SOME EXACT SOLUTIONS OF CHARGED FLUID SPHERES IN EINSTEIN-CARTAN THEORY

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In this paper the interior field of a static spherically symmetric charged fluid distribution in Einstein-Cartan theory has been studied. Assuming that the spins of the individual particles composing the fluid are all aligned in the radial direction, we have obtained a solution and the physical constants appearing in the solution have been evaluated by matching the solution to the Reissner-Nordstrom metric at the boundary. Unlike general relativity, p is discontinuous at the boundary of the fluid sphere.

1. *Introduction*

Recently Prasanna [1] considered the problem of static fluid spheres in the framework of Einstein-Cartan theory (E-C theory). Adopting Hehl's approach [2, 3] to E-C theory, Prasanna has obtained the solutions analogous to solutions obtained by Tolman [4] in general relativity. He has found that a space-time metric similar to the Schwarzschild interior solution will no longer represent a homogeneous fluid sphere in the presence of spin density. Nduka [5] has discussed the charged static fluid spheres in E-C theory and has found that the pressure is discontinuous at the boundary of the fluid sphere.

In the present paper we have solved the Einstein-Cartan equations for a charged fluid sphere by a different method. We have derived a general set of differential equations which the function $\nu(r)$ and $\lambda(r)$ of the metric coefficients must satisfy and have obtained the solution by adopting a technique similar to that of Adler [6] foı an uncharged fluid sphere in general relativity. The relevent differential equation reduces to Euler's equation which may be treated as a generalisation of the equation of Wyman [7].

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2. *The field equations and their solution*

The Einstein-Cartan-Maxwell equations for the perfect fluid [8, 9] are

$$R_{\mu\nu}-\tfrac{1}{2}Rg_{\mu\nu} = -8\pi T_{\mu\nu}, \tag{2.1}$$

$$[(-g)^{1/2}F^{\mu\nu}]_{,\nu} = 4\pi(-g)^{1/2}J^{\mu}, \tag{2.2}$$

$$F_{[\mu\nu,\lambda]} = 0, \tag{2.3}$$

where $T_{\mu\nu}$ is the energy momentum tensor, $F_{\mu\nu}$ is the electromagnetic field tensor and J^{μ} is the current four vector. Throughout this paper we set c and gravitational constant equal to unity.

For a static spherically symmetric system we take the metric as

$$ds^2 = e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta d\Phi^2), \tag{2.4}$$

where λ and ν are functions of r only.

For the system under study the energy momentum tensor T_{ν}^{μ} splits into two parts viz. t_{μ}^{ν} and E_{μ}^{ν} for matter and charges respectively [10] as

$$T_{\mu}^{\nu} = t_{\mu}^{\nu}+E_{\mu}^{\nu}. \tag{2.5}$$

The nonvanishing components to t_{μ}^{ν} are

$$t_4^4 = \varrho, \quad t_1^1 = t_2^2 = t_3^3 = -p.$$

Because of the spherical symmetry the only non-vanishing component of $F^{\mu\nu}$ is $F^{14} = -F^{41}$. Therefore the nonzero components of E_{μ}^{ν} are

$$E_4^4 = E_1^1 = -E_2^2 = -E_3^3 = -\frac{1}{8\pi}g_{44}g_{11}(F^{41})^2.$$

Equation (2.3) is obviously satisfied by this choice of $F^{\mu\nu}$ whereas (2.2) reduces to

$$F^{41} = \frac{Q(r)e^{-N}}{r^2}, \quad N = \frac{\lambda+\nu}{2}, \tag{2.6}$$

where $Q(r)$ is the charge up to radius r,

$$Q(r) = 4\pi\int_0^r J^4r^2e^N dr. \tag{2.7}$$

From equation (2.7) we see that outside the fluid sphere $Q(r)$ is a constant Q_0. That Q_0 is the total charge follows from (2.6) which gives the asymptotic form of the electric field as Q_0/r^2.

Then from (2.1) and (2.5) the field equations may be written as

$$8\pi\bar{\varrho}+8\pi E_4^4 = e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}, \tag{2.8}$$

$$8\pi\bar{p}-8\pi E_1^1 = e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \tag{2.9}$$

$$8\pi\bar{p}-8\pi E_2^2 = e^{-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\lambda'\nu'}{4}+\frac{\nu'-\lambda'}{2r}\right), \tag{2.10}$$

where following Hehl [2, 3] we have defined $\bar{p} = p-2\pi K^2$ and $\bar{\varrho} = \varrho-2\pi K^2$ with $K = A_1 e^{-\nu/2}$. Here A_1 is a constant of integration and dashes denote differentiation with respect to r.

Now we put

$$\nu = 2\log Y. \tag{2.11}$$

Then from equations (2.9) and (2.10) we get the second order differential equation in $Y(r)$ after eliminating $\bar{p}$ as follows:

$$Y''-\left(\frac{1}{r}+\frac{\lambda'}{r}\right)Y'+\left(\frac{e^{\lambda}}{r^2}-\frac{\lambda'}{2r}-\frac{1}{r^2}-\frac{2Q^2e^{\lambda}}{r^4}\right)Y = 0. \tag{2.12}$$

This is a generalisation of Wyman's equation [7].

Now we make the assumption

$$Q = Ar^{\eta}, \tag{2.13}$$

where A is constant of proportionality and η is a positive integer. Use of equation (2.13) in (2.12) yields

$$Y''-\left(\frac{1}{r}+\frac{\lambda'}{2}\right)Y'+\left(\frac{e^{\lambda}}{r^2}-\frac{\lambda'}{2r}-\frac{1}{r^2}-2A^2r^{2\eta-4}e^{\lambda}\right)Y = 0. \tag{2.14}$$

Now we define

$$e^{-\lambda} = \tau(r). \tag{2.15}$$

Then equation (2.14) may be written as a linear first order equation in $\tau(r)$ viz.

$$\tau'-\tau\left[\frac{2(Y+rY'-r^2Y'')}{r(Y+rY')}\right] = \frac{-2Y(1-2A^2r^{2\eta-2})}{r(Y+rY')}. \tag{2.16}$$

This has the solution

$$\tau(r) = \exp\left[-F(r)\right]\left\{\int^r \exp\left[F(r)\right]g(r)dr+C\right\}, \tag{2.17}$$

where

$$f = \frac{-2(Y+rY'-r^2Y'')}{r(Y+rY')},$$

$$g = \frac{-2Y(1-2A^2r^{2n-4})}{r(Y+rY')},$$

$$F(r) = \int^r f(r)dr$$

and C is a constant of integration to be fixed by the boundary conditions.

It is clear from the above equations that it is the $\bar{p}$ and not p which is continuous across the boundary $r = r_0$ of the fluid sphere. The continuity of $\bar{p}$ across the boundary ensures that of dg_{44}/dr i. e. $\nu' e^{\nu}$. Further, with $\bar{p}$ and $\bar{\varrho}$ replacing p and ϱ respectively we are assured that the metric coefficients are also continuous across the boundary. Hence we shall apply the usual boundary conditions to the solutions of equations (2.8), (2.9) and (2.10).

The exterior metric is taken as the usual Reissner-Nordstrom line-element given by

$$ds^2 = \left(1-\frac{2M}{r}+\frac{Q_0^2}{r^2}\right)dt^2-\left(1-\frac{2M}{r}+\frac{Q_0^2}{r^2}\right)dr^2$$
$$-r^2(d\theta^2+\sin^2\theta d\Phi^2), \tag{2.18}$$

where $Q_0 = Q(r_0)$ and M is the total mass of the sphere given by

$$M = 4\pi\int_0^{r_0}\varrho r^2 dr. \tag{2.19}$$

3. Specific analytic solution

Equation (2.14) may be solved by quadrature in a number of ways. We note that λ may be obtained if ν is given and vice versa. Once ν and λ are obtained, ϱ and p follow directly from equations (2.8) and (2.9). Nduka [5] has assumed that exp $(-\lambda)$ = constant and has obtained ν. We adopt Adler's technique [6] and choose ν in such a way that $f = g$. This is fulfilled by requiring that

$$r^2Y''-rY'-2A^2r^{2n-2}Y = 0. \tag{3.1}$$

We consider the case when $n = 1$ which reduces the equation (3.1) to

$$r^2Y''-rY'-2A^2Y = 0. \tag{3.2}$$

On putting $p = -1$ and $q = -2A^2$ (3.2) is transformed into

$$r^2Y''+prY'+qY = 0, \tag{3.3}$$

where p and q are constants. Equation (3.2) is Eulers' homogeneous equation. The solution of equation (3.2) may now be written down and the metric function $\nu(r)$ is obtained. Then equation (2.17) gives the other metric coefficient $\lambda(r)$ while density and pressure can be calculated from equations (2.8) and (2.9).

To solve equation (3.2) or (3.3) there are three possible cases [11] viz.

$$(i)\ 2A^2+1 > 0, \quad (ii)\ 2A^2+1 = 0, \quad (iii)\ 2A^2+1 < 0.$$

But $A^2 < 0$ leads to imaginary electromagnetic field and so in what follows we shall take $A^2 > 0$. Thus the solution of equation (3.2) is

$$Y = B_1 r^{1+\eta}+C_1 r^{1-\eta}, \tag{3.4}$$

where B_1 and C_1 are constants to be fixed by the boundary conditions and

$$\eta = \sqrt{1+2A^2}\,. \tag{3.5}$$

When equation (3.4) is used in conjunction with equation (2.17), $\tau(r)$ can be readily obtained as

$$\tau(r) = (2-\eta^2)^{-1}+D_1 r^{2(2-\eta^2)/(2-\eta)}[B_1(2+\eta)r^{2\eta}+C_1(2-\eta)]^{-2(2-\eta^2)/(4-\eta^2)}, \tag{3.6}$$

where D_1 is constant of integration to be fixed by boundary conditions.

Now Y and τ are known i. e. ν and λ are known. The electromagnetic energy is given by

$$8\pi r^2 E_1^1 = 8\pi r^2 E_4^4 = -8\pi r^2 E_2^2 = -8\pi r^2 E_3^3 = A^2. \tag{3.7}$$

By equation (2.8) and (2.9) pressure and density can be obtained in a straighforward manner

$$\begin{aligned} 8\pi r^2 \varrho(r) = {} & 1-\tau(r)-2\{(2-\eta^2)\tau(r)-1\}\,(B_1 r^{2\eta}+C_1) \\ & \{B_1(2+\eta)r^{2\eta}+C_1(2-\eta)\}^{-1}-A^2 \\ & +2\pi A_1^2(B_1 r^{1+\eta}+C_1 r^{1-\eta})^{-2}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} 8\pi r^2 p(r) = {} & \tau(r)\,\{B_1(3+2\eta)r^{2\eta}+C_1(3-2\eta)\}\,(B_1 r^{2\eta}+C_1)^{-1} \\ & -1+A^2+2\pi A_1^2(B_1 r^{1+\eta}+C_1 r^{1-\eta})^{-2}. \end{aligned} \tag{3.9}$$

Using the boundary conditions discussed in Section 2, the constants A_1, B_1, C_1 and D_1 are given by

$$A_1^2 = \frac{1}{2\pi}\left(\varrho_0-\frac{1}{4\pi r_0^2}\right)\{(B_1 r_0^{1+\eta}+C_1 r_0^{1-\eta})^{-2}-\mathscr{C}\}^{-1}, \tag{3.10}$$

where ϱ_0 is the value of density at the boundary $r = r_0$ and

$$\mathscr{C} = 1-\frac{2M}{r_0}+\frac{Q_0^2}{r_0^2},$$

$$B_1 = \frac{1}{2\eta}\left(1-2\mathscr{C}-\eta\mathscr{C}-\frac{Q_0^2}{r_0^2}\right)r_0^{-(1+\eta)}, \tag{3.11}$$

$$C_1 = -\frac{1}{2\eta}\left(1-2\mathscr{C}-\eta\mathscr{C}-\frac{Q_0^2}{r_0^2}\right)r_0^{-(1-\eta)}, \tag{3.12}$$

$$\begin{aligned} D_1 = {} & \{\mathscr{C}-(2-\eta^2)^{-1}\}r_0^{-2(2-\eta^2)/(2-\eta)} \\ & [B_1(2+\eta)r_0^{2\eta}+C_1(2-\eta)]^{2(2-\eta^2)/(4-\eta^2)}. \end{aligned} \tag{3.13}$$

The conditions $p > 0$ and $\varrho > 0$ will impose further restrictions on our solution.

By putting $A = 0$ in equations (3.4)–(3.13) we obtain the solution corresponding to the uncharged case of static fluid sphere in E-C theory. This uncharged case has been discussed recently by the present authors elsewhere [12] and hence it represents a new set of exact solution of the static fluid spheres in Einstein-Cartan theory.

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