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THE GEOMETRODYNAMICAL SUPERHAMILTONIAN AND SUPERMOMENTUM OF THE DIRAC FIELD IN CURVED SPACETIME

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The superhamiltonian and the supermomentum are constructed for the Dirac field in curved spacetime and it is shown that field dynamics and kinematics of spacelike hypersurfaces are consistent.

1. Introduction: Field dynamics in hyperspace

Since the investigations by Dirac, Tomonaga and Schwinger it is well known that the definition of canonical commutation relations and quantum states on curved spacelike hypersurfaces is useful even in the Minkowski spacetime. In a general Riemannian background this procedure is not only useful, but unavoidable because of the nonexistence of flat hypersurfaces.

A hypersurface Σ (normal vector n^μ, $n_\mu n^\mu = -1$) is given, if we know in which points X^μ of spacetime the points x^a of the hypersurface are situated[1],

$$\Sigma : X^\mu = X^\mu(x^a).$$

Two points with the same intrinsic coordinates x^a on two neighboured hypersurfaces Σ and Σ' are connected by an infinitesimal deformation $\delta X^\mu(x^a)$ which is composed of a normal deformation δX and a tangential one, δX^a, according to [1], [6]

$$\delta X^\mu = \delta X \cdot n^\mu + \delta X^a X^\mu_a \quad (X^\mu_a \equiv X^\mu_{|a}).$$

The intrinsic metric of the hypersurface ($\overset{(3)}{g}_{ab}$) and the metric of spacetime are connected by

$$\overset{(3)}{g}_{ab} = X^\mu_a X^\nu_b g_{\mu\nu} \quad \text{and} \quad g^{\mu\nu} = X^\mu_a X^\nu_b \overset{(3)}{g}{}^{ab} - n^\mu n^\nu.$$

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[1] Indices: $\mu, \nu = 1...4$; $a, b = 1...3$.

If we fix the time coordinate in such a manner that the hypersurfaces Σ are described by

$$\Sigma : t = \text{const}, \qquad n_\mu = n\delta_\mu^4 \quad (n = (-g^{44})^{-1/2}), \tag{1.1}$$

we have still freedom to perform kinemetrical transformations

$$x^{a'} = a^{a'}(x^b, x^4), \qquad x^{4'} = x^{4'}(x^4), \tag{1.2}$$

i.e. to transform the spatial coordinates on each hypersurface for themselves alone and to change the parametrisation of the hypersurfaces.

The dynamical problem of field theory consists in the investigation of the behaviour of the fields U_Ω and the canonically conjugate momenta π^Ω prescribed on an initial hypersurface Σ if this hypersurface is deformed through a Riemannian spacetime. The physical result, i.e. the change of U_Ω and π^Ω caused by the evolution from an initial to a final hypersurface, must be independent of the special sequence of hypersurfaces between the initial and final ones. From this principle of "path independence" of deformations in [1] and [6] the following Poisson brackets have been derived for fields the Lagrangian of which is of the form

$$\mathscr{L} = \mathscr{L}(U_\Omega, U_{\Omega|\sigma}, g_{\mu\nu}): \tag{1.3}$$

$$[\mathscr{T}(x), \mathscr{T}(\bar{x})] = [\mathscr{T}^a(x) + \mathscr{T}^a(\bar{x})]\delta_{|a}(x, \bar{x}), \tag{1.4a}$$

$$[\mathscr{T}_a(x), \mathscr{T}_b(\bar{x})] = \mathscr{T}_a(\bar{x})\delta_{|b}(x, \bar{x}) + \mathscr{T}_b(x)\delta_{|a}(x, \bar{x}), \tag{1.4b}$$

$$[\mathscr{T}^a(x), \mathscr{T}(\bar{x})] = \mathscr{T}(x)\delta^{|a}(x, \bar{x}) + 2\left[\frac{\delta\mathscr{T}(\bar{x})}{\delta g_{ab}(x)}\right]_{|b}. \tag{1.4c}$$

The validity of these Poisson brackets expresses the consistency of field dynamics and hypersurface deformations. The symbols used in (1.4) are constructed from the energy momentum tensor as follows:

$$\mathscr{T} = n^\mu \mathscr{T}_\mu, \qquad \mathscr{T}_a = X_a^\mu \mathscr{T}_\mu, \qquad \mathscr{T}_{ab} = \sqrt{\overset{(3)}{g}}\, T_{ab}$$

with

$$\mathscr{T}_\mu = \sqrt{\overset{(3)}{g}}\, T_\mu^\nu n_\nu \quad \text{and} \quad T^{\mu\nu} = -\frac{2}{\sqrt{\overset{(4)}{g}}}\frac{\delta\mathscr{L}}{\delta g_{\mu\nu}}.$$

These Poisson brackets play such a central role in the whole theory that we can postulate according to Schwinger [5]: $\mathscr{T}$ and $\mathscr{T}_a$ are to be constructed from the fields and the canonically conjugate momenta in such a way that the relations (1.4) are fulfilled. This must be valid for fields the Lagrangian of which has a more complicated structure than (1.3), too. An example is the Dirac field the Lagrangian of which contains derivatives of the metric tensor. In what follows we intend to determine the correct dependence of $\mathscr{T}$ and $\mathscr{T}_a$ from the canonically conjugate variables of the Dirac field.

2. *Elements of the kinemetrically invariant Dirac theory*

In [2] we formulated the Dirac theory with the aid of kinemetrical invariants, i.e. quantities which are invariant under kinemetrical transformations (1.2). We got the following main results:

a) The suitably defined spatial and time derivatives of the Dirac matrices are not covariantly constant in a kinemetrically invariant manner, explicitly

$$\gamma_{a\|b} = -\gamma K_{ab} \tag{2.1a}$$

$$\pounds\gamma_a = -\frac{n_{|a}}{n}\gamma + \overset{(3)}{\gamma}{}^b K_{ab}, \tag{2.1b}$$

$$\gamma_{\|a} = -\overset{(3)}{\gamma}{}^b K_{ab} \tag{2.1c}$$

$$\pounds\gamma = -\frac{n_{|a}}{n}\overset{(3)}{\gamma}{}^a. \tag{2.1d}$$

In these formulae the symbols "$\|$" and $\pounds \equiv -\underset{n^\mu}{\pounds}$ are used for spatial and time derivatives, respectively, the application of which to kinemetrical invariants generates kinemetrical invariants again (concerning the more detailed definition see [2]). Moreover the relation $\pounds g_{ab} = 2K_{ab}$ (extrinsic curvature of the hypersurface) is valid and the abbreviations

$$\gamma \equiv n_\mu \gamma^\mu \quad \text{and} \quad \overset{(3)}{\gamma}{}^a \equiv \overset{(3)}{g}{}^{ab}\gamma_b$$

are used. If we construct the γ_μ from the constant Dirac matrices $\gamma_{(\nu)}$ of the Minkowski spacetime with the aid of tetrad fields $\lambda_\mu^{(\nu)}$,

$$\gamma_\mu = \lambda_\mu^{(\nu)}\gamma_{(\nu)},$$

and fix the tetrad fields according to the choice (1.1) of the coordinate system in such a way that

$$\lambda_\mu^{(4)} = n_\mu, \quad n_\mu \lambda^{(a)\mu} = 0$$

holds, we get for the bispinorial connexion coefficients the expressions

$$\Gamma_a = \tfrac{1}{4}\gamma\overset{(3)}{\gamma}{}^b K_{ab} \tag{2.2a}$$

$$\Gamma \equiv n^\mu \Gamma_\mu = \tfrac{1}{4}\frac{n_{|a}}{n}\overset{(3)}{\gamma}{}^a\gamma. \tag{2.2b}$$

b) From the Lagrangian

$$\mathscr{L} = -n\sqrt{\overset{(3)}{g}}\,\frac{\hbar c}{2}\left[\bar\psi\left(\overset{(3)}{\gamma}{}^a\psi_{\|a} + \gamma\pounds\psi + \frac{m_0 c}{\hbar}\psi\right) - \left(\bar\psi_{\|a}\overset{(3)}{\gamma}{}^a + (\pounds\bar\psi)\gamma - \frac{m_0 c}{\hbar}\bar\psi\right)\psi\right] \tag{2.3}$$

the kinemetrically invariant Dirac equation results in

$$\overset{(3)}{\gamma^a}\psi_{\|a}+\gamma\pounds\psi+\frac{m_0c}{\hbar}\psi=0 \tag{2.4a}$$

by the usual variational method. The adjoint Dirac equation is

$$\bar{\psi}_{\|a}\overset{(3)}{\gamma^a}+(\pounds\bar{\psi})\gamma-\frac{m_0c}{\hbar}\bar{\psi}=0, \tag{2.4b}$$

where the adjoint bispinor is defined by

$$\bar{\psi}=\psi^\dagger\beta \quad \text{and} \quad \beta_{\|a}=\pounds\beta=0.$$

3. *The superhamiltonian and the supermomentum of the Dirac field*

In order to construct the superhamiltonian and the supermomentum of the Dirac field such that they fulfil the Poisson brackets (1.4) we start with the components of the energy momentum tensor [3]

$$\mathscr{T}=-\frac{\hbar c}{2}\sqrt{\overset{(3)}{g}}\,[\bar{\psi}\gamma\pounds\psi-(\pounds\bar{\psi})\gamma\psi],$$

$$\mathscr{T}_a=-\frac{\hbar c}{4}\sqrt{\overset{(3)}{g}}\,[\bar{\psi}(\gamma\psi_{\|a}-\gamma_a\pounds\psi)-(\bar{\psi}_{\|a}\gamma-(\pounds\bar{\psi})\gamma_a)\psi],$$

$$\mathscr{T}_{ab}=-\frac{\hbar c}{4}\sqrt{\overset{(3)}{g}}\,[\bar{\psi}(\gamma_a\psi_{\|b}+\gamma_b\psi_{\|a})-(\bar{\psi}_{\|b}\gamma_a+\bar{\psi}_{\|a}\gamma_b)\psi].$$

To give these expressions the correct structure we accomplish the following manipulations:

a) We express the kinemetrically invariant time derivatives by spatial ones with the aid of the Dirac equation (2.4).

b) We introduce the canonically conjugate momenta to the field ψ, which from (2.3) result in

$$\pi=-\frac{\hbar}{2}\sqrt{\overset{(3)}{g}}\,\bar{\psi}\gamma. \tag{3.1}$$

c) We arrange a maximum symmetry in the fields ψ, $\psi^\dagger$ and the momenta π, $\pi^\dagger$ (see [4]) and symmetrize products $\gamma_a\gamma_b$ of Dirac matrices.

d) With the aid of $\psi_{\|a}=\psi_{|a}+\Gamma_a\psi$, $\bar{\psi}_{\|a}=\bar{\psi}_{|a}-\bar{\psi}\Gamma_a$ and (2.2) we substitute kinemetrically invariant spatial derivatives by partial ones.

Having carried out these manipulations we get

$$\mathscr{T} = \frac{c}{2}\Big[\pi\gamma \overset{(3)}{\gamma^a}\psi_{|a} - \psi^\dagger\gamma \overset{(3)}{\gamma^a}\pi_{|a} + \psi^\dagger_{|a}\gamma \overset{(3)}{\gamma^a}\pi^\dagger - \pi_{|a}\gamma \overset{(3)}{\gamma^a}\psi$$

$$+2\frac{m_0 c}{\hbar}(\pi\gamma\psi - \psi^\dagger\gamma\pi^\dagger) + K(\pi\psi - \psi^\dagger\pi^\dagger)\Big], \tag{3.2a}$$

$$\mathscr{T}_a = \frac{c}{2}[\pi\psi_{|a} - \pi_{|a}\psi + \psi^\dagger_{|a}\pi^\dagger - \psi^\dagger\pi^\dagger_{|a}], \tag{3.2b}$$

$$\mathscr{T}_{ab} = \frac{c}{2}[-\pi\gamma\gamma_{(a}\psi_{|b)} + \psi^\dagger\gamma\gamma_{(a}\pi^\dagger_{|b)} - \psi^\dagger_{|(b}\gamma\gamma_{a)}\pi^\dagger + \pi_{|(b}\gamma\gamma_{a)}\psi - K_{ab}(\pi\psi - \psi^\dagger\pi^\dagger)]. \tag{3.2c}$$

The expressions (3.2a) and (3.2b) are the superhamiltonian and the supermomentum for the Dirac field, respectively. By tedious calculations one can show that these expressions fulfil the Poisson brackets (1.4). It is not possible to give these calculations explicitly here, for verification we only mention the useful formulae [1]

$$F(x)\delta_{|i}(x,\bar{x})G(\bar{x}) = F(\bar{x})\delta_{|i}(x,\bar{x})G(\bar{x}) - F_{|i}(x)\delta(x,\bar{x})G(x)$$

and

$$\pounds\sqrt{\overset{(3)}{g}} = K\sqrt{\overset{(3)}{g}},$$

$$\frac{\partial\gamma}{\partial g_{ab}} = 0, \qquad \frac{\partial \overset{(3)}{\gamma^a}}{\partial g_{bc}} = -\tfrac{1}{2}\overset{(3)}{g^{a(b}}\overset{(3)}{\gamma^{c)}}.$$

As in the case of the Klein–Gordon and Maxwell fields the supermomentum (3.2b) of the Dirac field is independent of the metric of the hypersurface, i.e.

$$\frac{\partial\mathscr{T}_a}{\partial g_{bc}} = 0.$$

The superhamiltonian indirectly contains the metric of the hypersurface through the Dirac matrices at one hand and the term being proportional to the extrinsic curvature $K \equiv \overset{(3)}{g^{ab}}K_{ab}$ at the other hand. In the special relativistic case (the hypersurfaces t = const are flat) the terms proportional to K and K_{ab} do not occur in (3.2a, c) and the Dirac matrices are constant.

4. *Summary*

The main result of this paper is the proof that the superhamiltonian and the supermomentum given by (3.2a) and (3.2b), respectively, fulfil the Poisson brackets (1.4). This result is of importance for the consistency of kinematics of hypersurfaces and the dynamics of the Dirac field in curved spacetime. This example shows that the relation (1.4) is valid for fields the Lagrangian of which has not the simple structure (1.3), too.

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