OCTONIONIC QUARK CONFINEMENT*

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1. Questions

Amongst the important questions one would like to answer are the following:

— Why are fractionally charged quarks not observed?

— Why does the color group SU(3) play such a special role?
Indeed, in the standard model [1], it is, together with the electric charge, the only exact symmetry.

— Why have leptons integer and quarks fractional charge?

The usual answer to the first question is confinement. This is achieved either by phenomenological bag models [2] or by invoking infrared slavery [3]. In both cases the answer is dynamical. But maybe one should look in a completely different direction. If one compares for example classical mechanics and quantum mechanics, one finds that the dynamics is the same, since the Hamiltonian, by the correspondence principle, is chosen to be the same. What changes is kinematics, i. e. the interpretation of states and observables. Passing from commuting to non commuting observables entails such profound changes as the uncertainty relations of Heisenberg or the quantization of energy. In usual quantum mechanics, states are described by vectors in a complex Hilbert space. The idea of Günaydin and Gürsey [4] was to generalize complex numbers to octonions (or Cayley numbers). In this way, they got two categories of states: usual ones with complex coefficients, which they identified with leptons, and new ones, with octonionic components, corresponding to quarks. Because of the non-associativity of octonionic multiplication, quark states would lack certain properties essential for observable particles. Furthermore, there arises naturally, from the properties of the octonion algebra, a group SU(3) under which complex numbers transform like singlets and the octonionic components as triplets and antitriplets. The consistency of the interpretation requires this group to be exactly conserved. The algebraic structure of this new mechanics suggests the use of exceptional groups for unifying symmetries of weak, electromagnetic and strong interactions. In such schemes [5] the charge of the electron is three times the charge of the down quark.

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2. *Beginnings*

Use of octonions in quantum mechanics was first suggested in 1932 by Jordan [6]. This attempt was not pursued, but gave rise to an extensive mathematical literature on the so-called Jordan algebra [7, 8]. We shall review this work in the following.

After the sixties, octonions made their appearance in physics again. Pais, Gamba and others tried to relate the octonions to various internal symmetry schemes in elementary particle physics [9]. Goldstine, Horwitz and Biedenharn studied a Clifford algebra made of octonionic multiplication operators [10]. As mentioned above, octonionic quantum mechanics was proposed again in Ref. [4]. A critical assessment of this work has been presented by Kosinski and Rembielinski [11]. Symmetry schemes based on exceptional groups are presented in Ref. [5]. It is clear that most of these attempts are still speculative, since it cannot be claimed yet that a complete and consistent octonionic mechanics exists. A preliminary question has been answered affirmatively by Günaydin, Piron and Ruegg: Can one construct an octonionic quantum mechanics which satisfies all axioms of usual one-particle quantum mechanics [12]. However, in our opinion, the important problem of multiparticle states has not yet been solved (see also Ref. [11]. Finally, a further generalization has been proposed by Gürsey [13].

3. *Octonions and Jordan matrices*

A general, real, octonion O can be written

$$O = r_0 e_0 + \sum_{A=1}^{7} r_A e_A, \tag{3.1}$$

where r_0, r_A are real numbers, e_0 the real unit, and e_A seven imaginary, anticommuting units:

$$e_A^2 = -1, \quad e_A e_B + e_B e_A = 0, \quad A \neq B \tag{3.2}$$

or, in general:

$$e_A e_B = -\delta_{AB} + \sum_C f_{ABC} e_C, \tag{3.3}$$

where f_{ABC} are completely antisymmetric symbols with $f_{123} = f_{246} = f_{435} = f_{367} = f_{651} = f_{572} = f_{714} = 1$. The multiplication is non associative, for example

$$(e_1 e_2) e_4 \neq e_1 (e_2 e_4).$$

One can define the associator

$$(O_1, O_2, O_3) \equiv (O_1 O_2) O_3 - O_1 (O_2 O_3). \tag{3.4}$$

The octonion algebra is alternative, which means that the associator is completely antisymmetric, and therefore equal to zero if two factors are equal. Conjugation is defined by

$$\bar{O} = r_0 e_0 - r_A e_A \tag{3.5}$$

and the norm by

$$n(O) = \bar{O}O. \tag{3.6}$$

Octonions share with real, complex and quaternion numbers the property that

$$n(O_1 O_2) = n(O_1)n(O_2). \tag{3.7}$$

The automorphism group of the octonion algebra is defined in the following way: consider the linear transformations on the imaginary units

$$e'_A = \sum_{B=1}^{7} S_{AB} e_B, \tag{3.8}$$

$$(e_A e_B)' = e_A{}' e_B{}'. \tag{3.9}$$

Then S belongs to the seven-dimensional fundamental representation of the fourteen parameter simple exceptional Lie group G2 [8].

The property of interest for us is that the subgroup which leaves one of the imaginary units (say e_7) invariant, is SU(3). The representation 7 reduces like:

$$7 = 1 \oplus 3 \oplus 3^*. \tag{3.10}$$

Jordan matrices [6] are hermitean 3×3 matrices over octonions:

$$J = \begin{pmatrix} \alpha_1 & O_3 & \bar{O}_2 \\ \bar{O}_3 & \alpha_2 & O_1 \\ O_2 & \bar{O}_1 & \alpha_3 \end{pmatrix}, \tag{3.11}$$

where α_i are real numbers and O_i octonions. These matrices form a non associative algebra under the product

$$J_1 \circ J_2 = \tfrac{1}{2}(J_1 J_2 + J_2 J_1). \tag{3.12}$$

This is called the exceptional Jordan algebra J_3^8. It satisfies the Jordan identity

$$(J_1 \circ J_2) \circ J_1^2 = J_1 \circ (J_2 \circ J_1^2). \tag{3.13}$$

Jordan, von Neumann and Wigner [7] found that this last equation could be fulfilled only by two kinds of algebras: special Jordan algebras which are matrices over associative division rings and the exceptional Jordan algebra J_3^8.

It was discovered by mathematicians [8] that the automorphism group of J_3^8 is the exceptional, 52 parameter simple Lie group F4. It acts on the 26 traceless, hermitean Jordan matrices, leaving the unit matrix I invariant. It has two invariants

$$(J_1, J_2) = \mathrm{tr}\,(J_1 \circ J_2), \tag{3.14}$$

$$[J_1, J_2, J_3] = \mathrm{tr}\,((J_1 \circ J_2) \circ J_3), \tag{3.15}$$

where the Freudenthal product [14]

$$J_1 \times J_2 = J_1 \circ J_2 - \tfrac{1}{2} J_1 \operatorname{tr} J_2 - \tfrac{1}{2} J_2 \operatorname{tr} J_1 + \tfrac{1}{2} I (\operatorname{tr} J_1 \operatorname{tr} J_2 - \operatorname{tr} J_1 \circ J_2) \tag{3.16}$$

defines the completely symmetric trilinear form (3.15).

4. *Complex Hilbert space and its octonionic generalization*

In usual quantum mechanics over complex Hilbert space $\mathscr{H}$ a physical state is given by the ray

$$c|\alpha\rangle \in \mathscr{H}, \qquad |c| = 1, \qquad c \in \mathscr{C}. \tag{4.1}$$

It obeys the superposition principle

$$|\alpha\rangle, |\beta\rangle \in \mathscr{H} \Rightarrow c_1|\alpha\rangle + c_2|\beta\rangle \in \mathscr{H}. \tag{4.2}$$

Probabilities are determined by the scalar product:

$$\Pi_{\alpha\beta} = |\langle\alpha|\beta\rangle|^2, \qquad \langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1. \tag{4.3}$$

Orthogonal states satisfy

$$\langle\alpha|\beta\rangle = 0 \Rightarrow \langle\alpha|c|\beta\rangle = 0. \tag{4.4}$$

A change of basis is given by

$$|\alpha\rangle = (\sum_\beta |\beta\rangle\langle\beta|)\,|\alpha\rangle = \sum_\beta |\beta\rangle(\langle\beta|\alpha\rangle) = \sum_\beta c_\beta|\beta\rangle. \tag{4.5}$$

If now, in (4.1), we replace the complex number c by the octonion O, then certain properties are lost. For example, (4.4) is not always true:

$$\langle\alpha|\beta\rangle = 0 \text{ does not imply } \langle\alpha|(O|\beta\rangle) = 0. \tag{4.6}$$

Also, because of non-associativity, (4.5) is not always fulfilled. From this it follows that the Hilbert space formulation of quantum mechanics breaks down. One can try several ways out. Here we describe the formulation of Günaydin and Gürsey [4], leaving for Section 6 the work of Günaydin, Piron and Ruegg [12].

The general octonion (3.1) can also be written, using (3.3),

$$O = (r_0 e_0 + r_7 e_7) + \sum_{i=1}^{3} (r_i + r_{i+3} e_7) e_i = c_0(e_7) + \sum_{i=1}^{3} c_i(e_7) e_i \tag{4.7}$$

c_0 and c_i are complex numbers over the imaginary unit e_7. We now consider the direct sum of four complex Hilbert spaces [10], [11], where the wave-functions have respectively the components c_0 and $c_i e_i$. The scalar product is the complex number [15]

$$(O^{(1)}, O^{(2)})c = c_0^{*(1)} c_0^{(2)} + \sum_i c_i^{(1)} c_i^{*(2)}, \qquad c_0^*(e_7) = c_0(-e_7), \qquad c_i^*(e_7) = c_i(-e_7). \tag{4.8}$$

This scalar product is invariant under the group U(4), whose intersection with the automorphism G2 is again SU(3), the subgroup which leaves e_7 invariant. If we identify e_7 with the usual i of quantum mechanics, it is clear that SU(3) has to be exactly conserved.

The four complex Hilbert spaces are related to each other by octonionic multiplication. This can also be implemented by 4×4 matrices, which, together with the SU(3) operations, form again the group U(4). Hence, the "octonionic space" is just the sum of four complex spaces with additional algebraic structure. Günaydin and Gürsey identify one-particle wave-functions having components c_0 with leptons, and those having components $c_i e_i$ with quarks and $c_i^* e_i$ with anti-quarks. The nice feature is that one can define quark--antiquark and three quark wave-functions which are complex numbers, and could be identified with ordinary hadrons [15]. First define

$$O^{(1)} * O^{(2)} = \tfrac{1}{2}\left[(O^{(1)}O^{(2)})e_7 - O^{(1)}(O^{(2)}e_7)\right], \tag{4.9}$$

$$O^{(1)} \circ O^{(2)} = \tfrac{1}{2}\left[(O^{(1)}O^{(2)})e_7 + O^{(1)}(O^{(2)}e_7)\right]. \tag{4.10}$$

Then

$$(c_i^{(1)}e_i) \circ (c_j^{(2)*}e_j) = -c_i^{(1)}c_i^{(2)}e_7, \tag{4.11}$$

$$(c_i^{(1)}e_i) \circ \left[(c_j^{(2)}e_j) * (c_k^{(3)}e_k)\right] = -\varepsilon_{ijk}c_i^{(1)}c_j^{(2)}c_k^{(3)}. \tag{4.12}$$

(4.11) and (4.12) no longer contain the "quark" imaginary units e_i, and are related to Clebsch-Gordon products

$$3 \otimes 3^* \to 1, \quad 3 \otimes 3 \otimes 3 \to 1.$$

However, it seems doubtful that they can be identified with quantum mechanical tensor products [10, 11].

5. *Jordan algebraic formulation of quantum mechanics and projective geometry*

Instead of representing a physical state (4.1) by a ray $c|\alpha\rangle$, it is equivalent to use a projection operator

$$P_\alpha = |\alpha\rangle\langle\alpha| = P_\alpha^2 = c|\alpha\rangle\langle\alpha|c^*, \qquad \mathrm{tr}\, P_\alpha = \mathrm{tr}\, |\alpha\rangle\langle\alpha| = \langle\alpha|\alpha\rangle = 1. \tag{5.1}$$

The hermiticity property as well as the algebraic definition of P_α

$$P_\alpha^2 = P_\alpha \tag{5.2}$$

is preserved by the Jordan product (3.12)

$$A \circ \mathrm{B} = \tfrac{1}{2}(\mathrm{AB}+\mathrm{BA}). \tag{5.3}$$

Probability (4.3) is given by

$$\Pi_{\alpha\beta} = \mathrm{tr}\, P_\alpha \circ P_\beta, \tag{5.4}$$

$$\Pi_{\alpha\beta} = \mathrm{tr}\, |\alpha\rangle\langle\alpha|\beta\rangle\langle\beta| = |\langle\alpha|\beta\rangle|^2. \tag{5.5}$$

Jordan [6] noticed that all observable quantities such as probabilities and expectation values can be expressed in terms of anticommutators (5.3) of operators. The algebra of all observables is a Jordan algebra satisfying (3.13). Of course, if one limits one-self to operators over complex numbers, i.e. to special Jordan algebras, one gets a formulation which is equivalent to complex Hilbert space quantum mechanics, as exemplified by (5.1) and (5.5).

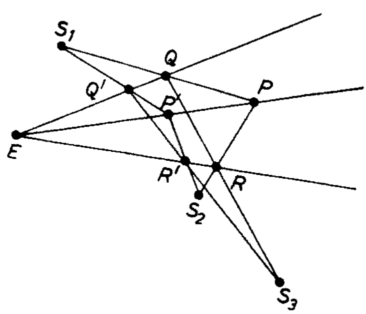
However, one may get something new if one chooses the exceptional Jordan algebra (3.11), keeping only the algebraic definitions (5.2) and (5.4). The discussion of paragraph 4 shows that there is no underlying Hilbert space if full use of the octonion algebra is made, which is the case in the definition (5.4), P_α belonging to (3.11). Then arises the question of the validity of the axioms of quantum mechanics in this case. This was not systematically done by Jordan. Clearly, before trying to find an answer, one has to formulate the quantum axioms in a language which does not make use of Hilbert space. Such a language was found by Birkhoff and von Neumann [16], who used proposition calculus and projective geometry. This approach was further elaborated in Ref. [17]. Its relation to the Jordan algebraic method is discussed in Ref. [18].

Given an $(n+1)$ dimensional vector space H $(n > 2)$ over an associative division ring K one can give it the structure of a projective space of dimension n by associating with every ray $|\alpha > q$ of H a point represented by the projection operator

$$P_\alpha = |\alpha\rangle q\bar{q}\langle\alpha| = |\alpha\rangle\langle\alpha|, \qquad q\bar{q} = 1, \qquad q \in K. \tag{5.6}$$

Then the subspace spanned by two orthogonal vectors $|\alpha\rangle$ and $|\beta\rangle$ will be associated to the line $P_\alpha + P_\beta$ passing through P_α and P_β. Similarly, one will get a projective plane from three orthogonal vectors, and so on. A linear superposition of the vectors $|\alpha\rangle$ and $|\beta\rangle$ will correspond to a point on the line joining the points P_α and P_β. The superposition principle of quantum mechanics just means that all points on this line correspond to physical states, if P_α and P_β are physical states.

Finally, Birkhoff and von Neumann remarked that projection operators have only eigenvalues 1 and 0, which corresponds to yes-no experiments (propositions). Any "com-



plete" experiment can be decomposed in such yes-no experiments. So they were able to show the equivalence of the algebra of projection operators, the associated projective geometry and the quantum mechanical properties of yes-no experiments. The theorem of interest to us is that any projective geometry of dimension $n > 2$ satisfies Desargue's theorem and can be represented by a $n+1$ dimensional vector space over an associative division ring K. Desargue's theorem states that the points S_1, S_2 and S_3 lie on a straight line:

On the other hand, Moufang [19] gave an example of a projective plane ($n = 2$) coordinatized by octonions which does not satisfy Desargue's theorem. Jordan [20] gave without proof an algebraic construction equivalent to the Moufang plane. This was independently rediscovered by Freudenthal and studied in great detail by him and Springer [21] (see also Jacobson [7]).

Günaydin, Piron and Ruegg [12] showed by elementary means that the axioms of one-particle quantum mechanics are satisfied by the Jordan construction [20]. In particular, they showed that

a) The axioms of the projective plane are satisfied and there exist non Desarguean configurations.
b) The plane is orthocomplemented.
c) One can define a unique probability function satisfying Gleason's axioms [22].
d) One recovers the usual quantum theory of measurements.

These results will now be summarized in the following section.

6. *The Moufang plane*

Jordan [20] showed that the most general one-dimensional projection operator belonging to the exceptional Jordan algebra J_3^8 is

$$P = \begin{pmatrix} a \\ b \\ c \end{pmatrix} (\bar{a}\,\bar{b}\,\bar{c}) = \begin{pmatrix} a\bar{a} & a\bar{b} & a\bar{c} \\ b\bar{a} & b\bar{b} & b\bar{c} \\ c\bar{a} & c\bar{b} & c\bar{c} \end{pmatrix}, \tag{6.1}$$

where a, b, c are octonions, one of them being real, and satisfy

$$\operatorname{tr} P = a\bar{a}+b\bar{b}+c\bar{c} = 1. \tag{6.2}$$

Using the alternativity property of the associator (3.4)

$$(aa)b = a(ab), \quad (a\bar{a})b = a(\bar{a}b), \tag{6.3}$$

from which follows the Moufang identity

$$a(bc)a = (ab)(ca) \tag{6.4}$$

one finds

$$P^2 = P. \tag{6.5}$$

P has the property that the Freudenthal product (3.16) vanishes:

$$P \times P = 0. \tag{6.6}$$

One can form a two-dimensional projection operator l with the Freudenthal product of two one-dimensional projectors P_1 and P_2:

$$l_{12} = I - \frac{P_1 \times P_2}{\operatorname{tr} P_1 \times P_2}, \tag{6.7}$$

$$l_{12}^2 = l_{12}, \quad \operatorname{tr} l_{12} = 2, \tag{6.8}$$

where I is the 3×3 unit matrix, with the important property

$$P_i \circ l_{12} = P_i, \quad i = 1, 2. \tag{6.9}$$

Indeed, from the symmetry of (3.15), it follows:

$$\operatorname{tr} (P_1 \times P_2) \circ P_1 = \operatorname{tr} (P_1 \times P_1) \circ P_2, \quad \operatorname{tr} (I - l) \circ P_1 = 0. \tag{6.10}$$

One can prove the lemma

$$\operatorname{tr} P \circ Q = 0 \Leftrightarrow P \circ Q = 0 \tag{6.11}$$

for one-dimensional projection operators P and Q. This shows that (6.10) implies (6.9).

We are now in a position to give the Jordan construction of the projective Moufang plane:

— points are represented by one-dimensional projectors P,
— lines are represented by two-dimensional projectors l,
— the plane is represented by the unit matrix I.

A point P is on a line l if and only if

$$P \circ l = P. \tag{6.12}$$

This shows that (6.7) is the line passing through the points P_1 and P_2. The intersection of the two lines I–P_1 and I–P_2 is the point P

$$P = \frac{P_1 \times P_2}{\operatorname{tr} P_1 \times P_2}. \tag{6.13}$$

In order to proceed, we need the exceptional group F4. Indeed, this is the automorphism group of the Jordan algebra, with the two invariants (3.14) and (3.15). It is also an automorphism group for the Freudenthal product. Hence, it is an automorphism group for the Jordan construction of the projective plane: points are transformed into points, lines into lines, and the relation $P \circ l = P$ into $P' \circ l' = P'$. This allows to simplify the one-dimensional projection operator P (6.1). Let us define

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.14}$$

Then the following lemmas can be proved [12].

Lemma 1: There exists always a transformation belonging to F4 which brings P given by (6.1) into the form E_1.
Lemma 2: Given P_1, P_2, P_3 such that $P_1 \circ P_2 = P_2 \circ P_3 = P_3 \circ P_1 = 0$, then there exists always a transformation of F4 which brings them to the form E_1, E_2, E_3.
Lemma 3: Given *any* two P_1 and P_2, then there exists always a transformation of F4 which brings them simultaneously into a real form.
Lemma 4: $\mathrm{tr}\, P_1 \circ P_2 = 0$ implies $P_1 \circ P_2 = 0$.
Lemma 5: Given two different P_1 and P_2, then P_3 satisfies $P_3 \circ P_1 = P_3 \circ P_2 = 0$ if and only if P_3 is a multiple of $P_1 \times P_2$.

Finally, we have a more general
Lemma 6: Any element of J_3^8 can be brought to diagonal form by an F4 transformation.

With the aid of these lemmas one can show that the Jordan construction satisfies the axioms of the projective plane:
1) Any two distinct points are contained in one and only one line.
2) The intersection of any two distinct lines is one point.
3) There exists four points no three of which are in the same line.

Explicit examples of non Desarguean configurations can be given [12].

7. *Orthocomplementation of the Moufang plane and compatible measurements*

In usual Hilbert space quantum mechanics, given a vector v there exists uniquely a subspace ε orthogonal to v such that the union of ε and v is the whole space. ε is called the orthocomplement of v, and denoted v'. More generally, it is clear that if $\varepsilon_1 \subset \varepsilon_2$, than $\varepsilon_1' \supset \varepsilon_2'$, and that $(\varepsilon')' = \varepsilon$. This concept is necessary for notions such as: eigenvectors belonging to different eigenvalues are orthogonal, projection operators on orthogonal spaces commute, i. e. are compatible, etc.

In the projective geometry language we consider linear varieties L (points, lines, etc.) and require for the orthocomplement L' of L:

$$(L')' = L, \qquad L_1 \supset L_2 \Rightarrow L_1' \subset L_2'. \tag{7.1}$$

Birkhoff asked the question: Can one orthocomplement a non Desarguean geometry? [23]. The answer for the Moufang plane is yes. Define orthogonality by

$$P' \perp P_2 \Leftrightarrow P_1 \circ P_2 = 0. \tag{7.2}$$

Then the orthocomplement is defined by

$$P_1 = I - P, \qquad l' = I - l. \tag{7.3}$$

Clearly, if P is a point, P' is a line, and if l is a line, l' is a point. Furthermore, P is not on the line P' ($P \circ P' = 0 \neq P$). The second implication (7.1) is also valid: if P is on the line l, l' is on the line P'. Indeed, $P \circ l = P'$ implies

$$P' \circ l' = (I-P) \circ (I-l) = I - P - l + P = l'. \tag{7.4}$$

If P_1 is orthogonal to P_2, then

$$P_1 \times P_2 = \tfrac{1}{2}(I-P_1-P_2)$$

and the line l_{12} through P_1 and P_2 is

$$l_{12} = P_1+P_2. \tag{7.5}$$

The orthocomplement of l_{12} is

$$P_3 = I-l_{12} = I-P_1-P_2. \tag{7.6}$$

Hence the plane contains three orthogonal points. Lemma 2 shows that they can be brought to the form E_1, E_2, E_3 (see Eq. (6.14)).

Finally, we discuss the important concept of compatible proposition, using the definition of Ref. [17]. L_1 and L_2 are compatible if, for example, $L_1 \subset L_2$, or if $L_1 \circ L_2 = 0$. More generally, L_1 ban be written as the sum of two orthogonal projectors P_1 and P_2, P_1 being on L_2 and P_2 orthogonal to L_2

$$L_1 = P_1+P_2, \quad P_1 \circ L_2 = P_1, \quad P_2 \circ L_2 = 0. \tag{7.7}$$

Consider two lines l_1 and l_2 (tr $l_i = 2$). Then, because of lemma 2, we can always transform l_2 to the form

$$l_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.8}$$

The most general solution of (7.7) is then

$$l_1 = \begin{pmatrix} a\bar{a} & a\bar{b} & 0 \\ b\bar{a} & b\bar{b} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.9}$$

l_1 and l_2 commute.

8. The probability function, measurements and observables

Suppose the system is in the state given by the one-dimensional projector P and we want to measure the proposition L (point or line). Then the probability is given by the unique function:

$$W_P(L) = \text{tr}\,(P \circ L). \tag{8.1}$$

Satisfying Gleason's axioms [22]

$$0 \leqslant W_P(L) \leqslant 1, \tag{8.2}$$

2)
$$W_P(P) = 1, \tag{8.3}$$

3)
$$W_P(L_1 \cup L_2) = W_P(L_1)+W_P(L_2), \quad L_1 \perp L_2. \tag{8.4}$$

(8.4) means that the probability of the union of two othogonal projectors is just the sum. The proof of (8.2) to (8.4) requires again the lemmas. The proof of the uniqueness is based on the fact that the Moufang plane contains a real projective subgeometry, the orthogonality condition $P_1 \circ P_2 = 0$ (Jordan product) being equivalent to $P_1 P_2 = 0$ (ordinary product). It was also shown in Ref. [12] that the result of successive, compatible measurements is independent of the order of the measurements, in spite of the non-associativity of octonion multiplications.

We now come to the question of observables. First, each projection operator defines an observable. In general, in the Moufang plane, an observable is defined by three one--dimensional projections. They can always be brought by an F4-transformation to the form E_1, E_2 and E_3 (lemma 2). The question arises if any Jordan matrix can be written as a linear superposition of three mutually orthogonal one-dimensional projection operators and interpreted as an observable. The answer is yes, since any element of J_3^8 can be diagonalized by an F4-transformation (lemma 6). In order to get a Schrödinger equation, we need a time evolution operator. This should transform orthogonal states into orthogonal states and hence belong to the automorphism group of the orthocomplemented Moufang plane, which is F4. Some generator in the Lie algebra of F4 will play the role of the Hamiltonian.

The subalgebra of complex Jordan matrices can be used to construct a Desarguean projective subgeometry of the Moufang geometry. The corresponding quantum mechanics can be realized in a tree-dimensional Hilbert space. The subgroup of F4 which leaves the complex subgeometry invariant is $SU(3)^c$. Jordan matrices whose elements comprise the six remaining octonionic units transform as 3 and $\bar{3}$ under $SU(3)^c$. More precisely, under the maximal subgroup $SU(3)^c \otimes SU(3)$ the representation $\underline{26}$ of F4 reduces as

$$\underline{26} = (3^c, 3) \oplus (\bar{3}^c, \bar{3}) \oplus (1^c, 8). \tag{8.5}$$

Hence, it is tempting to identify $SU(3)^c$ with the color group, complex Jordan matrices with lepton states, and the other with quark and antiquark states [24] (note that a projection operator belongs to the reducible representation $27 = 26 \oplus 1$). Since a non-Desarguean projective plane cannot be embedded in a irreducible projective geometry of higher dimensions, this would mean that quarks have no space properties, these latter requiring an infinite dimensional geometry. On the other hand, the lepton subgeometry can hopefully be embedded in a consistent way in such an infinite geometry. However, the definition of a consistent tensor product of quark states is still an open problem.

9. Discussion

We now have two different models of octonionic quantum mechanics. The first (Ref. [4, 10, 11]) is a direct sum of four complex Hilbert spaces, linked by octonionic multiplication. The second (Ref. [12], see also Ref. [13]) seems to us more genuinly octonionic. It allows only three degrees of freedom, and this may be linked to the non observability of quarks. In both cases, there exists a privileged, exactly conserved SU(3) group, and a natural division of states into a lepton and a quark sector.

10. Exceptional groups and symmetry schemes

We have seen that the exceptional groups G2 and F4 are the automorphism groups of the octonion, respectively the exceptional Jordan algebra. Furthermore, F4 is the automorphism group of the quantum mechanics defined by the Moufang plane. Therefore, one may ask what symmetry schemes arise from these groups. Of course, this makes only sense if the problem of the tensor product has been solved. Gürsey [13] has even gone further and speculated on the relation between the exceptional group E6 and E7 and quadratic Jordan algebras. The relation with quantum mechanics is, however, obscure so far. One can of course also consider exceptional groups in the context of conventional, complex valued, gauge field theory [5]. The aim is to find a spontaneously broken symmetry group for weak, electromagnetic and strong interactions.

All five exceptional groups have a maximal subgroup containing SU(3) as a factor, the other factor being a candidate for the flavor group, which should at least contain SU(2) ⊗ U (1).

$$G2 \supset SU(3), \quad F4 \supset SU(3) \otimes SU(3), \quad E6 \supset SU(3) \otimes SU(3) \otimes SU(3),$$
$$E7 \supset SU(3) \otimes SU(6), \quad E8 \supset SU(3) \otimes E6. \tag{10.1}$$

Obviously, G2 is too small. Under (10.1), the fundamental representation decomposes in the following way:

$$G2 : 7 = 3^c \oplus \bar{3}^c \oplus 1, \quad F4 : 26 = (3^c, 3) \oplus (\bar{3}^c, \bar{3}) \oplus (1^c, 8),$$
$$E6 : 27 = (3^c, 3, 1) \oplus (\bar{3}^c, 1, \bar{3}) \oplus (1^c, \bar{3}, 3), \quad E7 : 56 = (3^c, 6) \oplus (\bar{3}^c, \bar{6}) \oplus (1^c, 20),$$
$$E8 : 248 = (8^c, 1) \oplus (1^c, 78) \oplus (3^c, 27) \oplus (\bar{3}^c, \overline{27}). \tag{10.2}$$

The fundamental representation should accomodate the elementary Fermions, including quarks and leptons. E8 contains a color octet of Fermions, which is not wanted so far.

The adjoint representation, where one would like to put the vector Bosons, decomposes as

$$G2 : 14 = 8^c \oplus 3^c \oplus \bar{3}^c, \quad F4 : 52 = (8^c, 1) \oplus (1^c, 8) \oplus (3^c, 6) \oplus (\bar{3}^c, \bar{6}),$$
$$E6 : 78 = (8^c, 1, 1) \oplus (1^c, 8, 1) \oplus (1^c, 1, 8) \oplus (3^c, \bar{3}, \bar{3}) \oplus (\bar{3}^c, 3, 3),$$
$$E7 : 133 = (8^c, 1) \oplus (1^c, 35) \oplus (3^c, 15) \oplus (\bar{3}^c, \overline{15})$$
$$E8 : 248 = (8^c, 1) \oplus (1^c, 78) \oplus (3^c, 27) \oplus (\bar{3}^c, \overline{27}). \tag{10.3}$$

E8 has the amusing feature that the vector Bosons belong to the fundamental representation, which could be interesting for supersymmetries.

Limiting ourselves to F4, E6 and E7, we see that the fundamental representation contains 3 or 6 quarks, 8, 9 or 20 leptons. The adjoint representation contains vector Bosons which couple quarks and leptons. In order to avoid the decay of the proton into leptons, one has to break the symmetry and give a very high mass to these Bosons (10^{18} GeV!). This, in turn, will renormalize the coupling constants, after the symmetry has been further broken down to $SU(3)^c \otimes U(1)$ [25].

The representations of F4 and E7 are real (or pseudoreal), which leads naturally to vector-like theories [26]. Present experimental data on neutrino scattering put severe limits on the coupling to right handed protons. On the other hand, in E6 $27 \neq \overline{27}$, so that one could have a 27-plet of left-handed particles. The predictions [5] for the Weinberg angle is 3/4 for E7 and F4, 3/8 for E6. Renormalization effects discussed above could bring this down to 0.5, respectively 0.25. Again, E6 is rather favoured by experiment.

The Higgs structure for breaking the symmetry is very complicated for E7 [27], whereas it could be much simpler for E6 [28]. The particle structure of the E6 model is the following: one left-handed 27-plet with u, d, b quarks, $\bar{u}$, $\bar{d}$, $\bar{b}$ antiquarks, four charged leptons ($e^{\pm}$, $\tau^{\pm}$) and five neutral leptons. Another 27-plet would contain s, c and b′ quarks, $\mu^{\pm}$ and some new leptons. A Higgs particle belonging to 78 would give a very high mass to the Bosons in (3^c, $\bar{3}$, $\bar{3}$), ($\bar{3}^c$, 3, 3) and the strange Bosons in (1^c, 8, 1) and (1^c, 1, 8), breaking E6 down to $SU(3)^c \otimes SU(2) \otimes U(1) \otimes SU(2) \otimes U(1)$. Two or more Higgs particles in 27 leave only $SU(3)^c \otimes U(1)$ as exact symmetries. One general feature of this breaking is that the neutrino can have zero mass only if at least one quark and one charged lepton in the same 27-plet have zero mass [29]. One also finds that the τ-neutrino has about the same mass as the τ, unless the τ life-time is two orders of magnitude different from the simple $V-A$ calculation.

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