DERIVATION OF THE WEISSKOPF-WIGNER FORMULA FROM THE KRÓLIKOWSKI-RZEWUSKI EQUATION FOR DISTINGUISHED COMPONENT OF A STATE VECTOR

By W. Garczyński and K. Urbanowski

Institute of Theoretical Physics, University of Wrocław*

(Received August 10, 1977)

The known Weisskopf-Wigner formula for decay width is derived by the Królikowski-Rzewuski formalism. All approximations made in the course of the derivation are listed.

1. Introduction

Królikowski and Rzewuski have derived general formulae for the quasipotential governing time evolution of the projection of a state vector onto a selected subspace in the Hilbert space of states of a system [1]. They are rather complicated and will not be reproduced here. For a one-dimensional projecting subspace significant simplification may be achieved. This case is of particular use in the theory of unstable states [2, 3].

We shall demonstrate how the known Weisskopf–Wigner approximate formula for decay width may be derived from the Królikowski–Rzewuski formalism. Our approach gives a better understanding of the character of approximations one makes when deriving this formula.

2. Basic definitions

Let P be a projecting operator onto a one-dimensional subspace of the Hilbert space $\mathscr{H}$ determined by some normalized vector $|\alpha\rangle \in \mathscr{H}$

$$P = |\alpha\rangle \langle\alpha| \,. \tag{2.1}$$

Assuming that $|\alpha\rangle$ belongs to some orthogonal and complete set of states $\{|\beta\rangle\}$ we may write for the orthogonal complement of P

$$Q = \mathbf{1}-P = \sum_{\beta\neq\alpha} |\alpha\rangle \langle\alpha| \,. \tag{2.2}$$

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Wrocławski, Cybulskiego 36, 50-205 Wrocław, Poland.

If a state $|\psi(t)\rangle$ evolves according the Schrödinger equation with a Hamiltonian $H = H^\dagger$ then the projection onto $|\alpha\rangle$ direction

$$|\psi_{\|}(t)\rangle = |\alpha\rangle \langle\alpha|\psi(t)\rangle \equiv |\alpha\rangle\psi(t) \tag{2.3}$$

satisfies the equation [4]

$$\left[i\frac{d}{dt} - h - v(t, t_0)\right]\psi(t) = 0. \tag{2.4}$$

Here we denoted (cf. [4] formulae (2.2), (2.6), (2.7))

$$h = \langle\alpha|H|\alpha\rangle, \tag{2.5}$$

$$v(t, t_0) = i\frac{d}{dt}\ln\left[1 - ia(t, t_0)\right] \tag{2.6}$$

$$= -i\int_{t_0}^{\infty} d\tau k(t-\tau)e^{i(t-\tau)h}\,\frac{1-ia(\tau, t_0)}{1-ia(t, t_0)} \tag{2.7}$$

$$= v(t-t_0),$$

where

$$a(t, t_0) = \int_{t_0}^{\infty} d\tau r(t-\tau)e^{i(t-\tau)h} = a(t-t_0), \tag{2.8}$$

$r(t)$ is given by the expansion

$$r(t) = i\sum_{n=1}^{\infty}(-i)^n \underbrace{(g*k)*\ldots*(g*k)}_{n}(t) \tag{2.9}$$

and

$$k(t) = \Theta(t)\langle\alpha|PHQe^{-itQHQ}QHP|\alpha\rangle, \tag{2.10}$$

$$g(t) = -i\Theta(t)e^{-ith}. \tag{2.11}$$

The convolution operation is defined as follows

$$g*k(t) = \int_{t_0}^{\infty} d\tau g(t-\tau)k(\tau). \tag{2.12}$$

The time instant t_0 is such that

$$Q|\psi(t_0)\rangle = 0 \tag{2.13}$$

which means that at this moment the state vector $|\psi(t)\rangle$ is directed along $|\alpha\rangle$. In the course of deriving equation (2.4) it is assumed that $\psi(t_0)$ coincides at t_0 with a solution of a free equation

$$\psi(t_0) = \varphi(t_0), \tag{2.14}$$

$$\left(i\frac{d}{dt} - h\right)\varphi(t) = 0. \tag{2.15}$$

In the limiting case $t_0 \to -\infty$ one gets somewhat simpler formulae since basic quantities loose their dependence on time.

One can show that [4]

$$\lim_{t_0 \to -\infty} a(t, t_0) = \lim_{t \to \infty} a(t, t_0) = a = -i \tag{2.16}$$

while

$$\lim_{t_0 \to -\infty} v(t, t_0) = \lim_{t \to \infty} v(t, t_0) = v \tag{2.17}$$

is a time independent quantity.

The quasipotential v is a complex number in general and our task consists of expressing it in terms of the Hamiltonian and projecting vector $|\alpha\rangle$. Most important for us here is the imaginary part of v for which we shall write

$$h+v = m - \frac{i}{2}\gamma. \tag{2.18}$$

Hence

$$\gamma = i(v - v^*) \tag{2.19}$$

is a decay width of the state $\psi_{st}(t)$ calculated from the equation

$$\left(i\frac{d}{dt} - h - v\right)\psi_{st}(t) = 0. \tag{2.20}$$

Hence we get after integration

$$|\psi_{st}(t)|^2 = e^{-\gamma t}. \tag{2.21}$$

Taking into account that the initial condition (2.14) approaches in the limit $t_0 \to -\infty$, the asymptotic condition

$$\psi(t) - \varphi(t) \to 0 \quad \text{for} \quad t \to -\infty, \tag{2.22}$$

which means that v is switched off adiabatically when time increases, we get from (2.7)

$$v = -i \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} d\tau k(\tau) e^{i\tau(h+i\varepsilon)}. \tag{2.23}$$

Using (2.10) we easily find for v the formula

$$v = \langle\alpha|PHQ \frac{\mathscr{P}}{h - QHQ} QHP|\alpha\rangle - i\pi\langle\alpha|PHQ\delta(h - QHQ)QHP|\alpha\rangle. \tag{2.24}$$

Comparing it with (2.18) we get for γ

$$\gamma = 2\pi\langle\alpha|PHQ\delta(h - QHQ)QHP|\alpha\rangle. \tag{2.25}$$

One sees also that γ is a nonnegative number as a diagonal element of the nonnegative operator $\delta(h - QHQ)$. It means that the wave function $\psi_{st}(t)$ obtained from the stationary equation (2.20) is damped in time.

In the theory of unstable states one replaces the exact equation (2.4) by the stationary equation (2.20) supplemented by the same initial condition [4]. It is clearly an approvimation. One may show [4] that both wave functions agree in the limit of increasing time

$$|\psi(t)-\psi_{st}(t)| \sim \text{const}\, t^{-\varepsilon}, \qquad \varepsilon > 0 \tag{2.26}$$

while for the quasipotential the approximation is even better

$$|v(t, t_0)-v| \sim \text{const}\, t^{-\varepsilon-1}. \tag{2.27}$$

With this accuracy we may replace $v(t, t_0)$ by the number v obtained from (2.24).

3. *Derivation of the Weisskopf-Wigner formula*

Let us assume that the Hamiltonian H is splitted into two parts

$$H = \overset{0}{H}+\overset{1}{H}, \qquad \overset{1}{H} \ll \overset{0}{H}, \tag{3.1}$$

where $\overset{1}{H}$ is responsible for transitions between $|\alpha\rangle$ and other states from $\mathscr{H}$. Let $\{|\beta\rangle\}$ be a complete set of eigenstates of the free Hamiltonian $\overset{0}{H}$

$$\overset{0}{H}|\beta\rangle = E_\beta|\beta\rangle, \qquad E_\alpha \equiv h \tag{3.2}$$

and let our distinguished state $|\alpha\rangle$ belong to this set. Then, according to the Weisskopf-Wigner theory, we may write approximately [5], [6]

$$\langle\alpha|e^{-itH}|\alpha\rangle \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-ixt}\frac{\Gamma_\alpha}{(x-h-\Delta_\alpha)^2+\frac{1}{4}\Gamma_\alpha^2}$$

$$= e^{-it(h+\Delta_\alpha)}e^{-\frac{\Gamma_\alpha}{2}t}, \tag{3.3}$$

where the constants Γ_α and Δ_α are given by the formulae

$$\Gamma_\alpha = \langle\alpha|\overset{1}{H}\delta(\overset{0}{H}-h)\overset{1}{H}|\alpha\rangle, \tag{3.4}$$

$$\Delta_\alpha = -\langle\alpha|\overset{1}{H}\frac{\mathscr{P}}{\overset{0}{H}-h}\overset{1}{H}|\alpha\rangle. \tag{3.5}$$

We shall derive this result in a systematic way using our formula (2.24). Let us assume that

$$P\overset{0}{H}Q = P\overset{1}{H}P = 0. \tag{3.6}$$

In this case we have the relations

$$P\overset{0}{H} = \overset{0}{H}P = P\overset{0}{H}P = hP, \tag{3.7}$$

$$PHQ = P\overset{1}{H}, \tag{3.8}$$

$$Q\overset{0}{H}Q = \overset{0}{H} - P\overset{0}{H}P, \tag{3.9}$$

and as a result

$$k(t) = \Theta(t)\,\langle\alpha|\overset{1}{H}Qe^{-itQHQ}Q\overset{1}{H}|\alpha\rangle$$

$$= \Theta(t)\,\langle\alpha|\overset{1}{H}e^{-it\overset{0}{H}}\overset{1}{H}|\alpha\rangle + O(\overset{1}{H}{}^3). \tag{3.10}$$

Terms containing third and higher powers of the interaction Hamiltonian arise since in general $Q\overset{1}{H}Q \neq 0$. This leads to the following formula for the quasipotential

$$v = \langle\alpha|\overset{1}{H}\frac{\mathscr{P}}{h-\overset{0}{H}}\overset{1}{H}|\alpha\rangle - i\pi\langle\alpha|\overset{1}{H}\delta(h-\overset{0}{H})\overset{1}{H}|\alpha\rangle + O(\overset{1}{H}{}^3)$$

$$= \Delta_\alpha - \frac{i}{2}\Gamma_\alpha + O(\overset{1}{H}{}^3) \tag{3.11}$$

and for the decay width

$$\gamma = i(v-v^*) = \Gamma_\alpha + O(\overset{1}{H}{}^3). \tag{3.12}$$

In the particular case when

$$Q\overset{1}{H}Q = 0 \tag{3.13}$$

we get precisely the Weisskopf-Wigner formula for γ since terms containing third and higher powers of the interaction Hamiltonian vanish.

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