AN EXACT PERTURBATION SOLUTION TO THE EQUATION $\varepsilon XY \frac{dY}{dX} = X - Y(Y+1)$

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We obtain an exact perturbation solution to a certain nonlinear first-order differential equation which arises in the study of stellar structure.

In this letter we obtain an exact perturbation solution to the following nonlinear first-order differential equation

$$\varepsilon XY \frac{dY}{dX} = X - Y(Y+1), \tag{1}$$

where ε is a small positive parameter, i.e., $0 < \varepsilon \ll 0$. This equation arises in the study of stellar structure (Chandrasekhar 1958). A similar equation was considered by Jeans (1925, 1929).

To solve equation (1), we assume an asymptotic expansion, in the small parameter ε, of the form (Nayfeh 1973),

$$Y(X, \varepsilon) = \sum_{n}^{\infty} \varepsilon^n Y_n(X). \tag{2}$$

In addition, we assume that the functions $Y_n(X)$ are continuous and have a first derivative on the interval $0 \leqslant X < \infty$. The function $Y_0(X)$ is called the generating function and is gotten by setting $\varepsilon = 0$ in equation (1),

$$X - Y_0(Y_0+1) = 0. \tag{3}$$

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We choose the following solution of equation (3) as our generating function,

$$Y_0(X) = \sqrt{X+\tfrac{1}{4}} - \tfrac{1}{2}. \tag{4}$$

Our purpose will be to determine the unknown functions $Y_n(X)$. We will show that this can be done iteratively, i.e., the function $Y_k(X)$ is determined once we know $Y_0(X), Y_1(X), \ldots, Y_{k-1}(X)$. Consequently, the function $Y_k(X)$ may be expressed, functionally, in terms of the generating function $Y_0(X)$. Only the functions $Y_0(X)$ and $Y_1(X)$ are determined in the book of Chandrasekhar (1958).

First, we must determine the boundary condition satisfied by equation (1) with assumed solution, equation (2). Note that equation (1) is a singular first-order differential equation (Bellman 1964). This means that when $\varepsilon = 0$, we have an algebraic equation rather than a first-order differential equation. A consideration of equations (1) and (4), and the assumptions on the functions $Y_n(X)$ easily show that

$$Y(0, \varepsilon) = 0. \tag{5}$$

Equation (5) gives the following result,

$$Y_n(0) = 0, \quad n \geqslant 0. \tag{6}$$

Let us now determine the function $Y_n(X)$. This may be done by substituting equation (2) into equation (1) and setting the coefficients of the various powers of ε equal to zero. After some straightforward, but lengthly algebra, we obtain

$$X \sum_{k=0}^{n-1} Y_k \frac{dY_{n-k-1}}{dX} + Y_n + \sum_{k=0}^{n} Y_k Y_{n-k} = 0. \tag{7}$$

Solving for $Y_n(X)$, gives

$$\frac{Y_n(X)}{Y_0(X)} = -(1+2Y_0)^{-1}\left\{X\frac{dY_{n-1}}{dX} + \sum_{k=1}^{n-1}\left(\frac{Y_k}{Y_0}\right)\left(X\frac{dY_{n-k-1}}{dX} + Y_{n-k}\right)\right\}, \tag{8}$$

where $Y_0(X)$ is given by equation (4).

Note that $Y_1(X)$ is given in terms of $Y_0(X)$. Similarly, $Y_2(X)$ is determined by $Y_0(X)$ and $Y_1(X)$. In general, $Y_k(X)$ is determined by $Y_0(X), Y_1(X), \ldots, Y_{k-1}(X)$; consequently, $Y_k(X)$ is uniquely specified by the generating function $Y_0(X)$. Thus, we conclude that the assumed asymptotic expansion, equation (2), is a solution of equation (1) if the functions $Y_n(X)$ are given by equation (8). Since all the functions $Y_n(X)$ are proportional to $Y_0(X)$, the conditions given by equation (6) are satisfied.

Finally, the series, given by equation (2), is a uniformly valid asymptotic expansion (Nayfeh 1973). This follows from the fact that $Y_n(X)/Y_0(X)$ is bounded for all $X \geqslant 0$. This latter result may be easily obtained by induction from equation (8).

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REFERENCES

Bellman, R., *Perturbation Techniques in Mathematics, Physics, and Engineering*, Holt, New York 1964.
Chandrasekhar, S., *Stellar Structure*, Dover, New York 1958, p. 295.
Jeans, J. H., *Monthly Notices of the Royal Astronomical Society* **85**, 196 (1925).
Jeans, J. H., *Astronomy and Cosmogony*, London–Cambridge 1929, p. 85.
Nayfeh, A. H., *Perturbation Methods*, Wiley, New York 1973, p. 9.