RANDOM PHASE APPROXIMATION SCHEME IN MULTIHADRON PRODUCTION PROCESSES

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(Received September 1, 1977)

We investigate general features of an effective one-particle approximation to the description of hadronic matter produced in high energy collisions. This scheme is the consequence of the assumption that particles are emitted by random fluctuating sources. The fluctuations independently of the nature of the sources lead to strong correlations possessing both short and long range components, offering thus a unified description of the clustering phenomena and of the broadening of multiplicity distributions. We show that the long range components may give rise to a limiting hadron temperature.

1. Introduction

The aim of this paper is to describe a mathematical framework for a large class of statistical models for the production of particles in high energy collisions based on a general and powerful formalism of Gaussian stochastic processes. This paper presents basically a continuation and development of ideas contained in [1].

The fundamental physical assumptions upon which we shall rely are old and classical. We assume that the delicate intricacy of the many-body production amplitude generated by strong and complicated interactions in the hadronic final state, with formation of many overlapping resonances is somehow smeared due to the concurrence of a large number of mechanisms. Thus, there is some sense in using an effective one-particle approximation. However, this does not mean that particles within this scheme are independently produced similarly to bosons emitted by a classical current in Bremsstrahlung-type models [2, 3]. We shall not use a factorized form of the production amplitude, but rather a linearization scheme which is an exact analog of the Random Phase Approximation (RPA) in many-body theory. This approximation is supposed to describe well the average properties of the system of particles at high density and many degrees of freedom involved.

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These properties are strongly related to the implicitly assumed lack of coherence between the amplitudes. Summarizing our assumptions:

A. We shall work with the final state density matrix rather than with the S matrix or with the individual multiparticle amplitudes. By RPA we will understand the following rule:

$$\langle A_1^\dagger A_2^\dagger \dots A_n^\dagger A_{n+1} \dots A_{2n}\rangle = \sum_{\substack{\text{all}\\ \text{pairings}}} \langle A_i^\dagger A_j\rangle \dots \langle A_k^\dagger A_l\rangle, \tag{1.1}$$

where A and $A^\dagger$ are annihilation and creation operators and the average is performed over the ensemble described by our density matrix.

We stress that RPA should not be considered here as an assumption about the dynamics of the process, but rather as a description of final asymptotic states. More about RPA from the space-time point of view and its application to transport theory approach to multiparticle processes can be found in [4, 5].

B. We shall use the concept of the one-particle effective amplitude regarded as a random, fluctuating function of the particle momentum and eventually other quantum numbers. This concept will give us a formal substitution of the strongly interacting hadronic field emitted in the high energy scattering process by an effective free field radiated by some fluctuating, incoherent, phenomenological sources. The nature of the sources is irrelevant for the general properties of the scheme. In this framework the RPA will be expressed by the assumed Gaussian probability distribution of amplitudes. Of course the idea of describing a many boson system or a Bose quantum field with the aid of random function formalism is not an original one and is widely used now in quantum theory. But the physical motivation for our probabilistic or stochastic scheme comes from another branch of physics. The "fluctuations" inherent to this approach are closely related to the phenomenon known in low energy nuclear reactions as Ericson fluctuations [6]. It was found that cross sections as functions of energy in the region of overlapping excitations, (i. e. slightly above the first well separated resonances), instead of smearing out present highly irregular, ragged patterns due to irregular distributions of phases of individual excitations. These fluctuations are not "real" in the sense that the whole pattern is entirely reproducible when the positions and relative phases of the excitations are known. However, as the graphic of the cross section in this region looks like a realization of a random function, the application of stochastic methods proved to be extremely useful in the description of average quantities, such as correlations. We believe that also in high energy domain this phenomenon may play an extremely important if not the dominant rôle in establishing the properties of inclusive correlations. The possible rôle of Ericson fluctuations in high energy physics but in different context was noted by Frautschi [7]. Of course we do not expect to observe Ericson fluctuations in the total cross section as the energy is too large, but rapidly varying phase shifts within the subsystems of the many-particle final state give good physical ground to our random field model. Very suggestive may be a look into a compilation of πN or KN phase shifts. The Argand diagrams above the few lowest, nice resonances exhibit often a behaviour resembling the phase graphic of a noise. And as in a many-particle state the number of possible resonance contributions grows with the number of particles much faster than the number of particles itself, due to the combinatorics, and there is no reason to expect a particular, highly correlated distribution of phases, the arguments above suggest that the multiparticle amplitude should behave as a particular realization of a multidimensional stochastic function although in principle it is a fully deterministic object. It means that the present approach is suitable for the descriptions of inclusive and semi-inclusive measurements where we can forget about the details and simulate the complicated real amplitude by a simple, "truly" statistical density matrix. As we shall use the effective one-particle approximation we cannot pretend to describe individual resonances and other coherent phenomena such as diffractive scattering. Also the leading particles which remember well the initial configuration are not imbedded in the scheme. Our main aim is to describe the general properties of strong, positive correlations between outgoing particles which appear in our scheme due to incoherence and interference effects [8–13].

2. *Production process as a chaotic field emission*

In this Section we derive a closed formula for the density matrix describing the incoherent emission of a boson field by an ensemble of fluctuating sources. To the diagonal elements of our density operator $\hat{\varrho}$:

$$\sigma_n(k_1, k_2, \ldots, k_n) = \langle k_1, \ldots, k_n|\hat{\varrho}|k_1, \ldots, k_n\rangle \tag{2.1}$$

we shall simply refer as to the cross sections although the conservation laws are for the moment ignored and $\hat{\varrho}$ will be arbitrarily normalized:

$$\langle 0|\hat{\varrho}|0\rangle = 1. \tag{2.2}$$

By $k_i = (q_i, \boldsymbol{k}_i)$ we denote the full set of quantum numbers needed to describe the i-th particle and by $\int dk$ the summation over charges (and eventually other discrete quantum numbers) q as well as the Lorentz-invariant integration over momenta $\int d^3k/2E$.

According to our one-particle approximation we write the n-particle state vector contributing to the production amplitude in the following factorized form:

$$|\varphi_{S_1}, \ldots, \varphi_{S_n}\rangle = \frac{1}{\sqrt{n!}}\int dk_1 \ldots dk_n \varphi_{S_1}(k_1) \ldots \varphi_{S_n}(k_n) a^\dagger(k_1) \ldots a^\dagger(k_n)\,|0\rangle, \tag{2.3}$$

where $a^\dagger(k)$ is the particle creation operator (on mass shell) fulfilling the standard commutation relation $[a(k), a^\dagger(k')] = \delta(k, k') = \delta_{qq'}\, 2E\, \delta^3(\boldsymbol{k}-\boldsymbol{k}')$. $\varphi_S(k)$ denotes our one-particle effective amplitude generated by the phenomenological source of the field. S describes the state of the source. These sources can be considered as volume or surface elements of radiating fireball [8–12], or may be derived from a more dynamical input such as resonances [13, 14], multiperipheral chains, fluctuating currents in generalized Bremsstrahlung-type models [15], partons, etc. It is important to note here that, as can be seen from the formula

(2.3), each particle is produced by a different source. It is easy to visualise when we treat sources as small volume elements of a fireball, but much less if we want to describe sources by resonances. Our approximation consists in neglecting the contribution to the correlation between two (or more) particles given by the resonance which directly decays into these particles. Only resonances formed separately by each particle under consideration with the remaining particles are taken into account. In standard quantum many-body dynamics it would be called a Hartree-Fock approximation, we prefer, however, to avoid this name because we do not have any self-consistency procedure which in the genuine Hartree-Fock approximation permits the calculation of an effective one-particle Hamiltonian.

The fluctuations are now taken into account by performing the averaging over the distribution of source states. In order not to spoil the RPA the sources must be treated as independent. The resulting expression for the density matrix:

$$\hat{\varrho} = \sum_n \langle |\varphi_{S_1}, \ldots, \varphi_{S_n}\rangle \langle \varphi_{S_1}, \ldots, \varphi_{S_n}|\rangle_{S_1, \ldots, S_n} \tag{2.4}$$

can be easily expanded:

$$\hat{\varrho} = \sum_n \frac{1}{n!} \int dk_1 \ldots dk_n dk'_1 \ldots dk'_n u(k_1, k'_1) \ldots u(k_n, k'_n)$$

$$\times a^\dagger(k_1) \ldots a^\dagger(k_n) |0\rangle \langle 0| a(k'_1) \ldots a(k'_n). \tag{2.5}$$

The function

$$u(k, k') = \langle \varphi_S(k) \varphi_S^*(k') \rangle_S \tag{2.6}$$

represents a measure of the probability that the particle at k and independently the particle at k' may potentially be emitted by the same source. As we commented above, it does not mean that both particles at k and k' were emitted by the same source *actually*, as for example decay products of a resonance. The function $u(k, k')$ which we shall call *coherence function* based on the analogy of the definition (2.6) with optics [16, 17] is a free element of the model, the only requirement being its hermiticity: $u(k, k') = u^*(k', k)$. Of course, u can be determined after specifying the nature and the distribution of the sources, however, we shall show in the next Section that very general assumptions connected with the consistency of the scheme also strongly limit the properties of the coherence function.

As the distribution of sources is supposed to be mainly determined by final-state interactions, u should weakly depend on the nature of the incoming particles and also on their energy which corresponds well to the experimentally observed universality of hadronic multiparticle reactions. The differences between different reactions, in particular between non-annihilation and annihilation processes should be attributed to the conservation laws, the leading particles which influence strongly the kinematics, and also to diffraction. In that respect our model has exactly the same problems as any other statistical model of particle production.

Knowing $u(k, k')$ all the exclusive and inclusive cross sections can be calculated from the generating functional

$$Z(\{\lambda\}) = \sum_n \frac{1}{n!} \int dk_1 \ldots dk_n \lambda(k_1) \ldots \lambda(k_n) \sigma_n(k_1, \ldots, k_n). \tag{2.7}$$

In [1] we derived a closed formula for $Z(\{\lambda\})$ with the aid of some properties of permutation group with respect to its decomposition into cycles and some combinatorics. Here we shall give a simpler and more elegant functional derivation. We start from the well known concise formula [17–18]:

$$Z(\{\lambda\}) = \mathrm{Tr}\,[\hat{\varrho} : \exp\,[\textstyle\int dk (\lambda(k)-1) a^\dagger(k) a(k)]:], \tag{2.8}$$

where colons, as usual denote normal ordering. In the following we shall extensively use a simplified notation omitting the explicit dependence on k and the integrations. By u without parameters we shall denote the integral operator whose kernel is $u(k, k')$, and bilinear forms like $\alpha^* u \beta$ will mean $\int dk dk' \alpha^*(k) u(k, k') \beta(k')$, The expression (2.8) takes then the form

$$Z(\{\lambda\}) = \mathrm{Tr}\,[\hat{\varrho} : \exp\,[(\lambda-1) a^\dagger a]:]. \tag{2.9}$$

Due to the fact that our density operator is already in normal-ordered form it takes particularly simple form in the coherent state [2, 3, 15, 17, 19] representation. We use normalized coherent states

$$|\varphi\rangle = \exp\,[-\tfrac{1}{2}\,\varphi^* \varphi]\,\exp\,[\varphi a^\dagger]\,|0\rangle. \tag{2.10}$$

The matrix element of the density operator takes the form:

$$\langle \alpha | \hat{\varrho} | \beta \rangle = \langle \alpha | 0 \rangle \langle 0 | \beta \rangle \exp\,[\alpha^* u \beta] \tag{2.11}$$

and the calculation of the generating functional is straightforward since it involves only the computation of functional Gaussian integrals:

$$\begin{aligned} Z(\{\lambda\}) &= \textstyle\int (d^2\alpha)(d^2\beta) \langle \alpha | \hat{\varrho} | \beta \rangle \langle \beta | : \exp\,[(\lambda-1) a^\dagger a] : | \alpha \rangle \\ &= \textstyle\int (d^2\alpha)(d^2\beta) \exp\,[-\alpha^*\alpha - \beta^*\beta + \alpha^* u \beta + \lambda \beta^* \alpha] \\ &= \int (d^2\alpha) \exp\,[-\alpha^*(1-\lambda u)\alpha] = \frac{1}{\det(1-\lambda u)}. \end{aligned} \tag{2.12}$$

Here $(d^2\alpha) = (d\alpha)(d\alpha^*)$ is a translationally invariant integration measure in the function space with the following normalization:

$$\textstyle\int (d^2\alpha) \exp(-\alpha^*\alpha) = 1. \tag{2.13}$$

$1-\lambda u$ is an operator with the kernel $\delta(k, k') - \lambda(k) u(k, k')$ and the prescription for the evaluation of the determinant is the standard one:

$$\frac{1}{\det(1-\lambda u)} = \exp\,[-\mathrm{Tr}\ln(1-\lambda u)] = \exp\left[\sum_{m/1}^{\infty} \frac{1}{m} \mathrm{Tr}\,(\lambda u)^m\right]. \tag{2.14}$$

Formula (2.12) can be derived in a slightly different way which is also instructive. We start with a short historical comment.

The history of the like-pion positive correlations observed in not very high energy physics — the so called Goldhaber effect [8], and the history of bunching effects observed in beams of optical photons emitted by thermal sources — the Hanbury Brown and Twiss effect [20] have one curious aspect in common. As both are related to the clustering of identical bosons, their first explanations were concerned mainly with the Bose-Einstein symmetry of the system and much less with the incoherence, or the fluctuations of the field which are also a necessary ingredient [17] to obtain correlations since a Bose system without any correlations is well known: it is a field emitted by a classical current [2]. The fluctuations of the field are known to be in this case as weak as the quantum nature of bosons allows for, and the description of the field is given by a coherent state vector (2.10). Here, $\varphi(k)$ denotes a classical wave function of the field and can be explicitly calculated once the properties of the source are known. However the detailed properties are irrelevant as we now must average over an ensemble of amplitudes in order to take the fluctuations of the field into account. We should define an object which might play a rôle of the probability distribution of the field amplitudes. It suggests that the diagonal (the so called Glauber-Sudarshan) representation of the density matrix should be particularly simple and useful, and we know that it is complete. As we noted in the Introduction, we shall choose a Gaussian form for the probability distribution. It is equivalent to the RPA and can be justified via the Central Limit theorem on a basis of many, incoherent, independent contributions from the sources [17]. This assumption gives us a density matrix for a chaotic boson field:

$$\hat{\varrho} = \mathcal{N} \int (d^2\varphi) \exp\left[-\varphi^* B^{-1}\varphi\right] \exp\left[-\varphi^*\varphi\right] \exp\left[\varphi a^\dagger\right] |0\rangle \langle 0| \exp\left[\varphi^* a\right], \qquad (2.15)$$

where $\mathcal{N}$ is a normalization factor and B is the covariance operator of our multidimensional Gaussian distribution. The calculation of the generating functional is straightforward and we get the expression (2.12) again provided $\mathcal{N} = 1/\det(u)$ and $B = u/(1-u)$.

We should mention here that the application of stochastic processes to high energy physics appeared also in different contexts. Dushutin and Maltsev [21] developed a series of models in which the one-particle density rather than the field amplitude is treated as random function and Arnold and Thomas [22] proposed a stochastic field scheme which is based on the multiperipheral picture.

We discuss now in more details our solution. Since the logarithm of the generating functional Z generates the irreducible cluster integrals, these in our model have a very simple form:

$$\tau_m(\{\lambda\}) \equiv \mathrm{Tr}\,(\lambda u)^m = \int dk_1 \ldots dk_m \lambda(k_1) \ldots \lambda(k_m) u(k_1, k_2) u(k_2, k_3) \cdot \ldots \cdot u(k_m, k_1). \qquad (2.16)$$

The cyclical invariance of the clusters corresponds well to the fact that in the RPA in the many-body theory only ring graphs are considered. It should be noted that the exponential form of $Z(\{\lambda\})$ makes it identical to with the generating functional in the Independent Cluster Emission models with τ_m playing a rôle of the phase-space integral for the m-particle cluster. Here, however, cluster is obviously not a physical object in the standard sense but rather a formal object identical to a linked cluster in the Mayer-Ursell expansion in statistical mechanics. The particles "inside of the cluster" are obviously not uncorrelated, but they form a chain of pairwise connected links.

The exclusive n-particle cross sections are obtained by the differentiation of the generating functional over $\lambda(k)$ and putting all $\lambda = 0$:

$$\sigma_n(k_1, \ldots, k_n) = \sum_P u(k_1, k_{P_1})u(k_2, k_{P_2}) \cdot \ldots \cdot u(k_n, k_{P_n}). \tag{2.17}$$

The sum is performed over all permutations of the indices (1, 2,..., n). This expression can be regarded as a permutant (or a symmetrant, or a permanent) of the matrix

$$\begin{bmatrix} u(k_1, k_1) & \ldots & u(k_1, k_n) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ u(k_n, k_1) & \ldots & u(k_n, k_n) \end{bmatrix}, \tag{2.18}$$

which is a natural Bose analog of a determinant. Such form had to be expected since our density matrix had to factorize into one-particle contributions preserving at the same time the Bose-Einstein symmetry. From the formal point of view the system of hadrons in our model is almost equivalent to a free boson gas [23, 24]. The generalization consists in taking the one-particle density operator u non-diagonal in momenta and other physical observables. We shall try to justify it in the next Section. This generalization leads to nontrivial correlations between outgoing particles and, as our scheme is fairly general and describes effects which must be present if the final state interactions are sufficiently complicated, we believe that it is a serious alternative to independent cluster emission models. Of course we are not fighting against the existence of intermediate objects in multiparticle production, on the opposite, only the existence of strong resonances makes our incoherence assumption plausible.

All inclusive distributions may be obtained from $Z(\{\lambda\})$ by differentiation and putting $\lambda = 1$ afterwards. They all can be expressed through the "dressed" coherence function

$$U(k, k') = \sum_{m=0}^{\infty} \int dk_1 \ldots dk_m u(k, k_1)u(k_1, k_2) \ldots u(k_m, k') \tag{2.19}$$

which is the kernel of the operator $U = u/(1-u)$. It is easy to see that U fulfills the following integral equation:

$$U(k, k') = u(k, k') + \int dk'' u(k, k'')U(k'', k'). \tag{2.20}$$

This equation is identical to the Ornstein-Zernicke equation in classical statistical mechanics which relates the two-particle correlation function to a "direct" correlation function which is supposed to be simpler and to have the most (if not all) long range correlations removed. Similar equations have been proposed also in high energy physics [25–27] based on a Feynman-Wilson gas-fluid analogy or some bootstrap conjectures. Here, however, the equation is not fulfilled directly by the correlation function but rather by a possibly complex-valued object which plays an evident rôle of the one-particle Green's function of the theory, giving thus to (2.20) an interpretation of the Dyson equation.

The n-particle inclusive cross section take the form identical to (2.17) with U substituted for u. It may be worthwhile to note here that the permutant form of the inclusive cross section may correspond quite well to the observations (although in order to show that, measurements of higher differential correlation functions are necessary) in spite of the fact that in principle our formula (2.17) for exclusive cross sections is completely wrong, as no *amplitude* exists which could reproduce this expression. It can nevertheless provide a good description of the average quantities and can be used as a weight function in Monte-Carlo calculations.

Steinhoff [14] represents through a permutant form the amplitude, not the cross sections, using Breit-Wigner formula for the object which is a formal analog of our coherence function. This describes (approximately) a coherent production of resonaces (see also [28]) and is completely different from our approach. The coherence function, as we shall show, has properties very different from the Breit-Wigner amplitude since the resonances in our approach are "hidden" in the overwhelming fluctuation pattern.

The relations between U and the observable inclusive quantities are given below. The one-particle inclusive distribution normalized to the average multiplicity is

$$\varrho_1(k) = U(k, k) \tag{2.21}$$

and the two-particle correlation function:

$$C_2(k, k') = U(k, k') \cdot U(k', k) = |U(k, k')|^2. \tag{2.22}$$

The average multiplicity is

$$\langle n \rangle = \int dk \varrho_1(k) = \sum_{m=1}^{\infty} \tau_m \tag{2.23}$$

and the higher correlation integrals:

$$f_n = \sum_{m=n}^{\infty} (m-1)(m-2) \ldots (m-n+1)\tau_m, \tag{2.24}$$

where τ_m denotes here $\tau_m(\{1\}) = \mathrm{Tr}(u)^m$. Formulae for the integrated cross sections, i. e. multiplicity distribution become quickly very complicated as n grows, but the generating function is simple:

$$Z(\lambda) = \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} \lambda^m \tau_m\right]. \tag{2.25}$$

The analogous formulae for fermions are easy to derive. Starting from the expression (2.17) with an ordinary determinant substituted for the permutant or using functional techniques over the Grassmann algebra taking (2.5) as a basis it is easy to show that

$$Z_F(\{\lambda\}) = \det(1 + \lambda u_F). \tag{2.26}$$

The dressed coherence function in this case is equal to $u_F/(1+u_F)$ and the two-particle correlation function takes the form

$$C_{2_F}(k, k') = -|U_F(k, k')|^2. \tag{2.27}$$

The application of these formulae seems however to be rather limited due to the small density of baryons produced in high energy collisions. They may be in principle used to describe correlations between nucleons evaporated from the nucleus hit by a high energy projectile. Here however the RPA is probably too crude. Unfortunately, our formalism is not suitable for the description of correlations between fermions and bosons. However, the large number of resonances in baryon-meson systems is well known to exist and thus there is a dynamical mechanism responsible for the Ericson fluctuations in this case. The only thing which does not work is the RPA scheme. Its generalization would be highly desirable.

3. General properties of the coherence function

We begin our discussion with some properties which are independent of the specific model and follow directly from our formalism. Some of these properties must be postulated at the level of the formula (2.15) but may be justified provided some more sense is given to the definition (2.6). As the Gaussian integral exists only if the covariance operator is bounded and positive (roughly speaking, the eigenvalues of its inverse must be strictly positive) we conclude that both u and U must be bounded and positive. It can be easily checked by calculating the generating functional from the formula (2.15) which should be well defined object for $\lambda \in [0, 1]$. The positivity of U demands from the norm of u, which we denote by $\|u\|$, to be bounded by 1. The positivity requirement is fulfilled if we assume the existence of a positive and normalizable probability measure $\mu(S)$ which describes the distribution of source, so that

$$u(k, k') = \int d\mu(S)\varphi_S(k)\varphi_S^*(k'). \tag{3.1}$$

We have then obviously

$$u(k, k) \geqslant 0 \tag{3.2}$$

and the Schwarz inequality

$$|\mathrm{Re}\, u(k, k')| \leqslant \sqrt{u(k, k)u(k', k')}\,. \tag{3.3}$$

We shall not endeavour here to discuss the problem of the phase of $u(k, k')$ although this subject is extremely interesting and important both from the fundamental point of view and also for the phenomenological applications.

The problem of boundedness is more difficult. We shall here neglect the charges of particles as they only cause u to be a finite matrix instead of a single function. As u should depend on the dynamics only, the natural requirement would be its Lorentz invariance. Taking u as a function of the effective mass $M^2 = (k+k')^2$, where k denotes the four-momentum of the particle, is of course a bad choice from the phenomenological point of view as it does not take into account the well pronounced anisotropy of the final configuration. We shall discuss this case for simplicity.

In this case $u(k, k)$ is independent of k, the same property formally has also U and if the influence of the energy-momentum conservation was small in central region, a plateau would develope at asymptotic energies. In the next section we show that this possibility is probably ruled out by the actual data.

Because of Im $u(k, k) = 0$, in the region where k and k' are close to each other the function u possesses the property of a genuine phenomenological cluster function: it has its maximum at $k = k'$ which means that qualitatively the clustering of particles is well described by this model already on a very formal level.

Another conclusion is that in channels with high multiplicity at low energy where all particles are kinematically confined to region of phase space smaller than the effective coherence length, correlations of like charges should dominate over the correlations between unlike (Goldhaber effect) [8].

Obviously, for the Lorentz-invariant u and without energy-momentum conservation u is neither a Hilbert-Schmidt nor a trace-class operator and all the cluster integrals τ_m diverage since for every m one momentum variable is not limited. However, u treated as an operator in the Banach space of continuous functions may still be bounded. The equation

$$uf = \int dk' u(k, k') f(k') \tag{3.4}$$

for the Lorentz-invariant u is a peculiar convolution transformation and it is easy to check that

$$\|u\| = \int dk' u(k, k') \tag{3.5}$$

exists, has all the properties of a norm and is independent of k provided the coherence function exhibits short range behaviour only, which is a well justified assumption. The norm of u plays a role of a phenomenological coupling constant.

We discuss now some existing models and their relation to our scheme. Taking $u(k, k')$ very narrow, rapidly vanishing for $k \neq k'$ and not singular at $k = k'$ (i.e. not a Dirac delta behaviour) we retrieve a generalized statistical model with fully factorized $|S|^2$ matrix element but with arbitrary one-particle distribution given by $u(k, k)$. In this case all the cluster intergrals (2.16) vanish for $m > 1$ and we are left without correlations and with Poisson multiplicity distribution. This could be a not very bad approximation if the density of particles in the phase space was low and the overlap of the one-particle density operators could be neglected. However, it is known that it is not the case even at highest energies, since the average multiplicity grows as fast or even faster than the rapidity space available. This model is sometimes called Uncorrelated Jet Model, which may be confusing, for also the coherent versions of the independent emission models, based on a factorized S-matrix, use the same name. It is very important to realize, however, the difference between underlying physical pictures. While the statistical models describe a system which behaves like a gas or a fluid, the coherent emission models with the amplitude given by (2.10) based sometimes on a Bremsstrahlung picture describe an almost classical field. The influence of the conservation laws for non-commuting observables such as isospin is drastically different in these two cases as can be seen comparing the statistical isospin weights with formulae derived for the coherent case, see for example [3] or [18].

Assuming $u(k, k') = A\delta(k, k')$ we have a model of a free boson gas with $\|u\| = A$

playing a rôle of the absolute activity (or fugacity) and the singular object $\delta(k, k)$ having a meaning of the volume of the system which is supposed to be large. This volume should not be considered as the normalization volume necessary in all S-matrix calculations, but rather the interaction volume, or the volume of the fireball which is responsible for the distribution of momenta of the outgoing particles. The properties of this system were analysed in [23, 24]. This would be not a very good choice if we tried to apply it to the particle production, for it would require a translational symmetry, or an infinite volume for the fireball. The boundary conditions for a finite volume cause the momentum not to be a good quantum number for a meson inside the fireball and it is a main reason for nontrivial correlations between momenta. It was noted already in 1960 in the Goldhaber–Goldhaber–Lee and Pais model [8]. Its non-relativistic, illustrative version is obtained in our framework by taking for $\varphi_S(k)$ a plane wave $\exp(i\boldsymbol{kx})$ and calculating (3.1), assuming some distribution for the radiating matter. Taking for simplicity a Gaussian distribution we obtain

$$
\begin{aligned}
u(k, k') &= \int d^3x \exp(-x^2/2R^2) \exp(i\vec{k}\vec{x}) \exp(-i\vec{k}'\vec{x}) \\
&= (\sqrt{2\pi}\, R)^3 \exp\left[-\tfrac{1}{2} R^2(\vec{k}-\vec{k}')^2\right]. \qquad (3.6)
\end{aligned}
$$

The model of Kopylov and Podgoretzki [9–11] differs in that respect, that the emission takes place only from the surface of the fireball, which is considered to be opaque and correlations between energies are also included assuming that the irradiating power of the fireball exponentially falls off. However, their conclusions have one shortcoming: although their formulae for exclusive cross sections are formally identical to ours, i.e. they are expressed through a permutant form, Kopylov and Podgoretzki interpret their analog of $|u(k, k')|^2$ as two-particle inclusive correlation function. It would be true if the higher cluster integrals were negligible. This is probably a wrong assumption and Kopylov and Podgoretzki are aware of this fact discussing in [29] the influence of the three-particle clusters on the two-particle correlations. However, the correct account of higher clusters may be of crucial importance for the tentatives of determining the size of the fireball [12].

These models are based on the optical analogy and essentially translate to the high energy physics domain the classical Hanbury Brown and Twiss [17, 20] correlations well known in optics. A serious deficiency of these models is that they predict correlations between bosons with identical internal quantum numbers only. It produced a rather eclectic situation in the description of high energy inclusive correlations. In a number of papers the short range azimuthal correlations between like pions are attributed to the Bose–Einstein statistics while the correlations in rapidity which for unlike pions are even stronger than for the like ones are explained by clusters. Obviously the optical analogy is very fruitful and adequate to the quantum nature of the particles, but we cannot consider the fireball as an optical medium neutral with respect to the hadron internal quantum numbers. Hadrons themselves built up this medium and discussing its optical properties is a way of speaking about the phase-shifts whose strong dependence on the internal quantum numbers was well established long time ago. Attempts to investigate interference effects based on a dynamical rather than geometrical scheme, such as resonance production [13, 14, 28] or multiperipheral picture [30] seem to be very promising and they predict correlations also between different charges. Unfortunately it is not very clear yet how to include resonances and threshold effects into a one-particle approximation scheme in a consistent way.

4. Energy-momentum conservation, long range components and the existence of a maximum temperature

Although the coherence function should be a short-range object in a reasonable dynamical scheme for the production process, the inclusive correlation function (2.22) will contain also long-range components due to the fact that the effective correlation length given by the m-particle cluster contribution to (2.16) grows with m. We shall now find under what conditions the long-range components are expected to dominate and what will be the limiting shape of the multiplicity distributions.

We take the energy-momentum conservation into account by multiplying (2.1) by the appropriate delta-function $\delta^4(P-\sum_i k_i)$ where P is the total four-momentum. The density of states with the total four-momentum fixed can be expressed through an inverse Laplace transform

$$\sigma(P) = Z(\{1\}) = \frac{1}{(2\pi i)^4} \int_{c-i\infty}^{c+i\infty} d^4\beta \exp(p\beta)\Xi(\beta) \tag{4.1}$$

of the partition function

$$\Xi(\beta) = \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} \int dk_1 \ldots dk_m e^{-\beta(k_1+\ldots+k_m)} u(k_1, k_2) \ldots u(k_m, k_1)\right]. \tag{4.2}$$

In the limit $P \to \infty$, where large multiplicities are supposed to dominate, we assume that grand-cannonical description of the system is proper, provided we are far from the momentum space boundary. We now analyse the divergence of the expression (4.1) in the limit $P \to \infty$ due to the singularities of $\Xi(\beta)$ with β playing a rôle of a time-like inverse temperature four-vector and $|\beta| = \sqrt{\beta^2}$. Our effective coherence function

$$u^{(\beta)}(k, k') = e^{-\frac{1}{2}\beta k} u(k, k') e^{-\frac{1}{2}\beta k'} \tag{4.3}$$

is now a trace class operator, as the exponents ensure the convergence of the integral

$$\int dk e^{-\beta k} u(k, k) \tag{4.4}$$

provided the $u(k, k)$ is not singular. It is also a Hilbert–Schmidt class operator obviously, and all the integrals (2.14) exist. Nevertheless, the whole sum in (2.16) may diverge for some $|\beta| > 0$. Denoting by $\gamma_i^{(\beta)}$ the i-th eigenvalue of the operator $u^{(\beta)}$ which has now a discrete and bounded spectrum we can put the expression (4.2) in the form

$$\Xi(\beta) = \exp\left[\sum_{m=1}^{\infty} \frac{1}{m} \sum_i (\gamma_i^{(\beta)})^m\right] = \exp\left[-\sum_i \ln(1-\gamma_i^{(\beta)})\right]. \tag{4.5}$$

This expression, as had to be expected, has a form similar to the partition function for non-interacting bosons. The eigenvalues $\gamma_i^{(\beta)}$ are decreasing functions of $|\beta|$. In the limit $|\beta| \to 0$ the largest eigenvalue $\gamma_0^{(\beta)}$ is pushed towards the upper limit of the continuous spectrum of u which means that

$$\lim_{|\beta|\to 0} \gamma_0^{(\beta)} = \|u\| \tag{4.6}$$

and it means that for $\|u\| > 1$ the partition function diverges already at some $|\beta| > 0$. This is essentially a generalization of the result of Montvay and Satz [24]. They deal with the case of an ideal boson gas and thus are able to get a more detailed result which we briefly repeat here. In their case the operator $u^{(\beta)}$ is just a multiplication by function $Ae^{-\beta k}$ which is clear from the formula (3.4). Thus, the spectral radius is equal to

$$A \sup_k e^{-\beta k} = Ae^{-|\beta|\mu} \tag{4.7}$$

with μ being a mass of the particle. The critical temperature $T = 1/|\beta|$ occurs at $\mu/\ln A$.

If $\|u\| < 1$ the only divergence of (4.2) occurs at $|\beta| = 0$ due the divergence of the individual terms of the sum in the exponent which is equivalent to the divergence of the sum over the spectrum (4.5).

The generating function for the multiplicity distribution has the following form:

$$Z(\lambda) = \exp\left[-\sum_i \ln(1-\lambda\gamma_i^{(\beta)})\right] \tag{4.8}$$

and the moments are easily expressed by the eigenvalues

$$\langle n\rangle = \sum_i \frac{\gamma_i^{(\beta)}}{1-\gamma_i^{(\beta)}}, \tag{4.9}$$

$$f_m = (m-1)! \sum_i \left[\frac{\gamma_i^{(\beta)}}{1-\gamma_i^{(\beta)}}\right]^m. \tag{4.10}$$

At high energies, when $|\beta|$ approaches its critical values, one or more eigenvalues sufficiently close to 1 give a dominant contribution. We see that f_m is then proportional to $\langle n\rangle^m$ and the generating function is approximately equal to

$$Z(\lambda) \simeq \frac{1}{(1-\lambda\gamma)^{\kappa}}, \tag{4.11}$$

where κ denotes the degeneracy of the dominant eigenvalue. This expression is the generating function for the Polya distribution which was found [31–33] to describe very well the FNAL proton-proton data with κ being approximately 3. From the qualitative point of view, however, the shape of the distribution will be very similar if the largest eigenvalue is not exactly degenerated but there are few dominant eigenvalues. Obviously, with a proper choice of them an excellent fit is possible which does not seem to be particularly interesting here, since many good fits already have been given. The choice of a non-integer κ would correspond to a cut in the generating function rather than a multiple pole.

On the other way round, if all $\gamma_i^{(\beta)}$ would persist below 1 for $|\beta| \to 0$ all the ratios $f_m/\langle n\rangle$ are bounded by constants and the limiting multiplicity distribution is Poisson-like.

Thus, in order to have the agreement with the experiment in our scheme we must have $\|u\| > 1$. It is somehow analogous to the situation where the intercept of the "bare" Pomeron in Reggeon-calculus is greater than 1. If we believe that the essential dependence of the total cross section on the energy is given by (4.1) this analogy is even closer. Apart from the broadening of multiplicity distributions we also obtain in this case a violation of the Feynman scaling in the central region due to the dominant contribution given by "large clusters" which are, of course invisible as distinct physical objects, but give rise to long-range correlations. Whether this mechanism can give a reasonable description of the experimentally observed breakdown of the Feynman scaling in the central region, i.e. the 30% increase of $d\sigma/dy|_{y=0}$ at ISR energies, is rather difficult to say without detailed calculations.

Similar situation as in our scheme arises also in other statistical models, provided there is a mechanism which promotes the states with large number of particles. A classical example is, of course, the Statistical Bootstrap model, but also the Pomeranchuk statistical model [34] with the volume of the fireball proportional to the number of particles gives the similar result concerning the existence of a maximum temperature [35]. Roughly speaking, $(V \cdot N)^N$ behaves as $N!$ for large N and the Poisson series changes into a geometric series with a finite radius of convergence. In our case, the appropriate factor is provided by the combinatorics related to the Bose–Einstein symmetry as is clear from the formula (2.17). However, we want to point out that the similarity of our model or the model of Montvay and Satz to the Statistical Bootstrap in this respect is by no means an accidental one. The "gas" models can be supported on physical grounds by the presence of strong excitations in the system which give rise to the fluctuation pattern. If we expect from our fluctuation picture to work also at very high energies, we must agree to have a large spectrum of resonances. And the results of this section strongly suggest that in order to retrieve the features shown by experimental data, in particular wide multiplicity distribution, our scheme demands an exponentially increasing density of states in order to obtain a singularity in the partition function. However, since we have never assumed any kind of thermal eqilibrium, the physical meaning of the limiting temperature is not clear and there is no connection between it and the damping of transverse momenta. Besides, what makes the system behave critically here is the limitation of "longitudinal temperature" as only in longitudinal direction kinematics plays an important role. Formally our RPA model with respect to the critical properties of the hadronic system is equivalent to the ideal Bose gas undergoing a Bose–Einstein condensation due to the pushing of the activity out of the stability region. Such a system in standard statistical mechanics is an extremely pathological one and cannot be described by the equilibrium thermodynamics.

5. Concluding remarks

We must state that we do not consider our RPA scheme to be a correct dynamical description of the hadronic matter since the dynamical properties of many-hadron systems are complicated, with dominant non-linear phenomena, and a linearization scheme seems to be an extremely crude simplification. But, as it is well known in optics, the statistical properties of the radiation field depend rather weakly on the dynamics of the sources and in the black-body-radiation limit they become almost completely independent. And in this context we believe in the validity of the RPA.

The picture of multiparticle production which we have presented here has obviously many limitations and deficiencies both from the fundamental and the phenomenological point of view. In its present formalism our scheme predicts no cluster-like correlations between fermions and bosons and it is not clear how to include consistently in the model the transverse momentum cut-off without putting it by hand. But, even without going into details, in its "skeleton" presentation it gives a natural explanation of the clustering phenomena, predicts the correct shape of the limiting multiplicity distributions and also links the properties of correlations to the possible critical behaviour of the hadron system. It has one big advantage over the standard statistical models, namely it gives not only the density of states or the cross sections but also permits the overlap calculations which would use the non-diagonal elements of our density matrix.

We end the paper recalling what is our major assumption which justifies this and possibly more general statistical approach to multiparticle production: namely, hadrons produced in high energy collision due to complicated interactions form an *ergodic* system, permitting thus to calculate inclusive correlations with a model stochastic ensemble instead of integrating and summing over the multidimensional space of unobserved particles.

It is a pleasure to thank Professor H. Satz for his hospitality at the Bielefeld University and for his interest in this work. Special thanks are due to D. Miller for numerous and interesting discussions and also to J. Engels for helpful remarks about the manuscript.

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