STATIC FLUID SPHERES IN EINSTEIN-CARTAN THEORY

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Analytic solutions of the Einstein-Cartan field equations for the interior of a fluid sphere are obtained. Some of these solutions may be applicable to the investigation of stellar interiors where high central density and pressure are significant (i. e. massive bodies like non-rotating neutron stars).

1. *Introduction*

Stimulated by the successful geometrization of physics in Einstein's general theory of relativity, the great French mathematician E. Cartan suggested that a more general geometrical framework incorporating the notion of torsion as well as Riemannian curvature might be useful in the description of a continuum of spinning particles (Cartan [1, 2]). Nearly after half a century this idea has received a strong theoretical ground (both geometrical and physical) through the investigation of various authors (Trautman [3], Kerlick [4], Kuchowicz [5], [6]; Hehl [7, 8]; Hehl et al. [9] and Prasanna [10]) in the form of a viable rival theory to Einstein's theory of gravitation and which is now called the Einstein-Cartan theory (or E-C theory in brief).

Since the predictions of the E-C theory differ from those of general relativity only for matter-filled regions, therefore, besides cosmology, an important application field for the E-C theory is relativistic astrophysics dealing with the interiors of stellar objects like neutron stars with some alignment of spins of the constituent particles and under conditions when torsion may produce some observable effects. As such it seems desirable to understand the full implications of the E-C theory for finite distributions like fluid spheres with non-zero pressure. With this view many workers have considered the problem of static fluid spheres in the E-C theory (Prasanna [11]; Kerlick [4]; Kuchowicz [12, 13]; Skinner and Webb [14]). In this paper we have investigated the same problem and have developed a technique to obtain the solution in an analytic form by the method of quadratures. The application of the technique in special cases gives some exact solutions in a quite easy manner. Some other solutions have also been obtained under different assumptions.

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2. The field equations

The Einstein-Cartan field equations are

$$R_i^j - \tfrac{1}{2} R\delta_i^j = -\chi t_i^j, \tag{2.1}$$

$$Q^i_{jk} - \delta^i_j Q^l_{lk} - \delta^i_k Q^l_{jl} = -\chi S^i_{jk}, \tag{2.2}$$

where Q^k_{ij} is torsion tensor, t_i^j is the canonical, asymmetric energy-momentum tensor, S^k_{ij} is the spin tensor and $\chi = 8\pi$.

For a static spherically symmetric system an appropriate metric is

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\Phi^2), \tag{2.3}$$

λ and ν being functions of r. We use comoving coordinates with 4-velocity $u^i = \delta^i_4$. The orthonormal coframe is chosen as

$$\vartheta^1 = e^{\lambda/2}dr, \quad \vartheta^2 = rd\theta, \quad \vartheta^3 = r\sin\theta d\Phi, \quad \vartheta^4 = e^{\nu/2}dt.$$

When we assume a classical description of spin, we have

$$S^k_{ij} = S_{ij}u^k \quad \text{with} \quad S_{ij}u^j = 0, \tag{2.4}$$

where S_{ij} is the antisymmetric tensor of the density of spin. In the case of spherical symmetry, the tensor S_{ij} has the only nonvanishing independent component $S_{23} = K$ (say) and the non-zero components of S^i_{jk} are

$$S^4_{23} = -S^4_{32} = K. \tag{2.5}$$

Hence from the E-C field equations (2.2), the nonzero components of Q^i_{jk} are

$$Q^4_{23} = -Q^4_{32} = -\chi K. \tag{2.6}$$

Thus for a perfect fluid distribution with isotropic pressure p and matter density ϱ (Prasanna [11]) the field equations (2.1) finally reduce to

$$8\pi p = 16\pi^2 K^2 - \frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{2.7}$$

$$8\pi\varrho = 16\pi^2 K^2 + \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right). \tag{2.8}$$

$$\frac{e^\lambda}{r^2} = \frac{1}{r^2} - \frac{\nu'^2}{4} - \frac{\nu''}{2} + \frac{\nu'\lambda'}{4} + \frac{\nu'+\lambda'}{2r}, \tag{2.9}$$

where dashes denote differentiation with respect to r.

The conservation laws give us the relation

$$[p' + \tfrac{1}{2}(\varrho + p)\nu'] + \chi K(K' + \tfrac{1}{2}K\nu') = 0. \tag{2.10}$$

If we use the equation of hydrostatic equilibrium viz.

$$p' + \tfrac{1}{2}(\varrho + p)\nu' = 0, \tag{2.11}$$

we get

$$K' + \tfrac{1}{2} K\nu' = 0. \tag{2.12}$$

From (2.12) we have

$$K = A_1 e^{-\nu/2}, \tag{2.13}$$

where A_1 is a constant of integration.

In principle we have a completely determined system if an equation of state is specified. However, as is well known, in practice this set of equations is formidable to solve using a preassigned equation of state. Therefore other methods are applied under different mathematical assumptions.

Following Hehl [7, 8], if we define

$$\bar{\varrho} = \varrho - 2\pi K^2, \qquad \bar{p} = p - 2\pi K^2, \tag{2.14}$$

we find that the equations (2.7) and (2.8) take the usual general relativistic form for a static fluid sphere as given by

$$8\pi\bar{p} = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{2.15}$$

$$8\pi\bar{\varrho} = \frac{1}{r^2} + e^{-\lambda}\left(-\frac{1}{r^2} + \frac{\lambda'}{r}\right), \tag{2.16}$$

(2.9) remaining the same. The equation of continuity (2.10) now becomes

$$\frac{d\bar{p}}{dr} + \tfrac{1}{2}(\bar{\varrho} + \bar{p})\nu' = 0. \tag{2.17}$$

It is clear from these equations that it is the $\bar{p}$ not the p which is continuous across the boundary $r = r_0$ of the fluid sphere. The continuity of $\bar{p}$ across the boundary ensures that of $\nu' \exp(\nu)$. Futher, with $\bar{p}$ and $\bar{\varrho}$ replacing p and ϱ respectively we are assured that the metric coefficients are continuous across the boundary. Hence we shall apply the usual boundary conditions to the solutions of equations (2.9), (2.15) and (2.16).

We use the boundary conditions

$$[e^{-\lambda}]_{r=r_0} = [e^{\nu}]_{r=r_0} = \left(1 - \frac{2m}{r_0}\right), \tag{2.18}$$

$$\bar{p} = 0 \quad \text{at} \quad r = r_0,$$

where r_0 is the radius of the fluid sphere and m is the mass of the fluid sphere. The total mass, as measured by an external observer, inside the fluid sphere of radius r_0 is given by

$$m = 4\pi \int_0^{r_0} \bar{\varrho} r^2 dr$$

$$= 4\pi \int_0^{r_0} \varrho r^2 dr - 8\pi^2 \int_0^{r_0} K^2(r) r^2 dr. \tag{2.19}$$

Thus the total mass of the fluid sphere is modified by the correction

$$8\pi^2 \int_0^{r_0} K^2(r) r^2 dr.$$

3. The solution of the Einstein-Cartan field equations by quadrature

It is well known that the equation (2.9) may be solved by quadratures in a number of ways; e. g. Tolman [16] and Prasanna [11] specify various conditions on the functions ν and λ that simplify the equations and allow immediate integration. Once ν and λ are obtained $\bar{p}$ and $\bar{\varrho}$ follow directly from (2.15) and (2.16). We define

$$Y(r) = e^{\nu/2}, \qquad \tau(r) = e^{-\lambda}. \tag{3.1}$$

Then (2.9) may be written as

$$\tau' - \left[\frac{2(Y + rY' - r^2 Y'')}{r(Y + rY')}\right]\tau = \frac{-2Y}{r(Y + rY')}. \tag{3.2}$$

It has the solution

$$\tau(r) = \exp\left[-F(r)\right]\left\{\int^r \exp\left[F(u)\right] g(u) du + c\right\},$$

where

$$F(r) = \int^r f(u) du, \quad f(r) = \frac{-2(Y + rY' - r^2 Y'')}{r(Y + rY')},$$

$$g(r) = \frac{-2Y}{r(Y + rY')}, \tag{3.3}$$

c being a constant of integration to be fixed by the boundary conditions. The remaining equations (2.15) and (2.16) give $\bar{p}$ and $\bar{\varrho}$ as

$$8\pi \bar{p} r^2 = -1 + \tau\left(1 + \frac{rY'}{2Y}\right) \tag{3.4}$$

and

$$8\pi \bar{\varrho} r^2 = 1 - \tau\left(1 + \frac{r\tau'}{\tau}\right). \tag{3.5}$$

Exact solutions in terms of known functions are most easily obtained by requiring one of the field variables to satisfy some subsidiary condition which simplifies the full set of equations. Once the field equations are solved in this manner an equation of state can then be found. Such solutions may be useful in understanding a system in the extreme relativistic limit where we cannot specify a priori what the equation of state might be.

Further, there is no reason to expect that all solutions will be physically reasonable. Only a subclass of these solutions corresponding to certain choices of the function $\nu(r)$ will be physically reasonable and a still smaller subclass will correspond to physically reasonable equations of state. Thus a judicious choice of $\nu(r)$ is necessary for a physically interesting solution.

4. *Specific analytic solutions*

We note that the equation (3.2) is linear in τ if Y is a known function. This being the case, we choose Y in such a manner that equation (3.2) can be immediately integrated. We assume that Y satisfies the Cauchy equation

$$r^2Y''-rY'+(1-\alpha^2)Y = 0, \quad 0<\alpha\leqslant 1. \tag{4.1}$$

This equation gives

$$Y(r) = ar^i+br^j, \tag{4.2}$$

where $i = 1+\alpha$, $j = 1-\alpha$; a and b being constants of integration. When Y from (4.2) is used in (3.3), τ is

$$\tau(r) = s^{-1}+cr^{2s/l}[akr^{2\alpha}+bl]^{-2s/lk}, \tag{4.3}$$

where

$$s = 2-\alpha^2, \quad k = 2+\alpha, \quad l = 2-\alpha.$$

Since Y and τ are known, they may be used to calculate $\bar{p}$ and $\bar{\varrho}$ from equations (3.4) and (3.5) and the spin density K may also be calculated from (2.13). The pressure is given by

$$8\pi p = 16\pi^2K^2+\frac{\tau(r)}{r^2}[anr^{2\alpha}+bq]\,[ar^{2\alpha}+b]^{-1}-\frac{1}{r^2} \tag{4.4}$$

and the density by

$$8\pi\varrho = 16\pi^2K^2-\frac{1}{r^2}-\frac{\tau(r)}{r^2}-\frac{2}{r^2}(s\tau(r)-1)\,(ar^{2\alpha}+b)\,(akr^{2\alpha}+bl)^{-1}, \tag{4.5}$$

where $n = 3+2\alpha$, $q = 3-2\alpha$. The spin density K is given by

$$K = A_1e^{-\nu/2} = A_1Y^{-1} = A_1(ar^i+br^j)^{-1}. \tag{4.6}$$

The constants of integration a, b, c can now be determined by matching the solutions at the boundary $r = r_0$ to Schwarzschild exterior solution. They are given by

$$a = (1-qY_s^2)(4\alpha r_0^i Y_s)^{-1}, \tag{4.7}$$

$$b = -(1-nY_s^2)(4\alpha r_0^j Y_s)^{-1}, \tag{4.8}$$

$$c = \left(Y_s^2 - \frac{1}{s}\right)[(1+Y_s^2)(2Y_s r_0^3)] \tag{4.9}$$

with $Y_s^2 = 1-2m/r_0$. Also A_1 is determined from

$$8\pi\varrho(r_0) = 16\pi^2 A_1^2(ar_0^i+br_0^j)^{-2} - \frac{1}{r_0^2} - \frac{2}{r_0^2}(s\tau(r_0)-1)$$
$$\times(ar_0^{2\alpha}+b)(akr_0^{2\alpha}+bl)^{-1}, \tag{4.10}$$

where

$$\tau(r_0) = s^{-1} + r_0^{2s/l}[akr_0^{2\alpha}+bl]^{-2s/lk}.$$

Note that now $0 < \alpha \leqslant 1$. The solution with $\alpha = 0$ cannot be matched to the Schwarzschild solution with a finite boundary.

For the paiticular values of the parameter α and intergation constants a, b, c several previously known solutions for static fluid spheres are contained herein.

The limiting value of $\alpha = 1$ gives a solution similar to the one given by Adler [15]. This is the only solution of the family which does not diverge at the origin. Three solutions obtained by Prasanna [11] are also included in this family. These are the solutions corresponding to Tolman's I, V and VI solution [16].

As the solution for the case $\alpha = 1$ is significant for further investigation, we mention it explicitly. When $\alpha = 1$, Y satisfies the differential equation

$$r^2Y'' - rY' = 0. \tag{4.11}$$

Therefore

$$e^{\nu/2} = Y(r) = A + Br^2. \tag{4.12}$$

This gives

$$e^{-\lambda} = \tau(r) = 1 + \frac{cr^2}{(A+3Br^2)^{2/3}}, \tag{4.13}$$

where the constants A, B, C aie specified by matching the solution to the exterior Schwarzschild solution at the boundary $r = r_0$. They are given by

$$A = (1-2\varepsilon)^{-1/2}(1-\tfrac{5}{2}\varepsilon), \tag{4.14}$$

$$B = (1-2\varepsilon)^{-1/2}(\varepsilon/2r_0^2), \tag{4.15}$$

$$C = -\frac{2\varepsilon}{r_0^2}(1-2\varepsilon)^{-1/3}(1-\varepsilon)^{2/3}. \tag{4.16}$$

The spin density K is given by

$$K = A_1(A+Br^2)^{-1} = A_1(1-2\varepsilon)^{1/2}\left(1-\tfrac{5}{2}\varepsilon+\frac{\varepsilon}{2}y^2\right)^{-1}. \tag{4.17}$$

Also we have

$$\varrho = \left(\frac{\varepsilon}{4\pi r_0^2}\right)(1-\varepsilon)^{2/3}(1-\tfrac{5}{2}\varepsilon+\tfrac{3}{2}\varepsilon y^2)^{-2/3}[3-2\varepsilon y^2(1-\tfrac{5}{2}\varepsilon+\tfrac{3}{2}\varepsilon y^2)^{-1}]$$

$$+16\pi^2A_1^2(1-2\varepsilon)\left(1-\tfrac{5}{2}\varepsilon+\frac{\varepsilon}{2}y^2\right)^{-2} \tag{4.18}$$

and

$$p = \left(\frac{\varepsilon}{4\pi r_0^2}\right)[e^{-\lambda}(1-\tfrac{5}{2}\varepsilon+\tfrac{1}{2}\varepsilon y^2)^{-1}-(1-\tfrac{5}{2}\varepsilon+\tfrac{3}{2}\varepsilon y^2)^{-2/3}(1-\varepsilon)^{2/3}]$$

$$+16\pi^2A_1^2(1-2\varepsilon)(1-\tfrac{5}{2}\varepsilon+\tfrac{1}{2}\varepsilon y^2)^{-2}, \tag{4.19}$$

where $\varepsilon = m/r_0$ and $y = r/r_0$. The constant A_1 is given by

$$A_1^2 = \left(\frac{1-2\varepsilon}{16\pi^2}\right)\left[\frac{\varepsilon}{4\pi r_0^2}\left(\frac{3-5\varepsilon}{1-\varepsilon}\right)-\varrho(r_0)\right]. \tag{4.20}$$

When $A_1 = 0$, the solutions of this section reduce to the interior solution in general relativity obtained independently by Adler [15] and Kuchowicz [17] (when $\alpha = 1$) and to that of Whitman [18] when $\alpha \neq 1$.

5. *Another application of the technique*

Let us chose

$$Y(r) = e^{\nu/2} = (\tfrac{1}{2}k_1r^2+k_2)^{1/2}, \tag{5.1}$$

where k_1 and k_2 are constant. Then the differential equation (3.2), on intergation, gives

$$\tau(r) = e^{-\lambda} = \frac{(\tfrac{1}{2}k_1r^2+k_2)(1+2k_3r^2)}{(k_1r^2+k_2)} \tag{5.2}$$

i. e. $e^{\lambda} = (k_1r^2+k_2)/(\tfrac{1}{2}k_1r^2+k_2)(1+2k_3r^2)$, where k_3 is a constant of integration. Also

$$K = A_1(\tfrac{1}{2}k_1r^2+k_2)^{-1/2}. \tag{5.3}$$

The constants k_1, k_2 and k_3 are determined by the boundary conditions at $r = r_0$. The pressure and density are given by

$$p = 2\pi K^2+\frac{k_1}{16\pi}\left(\frac{1+2k_3r^2}{k_2+k_1r^2}\right)+\frac{k_3}{4\pi}, \tag{5.4}$$

$$\varrho = 2\pi K^2+\frac{1}{8\pi}\,\frac{-3k_1^2k_3r^4+(\tfrac{1}{2}k_1^2-7k_1k_2k_3)r^2+(\tfrac{3}{2}k_1k_2-6k_2^2k_3)}{(k_1r^2+k_2)^2}. \tag{5.5}$$

The constants k_1, k_2 and k_3 are determined by

$$1-\frac{2m}{r_0}=\frac{(\frac{1}{2}k_1r_0^2+k_2)(1+2k_3r_0^2)}{(k_1r_0^2+k_2)}, \tag{5.6}$$

$$1-\frac{2m}{r_0}=\tfrac{1}{2}k_1r_0^2+k_2, \tag{5.7}$$

$$k_3=-\frac{k_1}{4}\frac{(1+2k_3^2r_0^2)}{(k_1r_0^2+k_2)}. \tag{5.8}$$

From (5.6)–(5.8) we have

$$k_1=-4k_3=\frac{2m}{r_0^3},\qquad k_2=1-\frac{3m}{r_0}. \tag{5.9}$$

Here $k_1>0$ and $k_3<0$. Also A_1 is determined from

$$\begin{aligned} A_1^2 &= \{8\pi k_2\varrho_0-(\tfrac{3}{2}k_1-6k_2k_3)\}/16\pi^2,\\ &= \left\{8\pi\varrho_0\left(1-\frac{3m}{r_0}\right)-\frac{3m}{r_0^3}\left(2-\frac{3m}{r_0}\right)\right\}\Big/16\pi^2, \end{aligned} \tag{5.10}$$

where $\varrho_0=\varrho(r=0)$.

When $A_1=0$ i. e. spin is absent, the solutions go over (with some adjustment of the arbitrary constants) to the interior solution obtained by Krori et al. [19] and Tolman's solution IV [16].

6. *Some additional solutions*

Case 1. Let us make the assumption

$$\nu'=\beta re^{\lambda/2}, \tag{6.1}$$

where β is a constant. Then from the field equations (2.15)–(2.17), we obtain

$$e^{-\lambda}=1-dr^2-\frac{\beta^2r^4}{4}, \tag{6.2}$$

$$e^{\nu}=\frac{1}{\eta}\exp\left\{\sin^{-1}\left(\frac{\beta^2r^2+2d}{2\sqrt{\beta^2+d^2}}\right)\right\}, \tag{6.3}$$

$$8\pi p=16\pi^2K^2+\beta\left(1-dr^2-\frac{\beta^2r^4}{4}\right)^{1/2}-\frac{\beta^2r^2}{4}-d,$$

$$K=A_1\eta\exp\left\{-\sin^{-1}\left(\frac{\beta^2r^2+2d}{2\sqrt{\beta^2+d^2}}\right)\right\}, \tag{6.5}$$

$$8\pi\varrho=16\pi^2K^2+3d+\tfrac{5}{4}\beta^2r^2, \tag{6.6}$$

where the constants β, d, η and A_1 are given by

$$m = \frac{r_0^3}{2}\left(d + \frac{\beta^2}{4} r_0^2\right), \tag{6.7}$$

$$m = \frac{r_0}{2}\left[1 - \frac{1}{\eta}\sin^{-1}\left(\frac{\beta^2 r_0^2 + 2d}{2\sqrt{\beta^2+d^2}}\right)\right], \tag{6.8}$$

$$r_0^2 = 4[\sqrt{4d^2+5\beta^2} - 3d]/5\beta, \tag{6.9}$$

$$A_1^2 = \frac{\varrho(r_0)}{2\eta^2}\exp\left\{2\sin^{-1}\left(\frac{\beta^2 r_0^2+2d}{2\sqrt{\beta^2+d^2}}\right)\right\}. \tag{6.10}$$

From (6.9), $r_0^2 > 0$ only when $\beta > d$. From (6.7) and (6.8)

$$\eta = \frac{\exp\left[\sin^{-1}\{(\beta^2 r_0^2 + 2d)/2\sqrt{\beta^2+d^2}\}\right]}{\left(1 - d r_0^2 - \frac{\beta^2}{4} r_0^4\right)}.$$

Thus the value of η is fixed by the values of β, d and r_0 determined (6.7)–(6.9).

When $A_1 = 0$ the solutions go over to the interioı solution in general relativity due to Krori et al. [20].

Case 2. Let us assume

$$\bar{\varrho} = \bar{\varrho}_0\left(1 - \frac{r^2}{r_0^2}\right), \tag{6.11}$$

where $\bar{\varrho}_0$ is the value of $\bar{\varrho}$ at the centre $r = 0$. Then from the field equations (2.15)–(2.17) we have

$$e^{-\lambda} = 1 - \frac{8\pi\bar{\varrho}_0}{15}\left(5r^2 - \frac{3r^4}{r_0^2}\right), \tag{6.12}$$

$$e^{\nu} = [A_2\cos(\zeta/2) + A_3\sin(\zeta/2)]^2, \tag{6.13}$$

$$8\pi p = 16\pi^2 K^2 + \frac{4}{r_0}\left(\frac{2\pi\bar{\varrho}_0}{5}\right)^{1/2}\left[1 - \frac{8\pi r_0^2}{15}(5x - 3x^2)\right]$$

$$\times\left[\frac{A_3 - A_2\tan(\zeta/2)}{A_2 + A_3\tan(\zeta/2)}\right] - \frac{8\pi\bar{\varrho}_0}{15}(5 - 3x), \tag{6.14}$$

$$K = A_1[A_2\cos(\zeta/2) + A_3\sin(\zeta/2)]^{-1}, \tag{6.15}$$

$$\varrho = 2\pi K^2 + \bar{\varrho}_0\left(1 - \frac{r^2}{r_0^2}\right), \tag{6.16}$$

where A_2 and A_3 are constants to be fixed by the boundary conditions and

$$x = \frac{r^2}{r_0^2}, \quad \zeta = \log\left[x - \tfrac{5}{6} + \left(x^2 - \tfrac{5}{3}x + \frac{5}{8\pi r_0^2 \bar{\varrho}_0}\right)\right]. \tag{6.17}$$

The use of boundary conditions gives

$$A_2 = \left(1 - \frac{16\pi r_0^2 \bar{\varrho}_0}{15}\right)^{1/2} \cos\left(\tfrac{1}{2}\zeta_1\right) - \frac{2r_0}{3}\left(\frac{2\pi \bar{\varrho}_0}{5}\right) \sin\left(\tfrac{1}{2}\zeta_1\right), \tag{6.18}$$

$$A_3 = \left(1 - \frac{16\pi r_0^2 \bar{\varrho}_0}{15}\right)^{1/2} \sin\left(\tfrac{1}{2}\zeta_1\right) - \frac{2r_0}{3}\left(\frac{2\pi \bar{\varrho}_0}{5}\right) \cos\left(\tfrac{1}{2}\zeta_1\right), \tag{6.19}$$

$$A_1^2 = \frac{\varrho(r_0)}{2\pi}\left[A_2 \cos\left(\tfrac{1}{2}\zeta_1\right) + A_3 \sin\left(\tfrac{1}{2}\zeta_1\right)\right], \tag{6.20}$$

where

$$\zeta_1 = \log\left[\tfrac{1}{6} + \left(\frac{5}{8\pi r_0^2 \bar{\varrho}_0} - \tfrac{2}{3}\right)\right].$$

Therefore (6.13) and (6.15) go to the form

$$e^{\nu} = \left[\left(1 - \frac{16\pi r_0^2 \bar{\varrho}_0}{15}\right) \cos\left(\frac{\zeta_1 - \zeta}{2}\right) - \tfrac{2}{3} r_0 \frac{2\pi \bar{\varrho}_0}{5} \sin\left(\frac{\zeta_1 - \zeta}{2}\right)\right]^2, \tag{6.13a}$$

$$K = A_1\left[\left(1 - \frac{16\pi r_0^2 \bar{\varrho}_0}{15}\right) \cos\left(\frac{\zeta_1 - \zeta}{2}\right) - \tfrac{2}{3} r_0 \frac{2\pi \bar{\varrho}_0}{5} \sin\left(\frac{\zeta_1 - \zeta}{2}\right)\right]^{-1}. \tag{6.15a}$$

To get a real solution we must have

$$r_0^2 < \frac{15}{16\pi \bar{\varrho}_0} \quad \text{or} \quad \frac{m}{r_0} < \tfrac{1}{2}, \tag{6.21}$$

where

$$m = \frac{8\pi r_0^3}{15} \bar{\varrho}_0.$$

When we put $A_1 = 0$ this solution reduces to that of Mehra [21] in general relativity.

7. *Discussions and conclusions*

In general the dependence of the spin on the radial distance r is not determined in the absence of a magnetic field. This dependence can therefore be chosen arbitrarily. Prasanna [11] introduced an assumption (Eq. (2.11)) to determine the radial dependence of spin. In the present work also we have used the same assumption. Thus one may obtain for various choices of $K(r)$, an arbitrary number of solutions corresponding to each of the solutions of Tolman [16], Adler [15], Kuchowicz [17] and other solutions dealt with in this paper.

The family of solutions obtained in Section 4 of this paper, some of them though singular at the origin, may be useful in the investigation of massive stars. They allow to vary the equation of state in a continuous manner by changing the value of the parameter α. It is interesting to note that this is one of the rare families of interior solutions for relativistic fluid spheres in which none of the field variables is considered constant. Therefore in contrast with the other cases, these solutions may be of a better use as a model of non-rotating neutron stars.

If we consider the possible observable effects of the E–C theory for compact objects, a recent calculation by Kerlick [22] for the surface deformation in neutron stars seems to indicate that the torsion effect is extremely small in comparison with other types of deformations. The torsion effect becomes significant only when spin density begins to dominate mass density at the critical density

$$\varrho_N = 1.2\times 10^{54}\ \mathrm{g\,cm^{-3}} \quad \text{for neutrons.}$$

This is greater by far than any conceivable stellar density. The only possible occurrence of densities of the order of ϱ_N is in the early stages of the universe at the first split second of creation.

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