TETRAD PALATINI FORMALISM AND QUADRATIC LAGRANGIANS IN THE GRAVITATIONAL FIELD THEORY

By B. N. Frolov

The Moscow Institute of Graphic Arts*

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The gauge theory of gravitation based on a quadratic Lagrangian in the tetrad Palatini formalism is developed. In the general case the theory contains torsion. The field equations without torsion generalize the field equations in Yang's theory of gravitation. It is shown that the solutions of the exterior Schwarzschild problem and of the Friedman problem coincide completely with solutions of these problems in the Einstein theory.

1. Introduction. The simple quadratic theory

The general gauge field theory (GGFT) has been developed in [6–12] and the gauge (compensating) approach to the gravitational field, earlier developed in [1–3], has been obtained on this basis. The Lagrangian in the GGFT is quadratic in the gauge field tensor, because it contains the electromagnetic field theory as a particular case. In this paper we consider a quadratic Lagrangian for the gravitational field, which corresponds to the Poincaré group in the GGFT:

$$\mathscr{L}_1 = \tfrac{1}{4}\, hR_{abij}R^{abij}, \qquad h = \det \|h^a_\mu\|; \tag{1}$$

here R_{abij} is the Riemann-Christoffel curvature tensor (in general containing torsion), h^a_μ is a tetrad potential of the gravitational field (Lame's matrix). According to the GGFT one must vary Lagrangian (1) by the method generalising the Palatini method, i. e. with respect to the tetrad potential h^a_μ and gauge field potential A^m_a idependently [6–11]. A^m_a become Ricci rotation coefficients γ^{ij}_a in the gravitational case (in general they also contain torsion). The usual Palatini variational formalism for the Lagrangian (1) (without torsion and without the external sources) was developed in [18].

The corresponding field equations have been derived in [7] and in [20]:

$$R^{\nu}{}_{\mu\alpha\beta;\nu} - (Q_{\sigma\varrho\mu} + 2Q_{\nu[\sigma}{}^{\nu}g_{\varrho]\mu})R^{\sigma\varrho}{}_{\alpha\beta} = -\frac{\kappa}{\lambda_1} S^{(\mathrm{ext})}_{\mu\alpha\beta}, \tag{2a}$$

$$R_{\mu\sigma\alpha\beta}R_{\nu}{}^{\sigma\alpha\beta} - \tfrac{1}{4}\, g_{\mu\nu}R_{\sigma\varrho\alpha\beta}R^{\sigma\varrho\alpha\beta} = -\frac{\kappa}{\lambda_1} t^{(\mathrm{ext})}_{\mu\nu}. \tag{2b}$$

* Address: Mosk. Poligraficheskii Institut, Pryanishnikova 2-a, Moskva 127550, USSR.

Here $t^{(\text{ext})}_{\mu\nu}$ is a "canonical" energy-momentum tensor and $S^{(\text{ext})}_{\mu\alpha\beta}$ is a spin tensor of external fields. The set of equations (2) determines the tetrad potential h^a_μ and the components of the torsion tensoi $Q_{\sigma\varrho}{}^{\mu} = Q_{[\sigma\varrho]}{}^{\mu}$.

The condition

$$t^{(\text{ext})} = t^{(\text{ext})}{}_{\nu}{}^{\nu} = 0 \tag{3}$$

is a consequence of equation (2b).

The variational principles in the tetrad gravitational theory have been developed in [6–11] and it has been shown there that the condition (3) is a consequence of the full Lagrangian invariance with respect to the gauge scale transformation of tetrads; similarly the symmetry of the energy-momentum tensor $t^{(\text{ext})}_{\mu\nu}$ is a consequence of the invariance with respect to the 4-rotation of tetrads.

One can obtain the true energy-momentum tensor and its superpotential for Lagrangian (1) by means of the methods developed in [6–11]. Using this tensor and its superpotential one can derive the full gravitational energy and momentum inside a closed surface:

$$P_a = \frac{\lambda_1}{\kappa c} \oint_{\Sigma} R^{\alpha\beta}{}_{ij}\gamma_a{}^{ij} d\sigma_{\alpha\beta}. \tag{4}$$

Integral (4) does not have the shortcomings leading to the well-known Bauer paradox, since it contains the curvature tensor and therefore vanishes for the flat space-time. Besides, (4) does not depend on the choice of cooidinate system.

2. *The general quadratic theory*

Eqs. (2) cannot be the equations of the macroscopic theory of gravitation since the continuous medium energy-momentum tensor does not satisfy the condition (3). Therefore in the macroscopic theory of gravitation it is necessary to consider the linear Lagrangian

$$\mathscr{L}_{\text{E}} = \tfrac{1}{2} hR, \tag{5}$$

where R is the curvature scalar (in general also containing torsion).

We shall consider also two possible quadratic Lagrangians

$$\mathscr{L}_2 = \tfrac{1}{4} hR_{ab}{}^{a}{}_{c}R^{bdc}{}_{d}, \qquad \mathscr{L}_3 = \tfrac{1}{4} hR_{ab}{}^{ab}R_{cd}{}^{cd} \tag{6a, b}$$

together with (5) and (1).

Thus we have the full Lagrangian

$$\mathscr{L} = \Lambda h + \mathscr{L}_{\text{E}} + \sum_{i=1}^{3} \lambda_i \mathscr{L}_i + \kappa \mathscr{L}^{(\text{ext})}, \tag{7}$$

where $\mathscr{L}^{(\text{ext})}$ is the external field Lagrangian, $\kappa = 8\pi G/c^4$ and $\lambda_1, \lambda_2, \lambda_3$ are new coupling constants, which are proportional to the square of some new fundamental length l_0.

Note that there are no other quadratic Lagrangians except (1) and (6) in the theory, since in GGFT [8, 9] there are only three quadratic combinations

$$g_{mn}F_{ab}{}^{m}F^{abn}, \qquad F_{ab}{}^{m}F^{bcn}I_{m}{}^{a}{}_{d}I_{n}{}^{d}{}_{c}, \qquad F_{ab}{}^{m}I_{m}{}^{ab}F_{cd}{}^{n}I_{n}{}^{cd},$$

which can be constructed from the tensor of gauge field $F_{ab}{}^{m}$:

$$R_{ab}{}^{i}{}_{j} = F_{ab}{}^{m}I_{m}{}^{i}{}_{j}.$$

Here $g_{mn} = \frac{1}{2}c_{m}{}^{p}{}_{q}c_{n}{}^{q}{}_{p}$, $c_{m}{}^{p}{}_{q}$ are the structure constants of the gauge group and $I_{m}{}^{a}{}_{b}$ are the generators of the gauge group representation which transforms the indices a, b [7–9, 12].

The corresponding field equations, obtained by varying with respect to h^{a}_{μ} and $\gamma_{a}{}^{ij}$, have the form [8, 9]

$$\begin{aligned} &\lambda_1 R^{\nu}{}_{\mu\alpha\beta;\nu} - \tfrac{1}{2}\lambda_2 R_{\mu[\alpha;\beta]} - \tfrac{1}{2}\lambda_2 g_{\mu[\alpha}R^{\nu}{}_{\beta];\nu} - \lambda_3 g_{\mu[\alpha}R_{,\beta]} \\ &\quad - (Q_{\sigma\varrho\mu} + 2Q_{\nu[\sigma}{}^{\nu}g_{\varrho]\mu})(\delta^{\sigma}_{\alpha}\delta^{\varrho}_{\beta} + \lambda_1 R^{\sigma\varrho}{}_{\alpha\beta} + \lambda_2 R^{\sigma}{}_{[\alpha}\delta^{\varrho}_{\beta]} \\ &\quad + \lambda_3\delta^{\sigma}_{\alpha}\delta^{\varrho}_{\beta}R) = -\kappa S^{(\text{ext})}_{\mu\alpha\beta}, \end{aligned} \tag{8}$$

$$\begin{aligned} &R_{(\mu\nu)} - \tfrac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} + \lambda_1(R_{\mu\sigma\alpha\beta}R_{\nu}{}^{\sigma\alpha\beta} - \tfrac{1}{4}g_{\mu\nu}R_{\sigma\varrho\alpha\beta}R^{\sigma\varrho\alpha\beta}) \\ &\quad + \tfrac{1}{2}\lambda_2(R_{\mu\sigma}R_{\nu}{}^{\sigma} + R_{\mu\sigma\nu\varrho}R^{(\sigma\varrho)} - \tfrac{1}{2}g_{\mu\nu}R_{\sigma\varrho}R^{\sigma\varrho}) \\ &\quad + \lambda_3(RR_{(\mu\nu)} - \tfrac{1}{4}g_{\mu\nu}R^2) = -\kappa t^{(\text{ext})}_{\mu\nu}. \end{aligned} \tag{9}$$

If the torsion tensor vanishes, the relations

$$(\lambda_1 + \lambda_2 + 3\lambda_3)R_{,\nu} = 0, \qquad R + 4\Lambda = \kappa t^{(\text{ext})} \tag{10a, b}$$

are consequences of Eqs (8), (9). As (10a) must be an identity for any numerical value of the scalar curvature R, one has

$$\lambda_1 + \lambda_2 + 3\lambda_3 = 0 \tag{10c}$$

and therefore there are only two new coupling constants in the theory.

For $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Eqs (8), (9) become the Einstein-Cartan field equations [3–5]. When $\lambda_1 = -\frac{1}{4}\lambda_2 = \lambda_3$ and the torsion tensor $Q_{\sigma\varrho}{}^{\mu} = 0$, Eq. (8) is fulfilled because of Bianchi identities; in (9) all quadratic extra terms vanish because of the Bach-Lanczos identity [16, 17] for quadratic Langrangians

$$\delta\int(d^4x)h(R_{\sigma\varrho\alpha\beta}R^{\sigma\varrho\alpha\beta} - 4R_{\sigma\varrho}R^{\sigma\varrho} + R^2) = 0. \tag{11}$$

Therefore in this case one gets the usual Einstein theory.

If the coupling constants λ_i are not equal to each other, but the external fields are spinless and the torsion tensor $Q_{\sigma\varrho}{}^{\mu} = 0$, one can find out that there is only one coupling constant $\lambda = \frac{3}{4}(\lambda_1 - \lambda_3)$ in the theory because of the Bach-Lanczos identity and the identity (10c). In this case the field equations (8), (9) become [8, 9, 13, 14]

$$R^{\nu}{}_{\mu\alpha\beta;\nu} + \tfrac{1}{3}g_{\mu[\alpha}R_{,\beta]} = 0, \tag{12a}$$

$$\begin{aligned} &R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} + \lambda(R_{\mu\sigma\alpha\beta}R_{\nu}{}^{\sigma\alpha\beta} - \tfrac{1}{4}g_{\mu\nu}R_{\sigma\varrho\alpha\beta}R^{\sigma\varrho\alpha\beta} \\ &\quad - \tfrac{1}{3}RR_{\mu\nu} + \tfrac{1}{12}g_{\mu\nu}R^2) = -\kappa t^{(\text{ext})}_{\mu\nu}. \end{aligned} \tag{12b}$$

The set of equations (12) is overdetermined, but it can easily be verified with the help of the Bach-Lanczos identity that in vacuum any solution of the Einstein equations $R_{\mu\nu}+\Lambda g_{\mu\nu}=0$ is the solution of Eqs (12a, b).

With the help of the Bianchi identities Eqs (12a) become

$$R^{\mu}{}_{[\alpha;\beta]} = \tfrac{1}{6}\,\delta^{\mu}_{[\alpha}R_{,\beta]}. \tag{13}$$

This set of equations has an algebraic consequence

$$R_{\mu}{}^{\nu}{}_{\alpha\beta}R_{\gamma\nu}+R_{\mu}{}^{\nu}{}_{\gamma\alpha}R_{\beta\nu}+R_{\mu}{}^{\nu}{}_{\beta\gamma}R_{\alpha\nu} = 0. \tag{14}$$

3. The exterior Schwarzschild problem

Now let us suppose that gravitational field is spherically symmetric, described by a line element of the form

$$ds^2 = e^{\alpha(r,t)}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)-e^{\gamma(r,t)}dt^2,$$

$$x^1 = r,\ x^2 = \theta,\ x^3 = \varphi,\ x^4 = t \quad (c = 1). \tag{15}$$

With the help of the nonvanishing components of Eqs (12b) let us construct the expressions of the form $\binom{1}{4}$, $\frac{1}{2}\left[\binom{1}{1}-\binom{4}{4}\right]$, $\frac{1}{2}\left[\binom{1}{1}+\binom{4}{4}\right]$:

$$\frac{1}{r}\alpha_4 e^{-\alpha}\left[1+\tfrac{4}{3}\lambda\Lambda-\frac{\lambda}{r}(\alpha_1-\gamma_1)e^{-\alpha}\right] = 0, \tag{16a}$$

$$\frac{1}{r}(\alpha_1+\gamma_1)e^{-\alpha}\left[1+\tfrac{4}{3}\lambda\Lambda-\frac{\lambda}{r}(\alpha_1-\gamma_1)e^{-\alpha}\right] = 0, \tag{16b}$$

$$(1+\tfrac{4}{3}\lambda\Lambda)\left[-\Lambda+\frac{1}{r^2}(1-e^{-\alpha})+\frac{1}{2r}(\alpha_1-\gamma_1)e^{-\alpha}\right]$$

$$+\lambda\left[W^2e^{-2(\alpha+\gamma)}-\frac{1}{r^4}(1-e^{-\alpha})^2\right] = 0, \tag{16c}$$

and also consider the trace of (12b):

$$\tfrac{1}{2}(R+4\Lambda) = 2\Lambda+We^{-(\alpha+\gamma)}-\frac{1}{r^2}(1-e^{-\alpha})-\frac{1}{r}(\alpha_1-\gamma_1)e^{-\alpha} = 0. \tag{16d}$$

Here

$$\alpha_1 = \frac{\partial\alpha}{\partial r},\quad \alpha_4 = \frac{\partial\alpha}{\partial t},\quad \gamma_1 = \frac{\partial\gamma}{\partial r},\quad \gamma_4 = \frac{\partial\gamma}{\partial t},\quad \alpha_{44} = \frac{\partial^2\alpha}{\partial t^2},\quad \gamma_{11} = \frac{\partial^2\gamma}{\partial r^2},$$

$$W = R_{1414} = e^{\alpha}(-\tfrac{1}{2}\alpha_{44}-\tfrac{1}{4}\alpha_4^2+\tfrac{1}{4}\alpha_4\gamma_4)+e^{\gamma}(\tfrac{1}{2}\gamma_{11}+\tfrac{1}{4}\gamma_1^2-\tfrac{1}{4}\gamma_1\alpha_1). \tag{16e}$$

In the case

$$\frac{\lambda}{r}(\alpha_1-\gamma_1)e^{-\alpha} \neq 1+\tfrac{4}{3}\lambda\Lambda \tag{17}$$

Eqs (16a, b) yield

$$\frac{\partial\alpha}{\partial t} = 0, \quad \frac{\partial}{\partial r}(\alpha+\gamma) = 0, \quad \alpha+\gamma = f(t) = 0, \tag{18}$$

i. e. the Birkhoff theorem is true, and therefore Eq. (16d) leads to the usual Schwarzschild metric (with the cosmological term). If the condition (17) does not hold, one can prove that the set of equations (16) has no solutions at all.

Thus the set of solutions of the field equations (12) in the spherically symmetric case in vacuum coincides with the set of solutions of Einstein equations [8, 9, 14]. Some other cases of interest were considered in [15], where the solutions of the field equations (12) (without the cosmological term) weie analysed.

4. *The Friedman problem*

Consider now the field equations (12) for the isotropic homogeneous spinless perfect fluid. Let a line element have the Robertson-Walker form

$$ds^2 = e^{\alpha(r,t)}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)-e^{\gamma(r,t)}dt^2, \tag{19a}$$

$$e^{\alpha(r,t)} = \frac{a^2(t)}{\left(1+\frac{k}{4}r^2\right)^2}, \quad \gamma(r,t) = 0 \quad (c=1). \tag{19b}$$

The coordinates are supposed to be comoving with the fluid, so that the energy-momentum tensor of the spinless perfect fluid has only diagonal components

$$t_1^1 = t_2^2 = t_3^3 = p, \quad t_4^4 = -\varepsilon. \tag{20}$$

Denoting

$$H_\mu^\nu = R_{\mu\sigma\alpha\beta}R^{\nu\sigma\alpha\beta}-\tfrac{1}{4}\delta_\mu^\nu R_{\sigma\varrho\alpha\beta}R^{\sigma\varrho\alpha\beta} \tag{21}$$

we have from Eq. (12b)

$$R_4^1+\lambda(H_4^1-\tfrac{1}{3}RR_4^1) = 0, \tag{22a}$$

$$R_1^1-\tfrac{1}{2}R-\Lambda+\lambda(H_1^1-\tfrac{1}{3}RR_1^1+\tfrac{1}{12}R^2) = -\kappa p, \tag{22b}$$

$$R_2^2-\tfrac{1}{2}R-\Lambda+\lambda(H_2^2-\tfrac{1}{3}RR_2^2+\tfrac{1}{12}R^2) = -\kappa p, \tag{22c}$$

$$R_4^4-\tfrac{1}{2}R-\Lambda+\lambda(H_4^4-\tfrac{1}{3}RR_4^4+\tfrac{1}{12}R^2) = \kappa\varepsilon. \tag{22d}$$

Instead of Eqs (22) let us consider an equivalent set of equations obtained from the components of Eqs (22) of the form $\binom{1}{4}$, $\binom{1}{1}-\binom{2}{2}$, $\binom{1}{1}-\binom{4}{4}$, $\binom{1}{1}+\binom{2}{2}+\binom{3}{3}+\binom{4}{4}$:

$$R^1_4\left(1-\frac{\lambda}{3}R\right)+\lambda H^1_4 = 0, \tag{23a}$$

$$(R^1_1-R^2_2)\left(1-\frac{\lambda}{3}R\right)+\lambda(H^1_1-H^2_2) = 0, \tag{23b}$$

$$(R^1_1-R^4_4)\left(1-\frac{\lambda}{3}R\right)+\lambda(H^1_1-H^4_4) = -\kappa(\varepsilon+p), \tag{23c}$$

$$R+4\Lambda = -\kappa(\varepsilon-3p). \tag{23d}$$

Using the explicit form of the metric tensor (19b), one can find that all quadratic terms in Eqs (23) vanish because of identities

$$H^1_4 = \tfrac{1}{3}RR^1_4, \tag{24a}$$

$$H^1_1-H^2_2 = \tfrac{1}{3}R(R^1_1-R^2_2), \tag{24b}$$

$$H^1_1-H^4_4 = \tfrac{1}{3}R(R^1_1-R^4_4), \tag{24c}$$

which take place for the metric tensor (19b).

Eqs (13) for the general form of the line element (19a) reduce to

$$(R^4_4+\tfrac{1}{6}R)_{,1} = 0, \tag{25a}$$

$$(R^2_2+\tfrac{1}{6}R)_{,1}-\tfrac{1}{2}\left(\alpha_1+\frac{2}{r}\right)(R^1_1-R^2_2) = 0, \tag{25b}$$

$$(R^1_1+\tfrac{1}{6}R)_{,4}+\tfrac{1}{2}\alpha_4(R^1_1-R^4_4) = 0, \tag{25c}$$

$$(R^1_1-R^2_2)_{,4}+\tfrac{1}{2}\alpha_4(R^1_1-R^2_2) = 0. \tag{25d}$$

Here we put $R^1_4 = 0$ in a consequence of (22a) and (24a). It can easily be verified that all Eqs (25) are satisfied by the metric tensor (19b).

Thus we have demonstrated that the only solutions of the field equations (12) with a homogeneous isotropic spinless-perfect-fluid material distribution are the solutions of the Einstein equations (for any $p = f(\varepsilon)$).

5. *Conclusions*

It was shown that in the general quadratic theory the field equations for the Schwarzschild problem and for the Friedman problem have the same solutions as the Einstein equations. At the same time, one can show that the Nordström metric does not satisfy the set of equations (12).

In the general quadratic theory the condition (10c) is very important. Only in case (10c) the external spinless scalar field does not generate torsion and therefore torsion cannot be generated by gravitational field itself without the external field with nonzero spin.

Let us note that Eqs (12a) in vacuum coincide with the field equations in the theory of gravitation proposed by Yang [19]. There is a difference, however, between Yang's theory and ours: in our theory Eqs (12b) have to hold simultaneously with Yang's Eqs (12a).

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REFERENCES

[1] R. Utiyama, *Phys. Rev.* **101**, 1597 (1956).
[2] A. M. Brodski, D. Ivanenko, G. A. Sokolik, *Zh. Eksp. Teor. Fiz.* **41**, 1307 (1961).
[3] T. W. B. Kibble, *J. Math. Phys.* **2**, 212 (1961).
[4] A. Trautman, *On the Structure of the Einstein-Cartan Equations*, Istituto Nazionale di Alta Matematica, Symposia Mathematica, Vol. 12 (1973).
[5] A. Trautman, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **20**, 185 (1972); **20**, 503 (1972); **20**, 895 (1972); **21**, 345 (1973).
[6] B. N. Frolov, *Vestn. Mosk. Univ. Fiz. Astron.* No 6, 48 (1963); No 2, 56 (1964).
[7] B. N. Frolov, in *Problems of Gravitation*, Theses, Soviet Gravitational Conference II, Tbilisi 1965, pp. 154, 160.
[8] B. N. Frolov, in *The Modern Problems of Gravitation*, Proceedings of the Soviet Gravitational Conference II, Tbilisi 1965, ed. Tbilisi 1967, pp. 270, 311.
[9] B. N. Frolov, *The Variational Principles in the Tetrad and Compensating Treatments of the Gravitational Field*, Dissertation, MOPI, Moscow 1970.
[10] B. N. Frolov, in *Abstracts of the 5th International Conference on Gravitation and the Theory of Relativity*, Tbilisi 1968, p. 65.
[11] B. N. Frolov, in *Theses*, Soviet Gravitational Conference III, Yerevan 1972, p. 170.
[12] B. N. Frolov, G. N. Sardanashvili, *Izv. VUZ Fiz.* No 9, 47 (1974).
[13] B. N. Frolov, in Theses of the Conference *The Modern Theoretical and Experimental Problems in the General Relativity and Gravitation*, Minsk 1976, p. 276.
[14] B. N. Frolov, *Izv. VUZ Fiz.* No 3, 154, 155 (1977).
[15] V. M. Nikolaenko, *Acta Phys. Pol.* **B7**, 681 (1976).
[16] R. Bach, *Math. Z.* **9**, 110 (1921).
[17] C. Lanczos, *Ann. Math.* **39**, 842 (1938).
[18] G. Stephenson, *Nuovo Cimento* **9**, 263 (1958).
[19] C. N. Yang, *Phys. Rev. Lett.* **33**, 445 (1974).
[20] E. Fairchild, *Phys. Rev.* **D14**, 384, Err. 2833 (1976).