constants β_{0+} and β_{0-}, and denote by $\gamma_{0\pm}$ the corresponding curves (15).

It is easily shown that the closed curves γ_{0+} and γ_{0-} do not intersect, while Σ_{0+} and Σ_{0-} intersect along a closed curve Γ_0 passing through P_0 (see Fig. 5). The curve Γ_0 lies entirely in the open part of Σ_{0-} and in the compact part of Σ_{0+}.

Hence, a generic Kerr-Newman black hole can be considered as the final stage of two sets of isoareal transformations of the same kind: namely, either it could be obtained from an extreme Kerr-Newman black hole by irreversible evolution keeping the area of the antievent horizon constant, or else from a Schwarzschild black hole by reversible

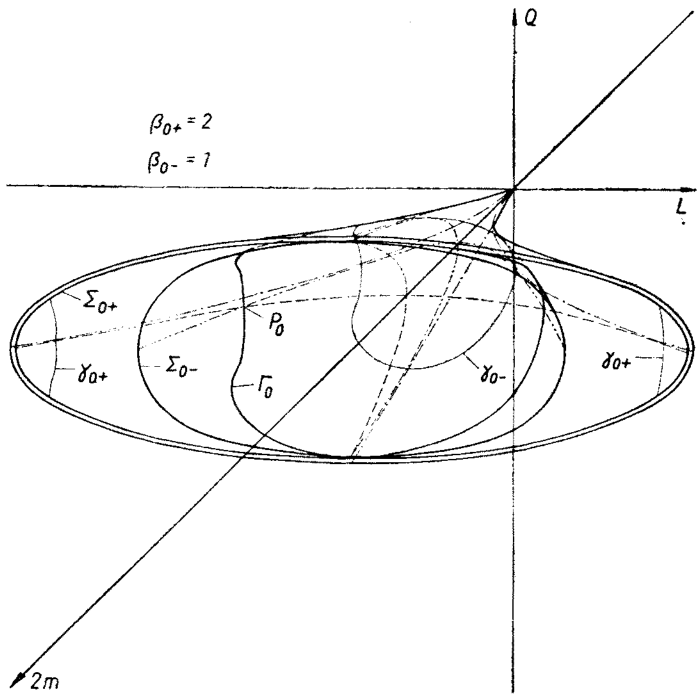

Fig. 5. The conoidal surface $m^4 = L^2 + Q^2 m^2$. Two isoareal surfaces are also indicated

evolution keeping constant the area of the event horizon[2]. Referring again to Fig. 5, we notice that the existence of a whole curve $\Gamma_0 = \Sigma_{0+} \cap \Sigma_{0-}$ reflects the existence of ∞^1 Kerr-Newman black holes having fixed areas A_+ and A_-.

3. *Extreme energies*

Let us now consider the following form of (10):

$$m^2 = \beta^2 + Q^2/2 + Q^4/16\beta^2 + L^2/4\beta^2. \tag{16}$$

From (11) and (12) it is easy to see that the following identity holds:

$$\beta_+^2 \beta_-^2 = L^2/4 + Q^2/16. \tag{17}$$

[2] This corresponds, in Fig. 5, to consider P_0 as the evolution of S_0 (reversible case) or as the evolution of anyone of the black holes lying on γ_{0+} (irreversible case).

Hence, we can write (10) in the following symmetrical form:

$$m^2 = \beta^2 + \beta_+^2 \beta_-^2 / \beta^2 + Q^2/2, \tag{18}$$

where β assumes either the value β_+ if A_+ is conserved or β_- if A_- is conserved, while (17) leads also to

$$m^2 = \beta_+^2 + \beta_-^2 + Q^2/2 = A_+/16\pi + A_-/16\pi + Q^2/2. \tag{19}$$

The physical meaning of the formula above is clear: the square of the energy of a black hole $P_0(m_0, Q_0, L_0)$ can be expressed as the sum of three contributions:
1) its irreducible mass squared β_+^2 (equivalently, apart numerical factors, the area of the event horizon), which corresponds to the mass of the unique Scharzschild black hole lying on the surface $\beta_+ =$ constant);
2) the "extreme energy" squared β_-^2 (equivalently, the area of the inner horizon, which is related to angular momentum and charge). This extreme energy is the irreducible mass of the extreme Kerr-Newman black holes lying on the curve γ_{0-} [3];
3) the purely electric energy $Q^2/2$.

Let us finally turn to derive an interesting equivalent form of (16). From the identity (17), one can easily recognize that:

$\beta_+^2 \beta_-^2 = Q^4/16$ in a Reissner-Nørdstrom geometry;

$\beta_+^2 \beta_-^0 = L^2/4$ in a Kerr geometry.

Hence, we can rewrite (16) as follows:

$$m^2 = \beta^2 + (\beta_+^2 \beta_-^2)_{\text{R.N.}}/\beta^2 + (\beta_+^2 \beta_-^2)_{\text{Kerr}}/\beta^2 + Q^2/2, \tag{20}$$

or equivalently

$$m^2 = m_{\text{ir}}^2 + E_{\text{re}}^2 + E_{\text{R.N. ext}}^2 + Q^2/2, \tag{21}$$

where we have taken $m_{\text{ir}} = \beta_+$, E_{re} being given by $L/2$ and $E_{\text{R.N.ext}}$ denoting $Q^4/16\, m_{\text{ir}}^2$.

The term $R_{\text{R.N.ext}}$ we shall call "extreme Reissner-Nørdstrom energy" since it coincides with the square of the irreducible mass of the unique extreme black hole lying on γ_{0+} and having zero angular momentum.

4. *Conclusions*

We have thus recognized that relation (1), which is the "fundamental formula of black holes dynamics", can be extended to be valid also when condition (2) is dropped. In this new range, the transformations governed by the generalized relation (10) are no longer reversible, since they do not keep constant the area of the event horizon. Hence, as we already pointed out in a previous paper, such transformations are physically observable, although they happen inside the infinite red-shift surface, since they involve radical

[3] When $Q = 0$, this "extreme energy" reduces to the concept already introduced in our first paper, as "extreme rotational energy" of an (uncharged) Kerr black hole, and there denoted by E_{re}.

changes in the configuration of the black hole itself. We can easily observe from (17) that in any isoareal transformation with $A_- =$ constant the area A_+ must necessarily increase.

It is known (Carter 1973) that the 1-form ω_+ given by (7) has a thermodynamical interpretation, where the area A_+ has the meaning of "entropy" of the black hole. It should be interesting to find the thermodynamical meaning of the area A_-. This is actually under investigation.

The results that we discussed above make clear the role of the conoidal surface (9) as a surface of "cosmic censorship". In fact it separates the physical region $\mathfrak{F}$ from the region of naked singularities; when a black hole lies on the aforementioned surface, it cannot undergo reversible transformations with increase of mass. This physically corresponds to the fact that, according to Hawking's theorem, along a reversible transformation the area A_+ (i.e. the entropy) has to be constant, while the area A_- has to change quadratically with the mass. Since for an extreme black hole lying on (9) the two horizons coalesce, a "reversible" increase of mass should correspond to an increase of A_- against $A_+ =$ constant, which in turn should imply the disappearance of both horizons. Hence, the cosmic censorship is reflected in the non-reversibility of each transformation governed by the law (10) after the black hole has reached an extreme configuration.

Editorial note. This article was proofread by the editors only, not by the authors.

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