THE TOTAL SUPERENERGY TENSORS IN GENERAL RELATIVITY AND IN EINSTEIN-CARTAN THEORY AND THE GLOBAL SUPERENERGETIC QUANTITIES OF A CLOSED SYSTEM

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In this paper we introduce the notion of the total superenergy tensors in the general relativity and in the Einstein-Cartan theory. We calculate the explicit forms of these tensors and give some remarks about the global superenergetic quantities of a closed system.

1. *The superenergy tensors*

In the paper [1] the definition of the superenergy tensor of a physical field Φ which possesses an energy-momentum tensor $T_{\mu}^{\cdot\nu}$ or pseudotensor and the definition of the Maxwellian superenergy tensor of the curvature tensor field $R_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}$ were given. The first definition given in [1] has the form[1]

$$ {}^{S}T_{\mu\cdot}^{\cdot\nu}(P) := \lim_{a\to 0} \frac{\iiint\limits_{-a}^{a} (T_{\mu\cdot}^{\cdot\nu} - \overset{0}{T}{}_{\mu\cdot}^{\cdot\nu})\,dy^1dy^2dy^3}{\frac{4}{3}a^5} = \lim_{\mathcal{R}_{\mathcal{X}}\to 0} \frac{\iiint\limits_{\mathcal{X}} (T_{\mu\cdot}^{\cdot\nu} - \overset{0}{T}{}_{\mu\cdot}^{\cdot\nu})\,d^3y}{\frac{2}{15}\pi\mathcal{R}_{\mathcal{X}}^5}. \tag{1}$$

For the tensor field $T_{\mu\cdot}^{\cdot\nu}$ of the class C^r, $r \geqslant 3$, it gives

$$(\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\overset{0}{T}{}_{\mu\cdot\alpha\beta}^{\cdot\nu\cdot\cdot} \underset{\mathrm{NCS}(P)}{\overset{*}{=}} \overset{0}{T}{}_{\mu\cdot,11}^{\cdot\nu} + \overset{0}{T}{}_{\mu\cdot,22}^{\cdot\nu} + \overset{0}{T}{}_{\mu\cdot,33}^{\cdot\nu}.$$

We put

$${}^{S}T_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) := (\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\overset{0}{T}{}_{\mu\cdot\alpha\beta}^{\cdot\nu\cdot\cdot}.$$

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[1] Eq. (1) differs from the definition given in [1] by the multiplier 2. Consequently ${}^{S}T_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) = (\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\overset{0}{T}{}_{\mu\cdot\alpha\beta}^{\cdot\nu\cdot\cdot}$ does not possess the multiplier $\frac{1}{2}$.

In (1) $T_{\mu\cdot}^{\cdot\nu}$ denotes an energy-momentum tensor or pseudotensor and ${}^{S}T_{\mu\cdot}^{\cdot\nu}$ — the corresponding superenergy tensor. The definition (1) has been introduced following Pirani's paper [2]. In this definition, the ${}^{S}T_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ is constructed by means of some kind of averaging of the differences $(T_{\mu\cdot}^{\cdot\nu} - \overset{0}{T}{}_{\mu\cdot}^{\cdot\nu})$ over a three-dimensional cube or sphere lying in the hypersurface $y^0 = 0$ of the NCS(P) of the connection $\left\{\begin{matrix}\alpha \\ \beta\gamma\end{matrix}\right\}$[2]. The point P (= the origin of the NCS(P)) is the geometric center of this cube (or sphere).

Now, we think that in the four-dimensional space-time we should average the differences $(T_{\mu\cdot}^{\cdot\nu} - \overset{0}{T}{}_{\mu\cdot}^{\cdot\nu})$ over a four-dimensional cube[3]. Thus, we modify the definition (1) of the superenergy tensor to the form

$$S_{\mu\cdot}^{\cdot\nu}(P) := \lim_{a\to 0} \frac{\int\!\!\int\!\!\int\!\!\int_{-a}^{a} (T_{\mu\cdot}^{\cdot\nu} - \overset{0}{T}{}_{\mu\cdot}^{\cdot\nu}) dy^0 dy^1 dy^2 dy^3}{\frac{8}{3} a^6}. \tag{2}$$

The origin, P, of the NCS(P) is the geometric center of the four-dimensional "cube" over which the averaging in the formula (2) is performed.

If the field $T_{\mu\cdot}^{\cdot\nu}$ is of the class C^r, $r \geq 3$, then we get from (2)

$$S_{\mu\cdot}^{\cdot\nu}(P) = S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) := (2\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\overset{0}{T}{}_{\mu\cdot,\alpha\beta}^{\cdot\nu} \underset{\mathrm{NCS}(P)}{\overset{*}{=}} \overset{0}{T}{}_{\mu\cdot,00}^{\cdot\nu} + \overset{0}{T}{}_{\mu\cdot,11}^{\cdot\nu} + \overset{0}{T}{}_{\mu\cdot,22}^{\cdot\nu} + \overset{0}{T}{}_{\mu\cdot,33}^{\cdot\nu}. \tag{3}$$

In the following we use the definition (2) of the superenergy tensor of a physical field Φ which possesses an energy-momentum tensor or pseudotensor. We shall calculate the total superenergy tensors of the gravitational field and matter in GRT and in ECT. These total superenergy tensors will be calculated according to the formula (2) from the energy-momentum complexes considered in GRT and in ECT.

2. *The energy-momentum complex in GRT and the energy-momentum complexes in ECT*

Let us write the Einstein equations in the form given by Trautman [3]

$$\tfrac{1}{2}\,\eta_{\mu\alpha\cdot}^{\cdot\cdot\sigma} \wedge \Omega^{\alpha\cdot}_{\cdot\sigma} = (-)\,\frac{8\pi G}{c^4}\,T_{\mu}. \tag{4}$$

$\Omega^{\alpha\cdot}_{\cdot\sigma}$ is here the 2-form of the Riemannian curvature, T_μ — the 3-form of the energy-momentum of matter: $T_\mu := T_{\mu\cdot}^{\cdot\nu}\eta_\nu$ where $T_{\mu\cdot}^{\cdot\nu}$ is the metric energy-momentum tensor [4] and $\eta_\nu := \frac{1}{6} v^\alpha \wedge v^\beta \wedge v^\gamma \eta_{\nu\alpha\beta\gamma}$. $\eta_{\mu\alpha\sigma}^{\cdot\cdot\cdot} := v^\gamma \eta_{\mu\alpha\sigma\gamma}^{\cdot\cdot\cdot\cdot} \equiv v^\gamma \sqrt{-g}\,\varepsilon_{\mu\alpha\sigma\gamma}$, where v^γ denotes the fixed field of the co-frames. After a direct calculation, we get from (4)

$$d\left(\frac{c^4}{16\pi G}\,\eta_{\mu\alpha\cdot}^{\cdot\cdot\sigma} \wedge \omega^{\alpha\cdot}_{\cdot\sigma}\right) = T_\mu + \frac{c^4}{16\pi G}\,(\eta_{\mu\alpha\cdot}^{\cdot\cdot\gamma} \wedge \omega^{\sigma\cdot}_{\cdot\gamma} \wedge \omega^{\alpha\cdot}_{\cdot\sigma} - \eta_{\gamma\alpha\cdot}^{\cdot\cdot\sigma} \wedge \omega^{\gamma\cdot}_{\cdot\mu} \wedge \omega^{\alpha\cdot}_{\cdot\sigma}). \tag{5}$$

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[2] In both theories, GRT and ECT, we use the NCS(P) of this connection.

[3] Sphere is not admissible now.

$\omega^{\sigma}_{\cdot\alpha}$ denotes here the 1-form of the Riemannian connection and d is the operator of the exterior differentiation.

In local coordinates ($\vartheta^{\gamma} \equiv dx^{\gamma}$) and in Schouten's language [5], we have from (5)

$$\partial_{\alpha}\left(\frac{c^4}{16\pi G}\,\varepsilon^{\nu\alpha\beta\gamma}g^{\sigma\kappa}\eta_{\mu\lambda\kappa\beta}^{\cdot\cdot\cdot\cdot}\left\{{\lambda \atop \sigma\gamma}\right\}\right) = \sqrt{-g}\,(T_{\mu\cdot}^{\cdot\nu}+{}_{\mathrm{E}}t_{\mu\cdot}^{\cdot\nu}), \tag{6}$$

where $T_{\mu\cdot}^{\cdot\nu}$ is the metric energy-momentum tensor of matter and

$${}_{\mathrm{E}}t_{\mu}^{\ \nu} = \frac{c^4}{16\pi G}\,\eta^{\nu\alpha\beta\gamma}_{\cdot\cdot\cdot\cdot}\left(g^{\delta\varepsilon}\eta_{\mu\varrho\varepsilon\alpha}^{\cdot\cdot\cdot\cdot}\left\{{\sigma \atop \delta\beta}\right\} - g^{\sigma\varepsilon}\eta_{\delta\varrho\varepsilon\alpha}^{\cdot\cdot\cdot\cdot}\left\{{\delta \atop \mu\beta}\right\}\right)\left\{{\varrho \atop \sigma\gamma}\right\}$$

is Einstein's energy-momentum pseudotensor of the gravitational field $\left\{{\alpha \atop \beta\gamma}\right\}$ (in a new representation).

${}_{\mathrm{E}}K_{\mu}^{\ \nu} := \sqrt{-g}\,(T_{\mu\cdot}^{\cdot\nu}+{}_{\mathrm{E}}t_{\mu\cdot}^{\cdot\nu})$ is Einstein's canonical energy-momentum complex of GRT (see, e. g., [4, 6, 7]). From (6) follow the continuity equations $\partial_{\nu}\,{}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu} = 0$ from which, with the help of the Stokes Theorem [8–10], we get in asymptotically Lorentzian coordinates, ALC, the integral conservation laws of the global quantities of a closed system[4].

${}_{\mathrm{E}}t_{\mu\cdot}^{\cdot\nu}$ is either a tensor or a more general geometric object. Consequently, the local distribution of the gravitational energy density and the local distribution of the total energy density do not have any physical meaning. Moreover, if we want to calculate the global quantities of a closed system, we have to work in asymptotically Lorentzian coordinates, ALC. Many attempts were done to avoid these difficulties: or by adding a proper curl to both sides of (6), or by using single index complexes, or by using tetrads or a second metric, see e. g., [13]. In our opinion, these attempts had not given anything better than we can get from (6) and from the single index complex of Bergmann [14] which is based on (6). Especially, addition of a curl constructed from the gravitational potentials to both sides of (6) changes the global quantities of a closed system [14] and leads to the wrong transformation properties of the global quantities of a closed system [6], [15]. For this reason we will use as the energy-momentum complex in GRT Einstein's canonical energy-momentum complex ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$.

The ECT equations can be written in the combined form, see, e. g., [16, 17] or in the form given by Trautman [3]. In the combined form they have the same analytic shape as (4) with the exception that T_{μ} — the 3-form of the energy-momentum of matter is constructed now from the so-called "combined energy-momentum tensor" ${}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\nu}$, [16, 17, 18] instead of the metric energy-momentum tensor. Consequently, the energy-momentum complex of the combined formulation of ECT has formally the same form as ${}_{\mathrm{E}}K_{\mu\cdot}^{\cdot\nu}$. The differences consist in the following 1°. $T_{\mu\cdot}^{\cdot\nu} \to {}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\nu}$, 2°. ${}_{\mathrm{E}}t_{\mu\cdot}^{\cdot\nu}$ is now the Einstein pseudotensor constructed from the Riemannian part of the full metric connection $\Gamma^{\alpha}_{\beta\gamma}$.

[4] The definition of a closed system and ALC in GRT was given by Møller [6], [11] and Trautman [12]. In the framework of ECT we define a closed system and ALC in the same way.

In Trautman's formulation [3], the ECT equations have the form

$$\tfrac{1}{2}\,\eta_{\mu\alpha\cdot}^{\;\cdot\cdot\sigma}\wedge{}^{U}\Omega^{\alpha}_{\cdot\sigma}=(-)\frac{8\pi G}{c^4}\,T_{\mu},\tag{7}$$

$$\eta_{\mu\alpha\sigma}^{\cdots}\wedge\Theta^{\sigma}=\frac{8\pi G}{c^4}\,S_{\mu\alpha}^{\cdot\cdot}.\tag{8}$$

${}^{U}\Omega^{\alpha\cdot}_{\cdot\sigma}$ is here the curvature 2-form of the U_4 geometry and Θ^{σ} — the torsion 2-form of this geometry. T_{μ} denotes the energy-momentum 3-form of matter constructed from the canonical energy-momentum tensor ${}_{\rm can}T_{\mu\cdot}^{\cdot\nu}$ [3], [16]: $T_{\mu}:={}_{\rm can}T_{\mu}^{\;\nu}\eta_{\nu}$. $S_{\mu\alpha}$ is the 3-form of spin of matter: $S_{\mu\alpha}:=\tau_{\mu\alpha\cdot}^{\cdot\cdot\nu}\,\eta_{\nu}$ and $\tau_{\mu\alpha\cdot}^{\cdot\cdot\nu}$ is the canonical spin-tensor of matter [3], [16]. The rest of the notation is the same as in (4). Trautman's formulation of ECT explicitly uses the geometric characteristics of the U_4 geometry.

(7) and (8) can be written in the following, equivalent form

$$d\left(\frac{c^4}{16\pi G}\,\eta_{\mu\alpha\cdot}^{\;\cdot\cdot\sigma}\wedge\omega^{\alpha\cdot}_{\cdot\sigma}\right)=T_{\mu}+\frac{c^4}{16\pi G}\,(\Theta^{\varrho}\eta_{\mu\alpha\cdot\varrho}^{\;\cdot\cdot\sigma\cdot}\wedge\omega^{\alpha\cdot}_{\cdot\sigma}$$

$$+\eta_{\mu\alpha\cdot}^{\;\cdot\cdot\gamma}\wedge\omega^{\sigma\cdot}_{\cdot\gamma}\wedge\omega^{\alpha\cdot}_{\cdot\sigma}-\eta_{\gamma\alpha\cdot}^{\;\cdot\cdot\sigma}\wedge\omega^{\gamma\cdot}_{\cdot\mu}\wedge\omega^{\alpha\cdot}_{\cdot\sigma}),\tag{7'}$$

$$d\left(\frac{c^4}{8\pi G}\,\eta_{\mu\alpha\sigma}^{\cdots}\wedge v^{\sigma}\right)=-S_{\mu\alpha}+\frac{c^4}{8\pi G}\,(\Theta^{\varrho}\eta_{\mu\alpha\sigma\varrho}^{\cdots\cdot}\wedge v^{\sigma}$$

$$+\omega^{\gamma\cdot}_{\cdot\mu}\wedge\eta_{\gamma\alpha\sigma}^{\cdots}\wedge v^{\sigma}+\omega^{\gamma\cdot}_{\cdot\alpha}\wedge\eta_{\mu\gamma\sigma}\wedge v^{\sigma}).\tag{8'}$$

$\omega^{\alpha\cdot}_{\cdot\sigma}$ is here the connection 1-form of the U_4 geometry and v^{σ} denotes the fixed field of the co-frames.

In local coordinates and in Schouten's language we have from (7') and (8')

$$\partial_{\alpha}\left(\frac{c^4}{16\pi G}\,\varepsilon^{\nu\alpha\beta\gamma}_{\cdots\cdot}g^{\sigma\kappa}\eta_{\mu\lambda\kappa\beta}^{\cdots\cdot}\Gamma^{\lambda}_{\sigma\gamma}\right)=\sqrt{-g}\,({}_{\rm can}T_{\mu\cdot}^{\cdot\nu}+t_{\mu\cdot}^{\cdot\nu}),\tag{9}$$

$$\partial_{\sigma}\left(\frac{c^4}{8\pi G}\,\varepsilon^{\nu\sigma\gamma\beta}_{\cdots\cdot}\eta_{\mu\alpha\beta\gamma}^{\cdots\cdot}\right)=\sqrt{-g}\,(\bar{\tau}_{\mu\alpha\cdot}^{\cdot\cdot\nu}-\tau_{\mu\alpha\cdot}^{\cdot\cdot\nu}).\tag{10}$$

In the formula (9)

$$t_{\mu}^{\;\nu}=\frac{c^4}{16\pi G}\,\eta^{\nu\alpha\beta\gamma}_{\cdots\cdot}(g^{\delta\varepsilon}\eta_{\mu\varrho\varepsilon\alpha}^{\cdots\cdot}\Gamma^{\sigma}_{\delta\beta}\Gamma^{\varrho}_{\sigma\gamma}-g^{\sigma\varepsilon}\eta_{\delta\varrho\varepsilon\alpha}^{\cdots\cdot}\Gamma^{\delta}_{\mu\beta}\Gamma^{\varrho}_{\sigma\gamma}+\tfrac{1}{2}\,g^{\sigma\tau}\eta_{\mu\kappa\tau\lambda}^{\cdots\cdot}\Gamma^{\lambda}_{[\alpha\beta]}\Gamma^{\kappa}_{\sigma\gamma})$$

is the energy-momentum pseudotensor of the field $\Gamma^{\alpha}_{\beta\gamma}$ and ${}_{\rm can}T_{\mu\cdot}^{\cdot\nu}$ is the canonical energy--momentum tensor of matter. In the formula (10)

$$\bar{\tau}_{\mu\alpha\cdot}^{\cdot\cdot\nu}=\frac{c^4}{8\pi G}\,\eta^{\nu\sigma\beta\gamma}_{\cdots\cdot}(\tfrac{1}{2}\,\Gamma^{\varrho}_{[\sigma\beta]}\eta_{\mu\alpha\gamma\varrho}^{\cdots\cdot}+\Gamma^{\varrho}_{\mu\sigma}\eta_{\varrho\alpha\gamma\beta}^{\cdots\cdot}+\Gamma^{\varrho}_{\alpha\sigma}\eta_{\mu\varrho\gamma\beta}^{\cdots\cdot})$$

and $\tau_{\mu\alpha\cdot}^{\cdot\cdot\nu}$ is the canonical spin-tensor of matter.

We may call the tensor $\bar{\tau}_{\mu\alpha\cdot}^{\cdot\cdot\nu}$ "the spin-tensor of the field $\Gamma^{\alpha}_{\beta\gamma}$". In vacuum $t_{\mu}^{\ \nu} \to {}_{\mathrm{E}}t_{\mu\cdot}^{\cdot\nu}$ and

$$\frac{c^4}{16\pi G}\,\varepsilon^{\nu\alpha\beta\gamma}g^{\sigma\kappa}\eta_{\mu\lambda\kappa\beta}^{\cdot\cdot\cdot\cdot}\Gamma^{\lambda}_{\sigma\gamma} \to {}_{\mathrm{F}}U_{\mu}^{[\nu\alpha]}.$$

${}_{\mathrm{E}}t_{\mu\cdot}^{\cdot\nu}$ denotes here Einstein's canonical pseudotensor and ${}_{\mathrm{F}}U_{\mu}^{[\nu\alpha]}$ denotes Freud's superpotentials.

From (9) and (10) follow the continuity equations

$$\partial_{\nu}[\sqrt{-g}\,({}_{\mathrm{can}}T_{\mu\cdot}^{\cdot\nu}+t_{\mu\cdot}^{\cdot\nu})] = 0,$$

$$\partial_{\nu}[\sqrt{-g}\,(\bar{\tau}_{\mu\alpha\cdot}^{\cdot\cdot\nu}-\tau_{\mu\alpha\cdot}^{\cdot\cdot\nu})] = 0,$$

from which we can get, in ALC, the integral conservation laws of the global quantities of a closed system.

Outside of matter (8) and therefore also (10) become trivial. As the result, we do not have non-trivial integral conservation laws for a closed system from the continuity equations following from (10)[5]. On the other hand, we do have non-trivial integral conservation laws for a closed system from the continuity equations following from (9). The global quantities of a closed system calculated from (9) in ALC are the same as the global quantities of the same closed system calculated from the energy-momentum complex of the combined formulation of ECT. Thus, we can put

$$K_{\mu}^{\ \nu} := \sqrt{-g}\,({}_{\mathrm{can}}T_{\mu\cdot}^{\cdot\nu}+t_{\mu\cdot}^{\cdot\nu})$$

as an energy-momentum complex of ECT in Trautman's formulation. This complex differs from the energy-momentum complex of ECT in the combined formulation of ECT by the curl[6]

$$C_{\mu\cdot}^{\cdot\nu} = \partial_{\alpha}\left(\frac{c^4}{16\pi G}\,\varepsilon^{\nu\alpha\beta\gamma}g^{\sigma\kappa}\eta_{\mu\lambda\kappa\beta}^{\cdot\cdot\cdot\cdot}K_{\sigma\gamma\cdot}^{\cdot\cdot\lambda}\right).$$

$K_{\sigma\gamma\cdot}^{\cdot\cdot\lambda}$ denotes here the $(-)$ contortion tensor [16]. The curl $C_{\mu\cdot}^{\cdot\nu}$ vanishes in vacuum. The complex $K_{\mu}^{\ \nu}$, similarly as the energy-momentum complex of ECT in the combined formulation, is either a tensor or a geometric object of the other kind. Consequently, we must use it with care (we must use it in ALC).

It is interesting how spin in ECT influences the metric and the global quantities of a closed system. It appears [19], in the linear approximation[7] of ECT, that spin gives a contribution to the metric identic in form with the contribution arising from the angular momentum of a closed system. Hence, the presence of spin has influence on the components

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[5] We get all the global quantities of a closed system equal to zero.

[6] We may put as well as the energy-momentum complex of ECT in Trautman's formulation the energy-momentum complex of ECT in combined formulation because any energy-momentum complex is determined up to the curl. This problem will be discussed elsewhere.

[7] This approximation is sufficient to calculate the global quantities of a closed system.

of the global angular momentum of a closed system calculated either in the framework of the combined formulation of ECT or in the framework of Trautman's formulation of ECT. Obviously, if we want to consider the angular momentum of a closed system in ECT, we have to pass in the same way as in GRT [13], [20], to the language of single index complexes and have to use the suitable, asymptotically Killing descriptors of the components of the angular momentum. The components of the global angular momentum of a closed system must be calculated in ALC.

3. The total superenergy tensors in GRT and in ECT

Using the analytic forms of the energy-momentum complexes of GRT and ECT given in Section 2 we can calculate, according to the formula (2), the analytic forms of the corresponding, total superenergy tensors. We get the following results:

The total superenergy tensor of GRT,

$$
\begin{aligned}
{}^{\mathrm{t}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) &= {}_{\mathrm{g}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) + {}_{\mathrm{m}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) \\
&= \frac{2k}{9}\,(2\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\,[\overset{0}{T}{}^{\nu\cdot\cdot\cdot}_{\cdot\mu\alpha\beta} + \overset{0}{\overline{T}}{}^{\nu\cdot\cdot\cdot}_{\cdot\mu\alpha\beta} - \tfrac{1}{2}\,\delta^{\nu}_{\mu}\overset{0}{K}{}^{\gamma\delta\varrho\cdot}_{\cdot\cdot\cdot\beta}(\overset{0}{K}{}^{\cdot\cdot\cdot\cdot}_{\gamma\delta\varrho\alpha} + \overset{0}{K}{}^{\cdot\cdot\cdot\cdot}_{\gamma\varrho\delta\alpha}) \\
&\quad + 2\delta^{\nu}_{\mu}\bar{k}^{2}\,{}_{\mathrm{E}}\overset{0}{T}{}^{\cdot\gamma}_{(\alpha\cdot}\,{}_{\mathrm{E}}\overset{0}{T}{}^{\cdot\cdot}_{|\gamma|\beta)} - 3\bar{k}^{2}\,{}_{\mathrm{E}}\overset{0}{T}{}^{\cdot\cdot}_{\mu(\alpha}\,{}_{\mathrm{E}}\overset{0}{T}{}^{\nu}_{\cdot\beta)} - 2\bar{k}\overset{0}{K}{}^{(\gamma\cdot\nu)\cdot}_{\cdot\mu\cdot(\alpha}\,{}_{\mathrm{E}}\overset{0}{T}{}^{\cdot\cdot}_{|\gamma|\beta)}] \\
&\quad + (2\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\,[\overset{*}{\nabla}_{(\alpha}\overset{*}{\nabla}_{\beta)}\overset{0}{T}{}^{\cdot\nu}_{\mu\cdot} + \tfrac{1}{3}\,\overset{0}{K}{}^{\lambda\cdot\cdot\cdot}_{\cdot(\alpha|\mu|\beta)}\overset{0}{T}{}^{\cdot\nu}_{\lambda\cdot} - \tfrac{1}{3}\,\overset{0}{K}{}^{\nu\cdot\lambda\cdot}_{\cdot(\alpha\cdot\beta)}\overset{0}{T}{}^{\cdot\cdot}_{\mu\lambda}].
\end{aligned}
\tag{11}
$$

In the above expression $k = \dfrac{c^4}{16\pi G} = \dfrac{1}{2\bar{k}}$; ${}_{\mathrm{g}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ denotes the superenergy tensor of the gravitational field $\left\{ {\alpha \atop \beta\gamma} \right\}$ calculated from ${}_{\mathrm{E}}t_{\mu\cdot}^{\cdot\nu}$ and ${}_{\mathrm{m}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ denotes the superenergy tensor of matter[8] calculated from the metric energy-momentum tensor $T_{\mu}{}^{\nu}$. $K_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}$ denotes the curvature tensor of the Riemannian connection, ${}_{\mathrm{E}}T_{\mu\nu}^{\cdot\cdot} := T_{\mu\nu}^{\cdot\cdot} - \frac{1}{2}g_{\mu\nu}T_{\alpha\cdot}^{\cdot\alpha}$ and the meaning of other symbols is the same as in [1].

The total superenergy tensor ${}^{\mathrm{t}}_{\mathrm{com}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ of ECT in the combined formulation of ECT

The total superenergy tensor ${}^{\mathrm{t}}_{\mathrm{com}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ of ECT in the combined formulation of ECT has formally the same analytic form as the total superenergy tensor of GRT, i. e., it is given by formula (11). The only difference is that if we use the formula (11) as representing the total superenergy tensor of ECT, then $T_{\mu\cdot}^{\cdot\nu}$ will denote the combined energy-momentum tensor [16—18] instead of metric one. The total superenergy tensor ${}^{\mathrm{t}}_{\mathrm{com}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ of ECT in the combined formulation of ECT can be rewritten in terms of $R_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}(\Gamma)$ and ∇

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[8] The formula representing ${}^{\phi}{}_{\mathrm{S}}T_{\mu\cdot\alpha\beta}^{\cdot\nu\cdot\cdot}$ given in [1] contains a mistake. The correct formula in Schouten's notation is ${}^{\phi}{}_{\mathrm{S}}T_{\mu\cdot\alpha\beta}^{\cdot\nu\cdot\cdot} = \overset{*}{\nabla}_{(\alpha}\overset{*}{\nabla}_{\beta)}T_{\mu\cdot}^{\cdot\nu} + \frac{1}{3}K_{\cdot(\alpha|\mu|\beta)}^{\lambda\cdot\cdot\cdot}T_{\lambda\cdot}^{\cdot\nu} - \frac{1}{3}K_{\cdot(\alpha\cdot\beta)}^{\nu\cdot\lambda\cdot}T_{\mu\lambda}^{\cdot\cdot}$.

which characterize the U_4 geometry. The corresponding expression is given in the Appendix. The expression (11) is much simpler than the expression given in the Appendix and should be used in the concrete calculations.

The total superenergy tensor ${}_{\mathrm{T}}^{\mathrm{t}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ of ECT in Trautman's formulation of ECT

The total superenergy tensor of ECT in Trautman's formulation of this theory differs from the total superenergy tensor of ECT in the combined formulation by the contribution arising from the additional tensor

$$ {}_{\mathrm{add}}T_{\mu\cdot}^{\cdot\nu} = \frac{1}{\sqrt{-g}}\, C_{\mu\cdot}^{\cdot\nu} = \frac{c^4}{16\pi G}\, \frac{1}{\sqrt{-g}}\, \partial_\alpha\left(\varepsilon^{\nu\alpha\beta\gamma}_{\cdot\cdot\cdot\cdot}\, g^{\sigma\kappa}\, \eta_{\mu\lambda\kappa\beta}^{\cdot\cdot\cdot\cdot}\, K_{\sigma\gamma\cdot}^{\cdot\cdot\lambda}\right). $$

This contribution has the following form in terms of Riemannian part $\left\{ {\alpha \atop \beta\gamma} \right\}$ of the connection $\Gamma^{\alpha}_{\beta\gamma}$ and combined energy-momentum tensor ${}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\nu}$

$$ \begin{aligned} {}_{\mathrm{add}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho}) = (-)\,(2\overset{0}{v}{}^{\alpha}\overset{0}{v}{}^{\beta} - \overset{0}{g}{}^{\alpha\beta})\,\delta^{\nu\varrho\gamma}_{\mu\lambda\kappa} \Bigg[& -\frac{\bar{k}}{4}\,(\overset{*}{\nabla}_{\beta}\, {}_{\mathrm{E}}\overset{0}{T}{}_{\alpha\varrho}^{\cdot\cdot})\overset{0}{\overline{K}}{}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot} \\ & -\frac{\bar{k}}{3}\, {}_{\mathrm{E}}\overset{0}{T}{}_{\varrho(\alpha}^{\cdot\cdot}\overset{*}{\nabla}_{\beta)}\overset{0}{\overline{K}}{}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot} + \tfrac{1}{2}\,\overset{*}{\nabla}_{(\beta\alpha\varrho)}\overset{0}{\overline{K}}{}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot} + \tfrac{1}{2}\,\overset{0}{K}{}_{\delta(\alpha\varrho\cdot}^{\cdot\cdot\cdot\kappa}\overset{*}{\nabla}_{\beta)}\overset{0}{\overline{K}}{}^{\delta\cdot\lambda}_{\cdot\gamma\cdot} \\ & -\tfrac{1}{2}\,\overset{0}{K}{}_{\delta(\alpha\varrho|\gamma|}^{\cdot\cdot\cdot\cdot}\overset{*}{\nabla}_{\beta)}\overset{0}{\overline{K}}{}^{\kappa\delta\lambda}_{\cdot\cdot\cdot} + \tfrac{1}{2}\,\overset{0}{K}{}_{\delta(\alpha\varrho\cdot}^{\cdot\cdot\cdot\lambda}\overset{*}{\nabla}_{\beta)}\overset{0}{\overline{K}}{}^{\kappa\cdot\delta}_{\cdot\gamma\cdot} \\ & +\tfrac{1}{4}\,\overset{*}{\nabla}_{(\beta}\overset{0}{K}{}_{|\delta|\alpha\varrho)\cdot}^{\cdot\cdot\cdot\kappa}\overset{0}{\overline{K}}{}^{\delta\cdot\lambda}_{\cdot\gamma\cdot} - \tfrac{1}{4}\,\overset{*}{\nabla}_{(\beta}\overset{0}{K}{}_{|\delta|\alpha\varrho)\gamma}^{\cdot\cdot\cdot\cdot}\overset{0}{\overline{K}}{}^{\kappa\delta\lambda}_{\cdot\cdot\cdot} + \tfrac{1}{4}\,\overset{*}{\nabla}_{(\beta}\overset{0}{K}{}_{|\delta|\alpha\varrho)\cdot}^{\cdot\cdot\cdot\lambda}\overset{0}{\overline{K}}{}^{\kappa\cdot\delta}_{\cdot\gamma\cdot} \Bigg]. \end{aligned} \tag{12} $$

In (12)

$$ \overline{K}_{\mu\nu\cdot}^{\cdot\cdot\lambda} := \tau_{\mu\nu\cdot}^{\cdot\cdot\lambda} - \tau_{\nu\cdot\mu}^{\cdot\lambda\cdot} + \tau^{\lambda\cdot\cdot}_{\cdot\mu\nu} + \delta^{\lambda}_{\mu}\tau_{\nu\alpha\cdot}^{\cdot\cdot\alpha} - g_{\mu\nu}\tau^{\lambda\cdot\alpha}_{\cdot\alpha\cdot} $$

is the conspin tensor. $\tau_{\mu\nu\cdot}^{\cdot\cdot\lambda}$ denotes here the canonical spin-tensor of matter [13]. The expression of the tensor ${}_{\mathrm{add}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ in terms of ∇ and $R_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}(\Gamma)$ is given in the Appendix. The sum (11)+(12) gives the total superenergy tensor ${}_{\mathrm{T}}^{\mathrm{t}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ of ECT in Trautman's formulation of that theory expressed in terms of $\left\{ {\alpha \atop \beta\gamma} \right\}$ and ${}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\nu}$. The expression of the total superenergy tensor ${}_{\mathrm{T}}^{\mathrm{t}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ in terms of $\left\{ {\alpha \atop \beta\gamma} \right\}$ and ${}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\nu}$ is much simpler than the expression of this tensor in terms of $\Gamma^{\alpha}_{\beta\gamma}$ and ${}_{\mathrm{com}}T_{\mu\cdot}^{\cdot\nu}$.

4. The global superenergetic quantities of a closed system in GRT and in ECT

For a closed system and in ALC the "energetic" integrals expressing the global energy, E_c, and the components P_i of the global, linear momentum are convergent and the Einstein-Klein Theorem is true [12].

Let us consider, in analogy to the "energetic" integrals, the following "superenergetic" integrals

$$S_\mu(\Sigma) := \int_\Sigma {}^tS_{\mu\cdot}^{\cdot\nu} \sqrt{-g}\, d\sigma_\nu, \tag{13}$$

where ${}^tS_\mu{}^\nu$ denotes a total superenergy tensor of matter and gravitation and Σ means a space-like hypersurface which is asymptotically flat.

The vector field v^α appearing in ${}^tS_{\mu\cdot}^{\cdot\nu}(P; v^\varrho)$ is fixed in the following way: we put the time-like basic vector of every NCS(P) proportional to the time-like vector of the natural frame in the point P of the global coordinates that are used. This means physically that we use as the vector field v^α the field of the four-velocities of the observers which are at rest with respect to the global coordinates. This is a natural choice of the field v^α in fixed, global coordinates if we want to have a uniquely determined field ${}^tS_{\mu\cdot}^{\cdot\nu}$ to calculate the global superenergetic quantities of a closed system.

In ALC, (ct, x, y, z), when Σ is x^0 = const, the integrals (13) take the form

$$S_\mu = \int_{x^0=\mathrm{const}} {}^tS_{\mu\cdot}^{\cdot 0} \sqrt{-g}\, dxdydz. \tag{14}$$

Proposition: For a closed system, the integrals (14) are convergent.

Let us write the integrals (14) in the form

$$S_\mu = \underbrace{\int_{x^0=\mathrm{const}} {}_{\mathrm{m}}S_{\mu\cdot}^{\cdot 0} \sqrt{-g}\, dxdydz}_{\text{(I)}} + \underbrace{\int_{x^0=\mathrm{const}} {}_{\mathrm{g}}S_{\mu\cdot}^{\cdot 0} \sqrt{-g}\, dxdydz}_{\text{(II)}}. \tag{15}$$

The integrand in (I) does not vanish only in the domain occupied by matter, hence it is finite. In the integral (II), the contribution depending on the presence of matter also gives finite result. On the other hand, we have for a closed system and in ALC

$$\left|\begin{array}{l}\text{contribution to (II)}\\ \text{arising only from curvature}\\ \text{tensor } K_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}\end{array}\right| \sim O(r^{-6}) < |{}_{\mathrm{E}}t_\mu{}^0| \sim O(r^{-4}), \tag{16}$$

where $r := \sqrt{x^2+y^2+z^2}$ and the integrals $\int_{x^0=\mathrm{const}} |{}_{\mathrm{E}}t_\mu{}^0| \sqrt{-g}\, dxdydz$ are convergent. Thus for a closed system and in ALC, the integrals (14) do converge.

We call S_0 the global superenergy of a closed system and S_k — the components of its global, linear supermomentum.

For a stationary closed system, e.g., for Kerr's or Schwarzschild's space-time, the global superenergetic quantities, like energetic, are independent of time. Thus, in this case, we have a sort of conservation laws of the global quantities. Of course, these conservation laws do not follow from Noether and Stokes Theorems. For the more general closed systems the integrals (14) are, in general, dependent on time[9].

The integrals $S_\mu(\Sigma)$, for a fixed Σ, form a free-vector with respect to GL(4; R), moreover, the integral $S_0(V) = \int_V {}^tS_0{}^0 \sqrt{-g}\, dxdydz$ is invariant with respect to the coordinate transformations $x^{0'} = x^0, x^{k'} = x^{k'}(x^k)$, performed within the hypersurface

[9] This problem will be discussed elsewhere.

x^0 = const. From the last remark it follows that the "amount of the superenergy inside the volume V" has a physical meaning.

In the framework of ECT, the superenergetic integral quantities of a closed system calculated from ${}_{\text{com}}{}^{t}S_{\mu}{}^{\nu}$ and from ${}_{\text{T}}^{t}S_{\mu}{}^{\cdot\nu}_{\cdot}$ are, in general, different. This fact produces some difficulties[9].

5. Concluding remarks

The superenergy is a quantity which possesses in GRT and in ECT, locally and globally, some satisfactory properties. Moreover, it follows from Pirani's work [2] and from our definition of the superenergy tensors that the local superenergy flux of the gravitational wave may be properly defined.This local superenergy flux is determined by the components ${}_{g}S_{0}{}^{\cdot k}_{\cdot}$ of the superenergy tensor of the gravitational field. Consequently, we may call ${}_{g}S_{0}{}^{\cdot 0}_{\cdot}$ the density of the superenergy of the gravitational field. On the other hand, it is known that the local energy flux and the energy density of the gravitational wave cannot be satisfactorily defined because ${}_{\text{E}}t_{\mu}{}^{\nu}$ is not a tensor. We cannot also satisfactorily define the global energy flux of the gravitational wave, compare e.g. the problem of the energy flux of the Einstein–Rosen gravitational wave. Thus, in the gravitational radiation theory, the superenergy behaves better from the physical point of view, than the energy.

We can also consider the superenergy tensors in the framework of SRT. In SRT and in Lorentzian coordinates L, we get from the formula (2)[10]

$$S_{\mu\cdot}^{\cdot\nu}(x^{\varrho}; v^{\alpha}) = (2v^{\alpha}v^{\beta} - g^{\alpha\beta})T_{\mu\cdot,\alpha\beta}^{\cdot\nu}(x^{\varrho}) \underset{\text{L}}{\stackrel{*}{=}} \partial_0^2 T_{\mu\cdot}^{\cdot\nu} + \Delta T_{\mu\cdot}^{\cdot\nu};$$

$$v^{\alpha} \underset{\text{L}}{\stackrel{*}{=}} \delta_0^{\alpha}, \qquad g^{\alpha\beta} \underset{\text{L}}{\stackrel{*}{=}} \eta^{\alpha\beta}. \tag{17}$$

Consequently, the global superenergetic quantities of a closed system are equal to zero.

We will finish this paper with some remarks about the problems connected with the total superenergy tensors of GRT and ECT which still have to be solved. The first problem is the problem of uniqueness of the superenergetic global quantities of a closed system. This problem arises because: 1. there is some arbitrariness in the choice of the field v^{α} which has to be present in ${}^{t}S_{\mu}{}^{\cdot\nu}_{\cdot}$ $(P; v^{\varrho})$, 2. the energy-momentum complex of GRT and the energy-momentum complex ECT are not uniquely determined. Consequently, the total superenergy tensor of GRT and the total superenergy tensor of ECT are not uniquely determined. Probably, different possible fields of the total superenergy tensors will give different values of the global superenergetic quantities for a given closed system. In this paper we have considered only the energy-momentum complexes which naturally follow from GRT and from ECT equations and the total superenergy tensors corresponding to them with the natural fixing of the vector field v^{α}. The second problem is: what is the physical meaning of the superenergy in nature? These problems will be examined elsewhere.

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[10] In the framework of SRT we can perform the averaging and limiting processes in different points of space-time in the same, fixed Lorentzian coordinates L.

APPENDIX

We give here the expressions of the tensors ${}_{\text{com}}{}^{t}S_{\mu\cdot}^{\cdot\nu}(P;v^\varrho)$ and ${}_{\text{add}}S_{\mu\cdot}^{\cdot\nu}(P;v^\varrho)$ in terms of ∇ and $R_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}$ which characterize U_4 geometry of the Einstein–Cartan space-time.

$$
\begin{aligned}
{}_{\text{com}}{}^{t}S_{\mu\cdot}^{\cdot\nu}(P;v^\varrho) &= (2v^\alpha v^\beta - g^{\alpha\beta})\Big\{\frac{8k}{9}\big(R^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot(\alpha}R^{\cdot\cdot\cdot\cdot}_{|\mu(\lambda\sigma)|\beta)} - \tfrac{1}{2}\,\delta^\nu_\mu R^{\lambda\sigma\varrho\cdot}_{\cdot\cdot\cdot\alpha}R^{\cdot\cdot\cdot\cdot}_{\lambda(\sigma\varrho)\beta}\big) \\
&+ \tfrac{8}{9}\,R^{\nu\lambda\sigma\cdot}_{\cdot\cdot\cdot(\alpha|}\nabla_{[(\lambda}\overline{K}^{\cdot}_{|\mu]|\sigma)|\beta)} + \tfrac{8}{9}\,\nabla^{[\lambda}\overline{K}^{\nu]\sigma}_{\cdot\cdot\cdot(\alpha}R^{\cdot\cdot\cdot\cdot}_{|\mu(\lambda\sigma)|\beta)} - \delta_\mu^{\;\nu}\big(\tfrac{4}{9}\,R^{\lambda\sigma\varrho\cdot}_{\cdot\cdot\cdot\alpha}\nabla_{[(\sigma}\overline{K}^{\cdot}_{|\lambda]|\varrho)\beta} \\
&+ \tfrac{4}{9}\,\nabla^{[\sigma}\overline{K}^{\lambda]\varrho\cdot}_{\cdot\cdot\cdot\alpha}R^{\cdot\cdot\cdot\cdot}_{\lambda(\sigma\varrho)\beta}\big) - \tfrac{2}{9}\,R^{(\gamma\cdot\nu)\cdot}_{\cdot\mu\cdot(\alpha}\,{}_{\text{E}}T^{\cdot\cdot}_{|\gamma|\beta)} + \nabla_{(\alpha}\nabla_{\beta)}\,{}_{\text{com}}T_{\mu\cdot}^{\cdot\nu} + \tfrac{1}{3}\,R^{\lambda\cdot\cdot\cdot}_{\cdot(\alpha|\mu|\beta)}\,{}_{\text{com}}T_{\lambda\cdot}^{\cdot\nu} \\
&- \tfrac{1}{3}\,R^{\nu\cdot\lambda\cdot}_{\cdot(\alpha\cdot\beta)}\,{}_{\text{com}}T^{\cdot\cdot}_{\mu\lambda} + \bar{k}\big[\tfrac{8}{9}\,R^{\nu\lambda\sigma}_{\cdot\cdot\cdot(\alpha|}\overline{S}^{\cdot\cdot\varrho}_{(\lambda|\mu\cdot}\overline{K}^{\cdot}_{\varrho|\sigma)|\beta)} + \tfrac{8}{9}\,R^{\nu\lambda\sigma}_{\cdot\cdot\cdot(\alpha|}\overline{K}^{\cdot\cdot}_{[\mu|\varrho|\beta)|}\overline{K}^{\cdot\cdot\varrho}_{(\lambda]\sigma)\cdot} \\
&+ \nabla_\alpha\big[\overline{K}_\beta({}_{\text{com}}T_{\mu\cdot}^{\cdot\nu})\big] + \overline{K}_\alpha(\nabla_\beta\,{}_{\text{com}}T_{\mu\cdot}^{\cdot\nu}) + \tfrac{8}{9}\,\overline{K}^{[\nu\cdot\cdot}_{\cdot\varrho(\alpha|}\overline{K}^{\lambda]\sigma\varrho}_{\cdot\cdot\cdot}R^{\cdot\cdot\cdot\cdot}_{\mu(\lambda\sigma)|\beta)} \\
&+ \tfrac{16}{9}\,\nabla^{[\lambda}\overline{K}^{\nu]\sigma}_{\cdot\cdot\cdot(\alpha|}\nabla_{[(\lambda}\overline{K}^{\cdot}_{|\mu]|\sigma)|\beta)} + \tfrac{8}{9}\,\overline{S}^{\lambda\nu\varrho}_{\cdot\cdot\cdot}\overline{K}^{\cdot\sigma}_{\varrho\cdot(\alpha|}R^{\cdot\cdot\cdot\cdot}_{\mu(\lambda\sigma)|\beta)} \\
&- \delta^\nu_\mu\big(\tfrac{4}{9}\,R^{\lambda\sigma\varrho\cdot}_{\cdot\cdot\cdot\alpha}\overline{K}^{\cdot\cdot\cdot}_{[\lambda|\gamma\beta|}\overline{K}^{\cdot\cdot\gamma}_{(\sigma]\varrho)\cdot} + \tfrac{4}{9}\,R^{\lambda\sigma\varrho\cdot}_{\cdot\cdot\cdot\alpha}\overline{S}^{\cdot\cdot\gamma}_{(\sigma|\lambda}\overline{K}^{\cdot\cdot}_{\gamma|\varrho)\beta} + \tfrac{4}{9}\,\overline{K}^{[\lambda\cdot\cdot}_{\cdot\gamma\alpha}\overline{K}^{\sigma]\varrho\gamma}_{\cdot\cdot\cdot}R^{\cdot\cdot\cdot\cdot}_{\lambda(\sigma\varrho)\beta} \\
&+ \tfrac{8}{9}\,\nabla^{[\sigma}\overline{K}^{\lambda]\varrho\cdot}_{\cdot\cdot\cdot\alpha}\nabla_{[(\sigma}\overline{K}^{\cdot}_{|\lambda]|\varrho)\beta} + \tfrac{4}{9}\,\overline{S}^{\sigma\lambda\gamma}_{\cdot\cdot\cdot}\overline{K}^{\cdot\varrho\cdot}_{\gamma\cdot\alpha}R^{\cdot\cdot\cdot\cdot}_{\lambda(\sigma\varrho)\beta}\big) + \tfrac{2}{9}\,\delta^\nu_\mu\,{}_{\text{E}}T^{\cdot\gamma}_{\alpha\cdot}\,{}_{\text{E}}T^{\cdot\cdot}_{\gamma\beta} \\
&- \tfrac{1}{3}\,{}_{\text{E}}T^{\cdot\cdot}_{\mu(\alpha}\,{}_{\text{E}}T^{\nu\cdot}_{\cdot\beta)} - \tfrac{1}{9}\,\nabla_\mu\overline{K}^{\gamma\nu}_{\cdot\cdot(\alpha}\,{}_{\text{E}}T^{\cdot\cdot}_{|\gamma|\beta)} + \tfrac{1}{9}\,\nabla^\gamma\overline{K}^{\cdot\nu}_{\mu\cdot(\alpha}\,{}_{\text{E}}T^{\cdot\cdot}_{|\gamma|\beta)} \\
&- \tfrac{1}{9}\,\nabla_\mu\overline{K}^{\nu\gamma}_{\cdot\cdot(\alpha}\,{}_{\text{E}}T^{\cdot\cdot}_{|\gamma|\beta)} + \tfrac{1}{9}\,\nabla^\nu\overline{K}^{\cdot\gamma}_{\mu\cdot(\alpha}\,{}_{\text{E}}T^{\cdot\cdot}_{|\gamma|\beta)} + \tfrac{1}{3}\,\nabla_{(\alpha}\overline{K}^{\lambda\cdot\cdot}_{\cdot|\mu|\beta)}\,{}_{\text{com}}T_{\lambda\cdot}^{\cdot\nu} \\
&- \tfrac{1}{3}\,\nabla^\lambda\overline{K}^{\cdot\cdot\cdot}_{(\alpha|\mu|\beta)}\,{}_{\text{com}}T_{\lambda\cdot}^{\cdot\nu} - \tfrac{1}{3}\,\nabla_{(\alpha}\overline{K}^{\nu\lambda\cdot}_{\cdot\cdot\beta)}\,{}_{\text{com}}T^{\cdot\cdot}_{\mu\lambda} + \tfrac{1}{3}\,\nabla^\nu\overline{K}^{\cdot\lambda\cdot}_{(\alpha\cdot\beta)}\,{}_{\text{com}}T^{\cdot\cdot}_{\mu\lambda}\big] \\
&+ \bar{k}^2\big[\tfrac{16}{9}\big(\overline{K}^{[\nu\cdot\cdot}_{\cdot\varrho(\alpha|}\overline{K}^{\lambda]\sigma\varrho}_{\cdot\cdot\cdot}\nabla_{[(\lambda}\overline{K}^{\cdot}_{|\mu]|\sigma)|\beta)} + \nabla^{[\lambda}\overline{K}^{\nu]\sigma}_{\cdot\cdot\cdot(\alpha|}\overline{K}^{\cdot\cdot\cdot}_{[\mu|\varrho|\beta)|}\overline{K}^{\cdot\cdot\varrho}_{(\lambda]\sigma)\cdot} \\
&+ \nabla^{[\lambda}\overline{K}^{\nu]\sigma}_{\cdot\cdot\cdot(\alpha|}\overline{S}^{\cdot\cdot\varrho}_{(\lambda|\mu\cdot}\overline{K}^{\cdot}_{\varrho|\sigma)|\beta)} + \overline{S}^{\lambda\nu\varrho}_{\cdot\cdot\cdot}\overline{K}^{\cdot\sigma}_{\varrho\cdot(\alpha|}\nabla_{[(\lambda}\overline{K}^{\cdot}_{|\mu]|\sigma)|\beta)}\big) \\
&- \tfrac{8}{9}\,\delta^\nu_\mu\big(\overline{K}^{[\lambda\cdot\cdot}_{\cdot\gamma\alpha}\overline{K}^{\sigma]\varrho\gamma}_{\cdot\cdot\cdot}\nabla_{[(\sigma}\overline{K}^{\cdot}_{|\lambda]|\varrho)\beta} + \nabla^{[\sigma}\overline{K}^{\lambda]\varrho\cdot}_{\cdot\cdot\cdot\alpha}\overline{K}^{\cdot\cdot\cdot}_{[\lambda|\gamma\beta|}\overline{K}^{\cdot\cdot\gamma}_{(\sigma]\varrho)\cdot} \\
&+ \nabla^{[\sigma}\overline{K}^{\lambda]\varrho\cdot}_{\cdot\cdot\cdot\alpha}\overline{S}^{\cdot\cdot\gamma}_{(\sigma|\lambda\cdot}\overline{K}^{\cdot\cdot}_{\gamma|\varrho)\beta} + \overline{S}^{\sigma\lambda\gamma}_{\cdot\cdot\cdot}\overline{K}^{\cdot\varrho\cdot}_{\gamma\cdot\alpha}\nabla_{[(\sigma}\overline{K}^{\cdot}_{|\lambda|]\varrho)\beta}\big) + \overline{K}_\alpha\big[\overline{K}_\beta({}_{\text{com}}T_{\mu\cdot}^{\cdot\nu})\big] \\
&- \tfrac{1}{9}\,\overline{K}^{\gamma\cdot\cdot}_{\cdot\varrho(\alpha|}\overline{K}^{\cdot\nu\varrho}_{\mu\cdot\cdot}\,{}_{\text{E}}T^{\cdot\cdot}_{\gamma|\beta)} + \tfrac{1}{9}\,\overline{K}^{\cdot\cdot\cdot}_{\mu\varrho(\alpha|}\overline{K}^{\gamma\nu\varrho}_{\cdot\cdot\cdot}\,{}_{\text{E}}T^{\cdot\cdot}_{\gamma|\beta)} - \tfrac{2}{9}\,\overline{S}^{\cdot\gamma\varrho}_{\mu\cdot\cdot}\overline{K}^{\cdot\nu\cdot}_{\varrho\cdot(\alpha}\,{}_{\text{E}}T^{\cdot\cdot}_{|\gamma|\beta)} \\
&- \tfrac{1}{9}\,\overline{K}^{\nu\cdot\cdot}_{\cdot\varrho(\alpha|}\overline{K}^{\cdot\gamma\varrho}_{\mu\cdot\cdot}\,{}_{\text{E}}T^{\cdot\cdot}_{\gamma|\beta)} + \tfrac{1}{9}\,\overline{K}^{\cdot\cdot\cdot}_{\mu\varrho(\alpha|}\overline{K}^{\nu\gamma\varrho}_{\cdot\cdot\cdot}\,{}_{\text{E}}T^{\cdot\cdot}_{\gamma|\beta)} - \tfrac{2}{9}\,\overline{S}^{\mu\nu\varrho}_{\cdot\cdot\cdot}\overline{K}^{\cdot\gamma\cdot}_{\varrho\cdot(\alpha}\,{}_{\text{E}}T^{\cdot\cdot}_{|\gamma|\beta)} \\
&+ \tfrac{1}{3}\,\overline{K}^{\lambda\cdot\cdot}_{\cdot\varrho(\beta}\overline{K}^{\cdot\cdot\varrho}_{\alpha)\mu\cdot}\,{}_{\text{com}}T_{\lambda\cdot}^{\cdot\nu} - \tfrac{1}{3}\,\overline{K}^{\cdot\cdot\cdot}_{(\alpha|\varrho|\beta)}\overline{K}^{\lambda\cdot\varrho}_{\cdot\mu\cdot}\,{}_{\text{com}}T_{\lambda\cdot}^{\cdot\nu} + \tfrac{2}{3}\,\overline{S}^{\cdot\lambda\varrho}_{(\alpha|\cdot\cdot}\overline{K}^{\cdot\cdot}_{\varrho\mu|\beta)}\,{}_{\text{com}}T_{\lambda\cdot}^{\cdot\nu} \\
&- \tfrac{1}{3}\,\overline{K}^{\nu\cdot\cdot}_{\cdot\varrho(\beta}\overline{K}^{\cdot\lambda\varrho}_{\alpha)\cdot\cdot}\,{}_{\text{com}}T^{\cdot\cdot}_{\mu\lambda} + \tfrac{1}{3}\,\overline{K}^{\cdot\cdot\cdot}_{(\alpha|\varrho|\beta)}\overline{K}^{\nu\lambda\varrho}_{\cdot\cdot\cdot}\,{}_{\text{com}}T^{\cdot\cdot}_{\mu\lambda} - \tfrac{2}{3}\,\overline{S}^{\cdot\nu\varrho}_{(\alpha|\cdot\cdot}\overline{K}^{\cdot\lambda\cdot}_{\varrho\cdot|\beta)}\,{}_{\text{com}}T^{\cdot\cdot}_{\mu\lambda}\big] \\
&+ \bar{k}^3\big[\tfrac{16}{9}\big(\overline{K}^{[\nu\cdot\cdot}_{\cdot\varrho(\alpha|}\overline{K}^{\lambda]\sigma\varrho}_{\cdot\cdot\cdot}\overline{K}^{\cdot\cdot\cdot}_{[\mu|\delta|\beta)|}\overline{K}^{\cdot\cdot\delta}_{(\lambda]\sigma)\cdot} + \overline{K}^{[\nu\cdot\cdot}_{\cdot\varrho(\alpha|}\overline{K}^{\lambda]\sigma\varrho}_{\cdot\cdot\cdot}\overline{S}^{\cdot\cdot\delta}_{(\lambda|\mu\cdot}\overline{K}^{\cdot}_{\delta|\sigma)|\beta)} \\
&+ \overline{S}^{\lambda\nu\varrho}_{\cdot\cdot\cdot}\overline{K}^{\cdot\sigma}_{\varrho\cdot(\alpha|}\overline{K}^{\cdot\cdot\cdot}_{[\mu|\delta|\beta)|}\overline{K}^{\cdot\cdot\delta}_{(\lambda]\sigma)\cdot} + \overline{S}^{\lambda\nu\varrho}_{\cdot\cdot\cdot}\overline{K}^{\cdot\sigma}_{\varrho\cdot(\alpha|}\overline{S}^{\cdot\cdot\delta}_{(\lambda|\mu\cdot}\overline{K}^{\cdot}_{\delta|\sigma)|\beta)}\big)
\end{aligned}
$$

$$-\tfrac{8}{9}\,\delta^{\nu}_{\mu}(\overline{K}^{[\lambda\cdot\cdot}_{\cdot\gamma\alpha}\overline{K}^{\sigma]\varrho\gamma}_{\cdot\cdot\cdot}\overline{K}_{[\lambda|\delta\beta|}^{\cdot\cdot\cdot}\overline{K}_{(\sigma]\varrho)}^{\cdot\cdot\cdot\delta}+\overline{K}^{[\lambda\cdot\cdot}_{\cdot\gamma\alpha}\overline{K}^{\sigma]\varrho\gamma}_{\cdot\cdot\cdot}\overline{S}_{(\sigma|\lambda\cdot}^{\cdot\cdot\cdot\delta}\overline{K}_{\delta|\varrho)\beta}^{\cdot\cdot\cdot}$$

$$+\overline{S}^{\sigma\lambda\gamma}_{\cdot\cdot\cdot}\overline{K}_{\gamma\cdot\alpha}^{\cdot\varrho\cdot}\overline{K}_{[\lambda|\delta\beta|}^{\cdot\cdot\cdot}\overline{K}_{(\sigma]\varrho)}^{\cdot\cdot\cdot\delta}+\overline{S}^{\sigma\lambda\gamma}_{\cdot\cdot\cdot}\overline{K}_{\gamma\cdot\alpha}^{\cdot\varrho\cdot}\overline{S}_{(\sigma|\lambda\cdot}^{\cdot\cdot\cdot\delta}\overline{K}_{\delta|\varrho)\beta}^{\cdot\cdot\cdot})]\Big\}\,. \tag{A1}$$

$$_{\mathrm{add}}S_{\mu\cdot}^{\cdot\nu}(P;\,v^{\varrho}) = (2v^{\alpha}v^{\beta}-g^{\alpha\beta})\varepsilon^{\nu\varrho\delta\gamma}_{\cdot\cdot\cdot\cdot}\varepsilon_{\mu\lambda\kappa\delta}^{\cdot\cdot\cdot\cdot}\{\tfrac{1}{2}\,\nabla_{(\alpha}\nabla_{\beta}\nabla_{\varrho)}\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot}$$

$$+\tfrac{1}{2}\,R_{\delta(\alpha\varrho\cdot}^{\cdot\cdot\cdot\kappa}\nabla_{\beta)}\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}-\tfrac{1}{2}\,R_{\delta(\alpha\varrho|\gamma|}^{\cdot\cdot\cdot\cdot}\nabla_{\beta)}\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}+\tfrac{1}{2}\,R_{\delta(\alpha\varrho|\cdot|}^{\cdot\cdot\cdot\lambda}\nabla_{\beta)}\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}$$

$$+\tfrac{1}{4}\,\nabla_{(\beta}R_{|\delta|\alpha\varrho)\cdot}^{\cdot\cdot\cdot\kappa}\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}-\tfrac{1}{4}\,\nabla_{(\beta}R_{|\delta|\alpha\varrho)\gamma}^{\cdot\cdot\cdot\cdot}\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}+\tfrac{1}{4}\,\nabla_{(\beta}R_{|\delta|\alpha\varrho)\cdot}^{\cdot\cdot\cdot\lambda}\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}$$

$$+\bar{k}[-\tfrac{1}{4}\,(\nabla_{\beta}\,{}_{\mathrm{E}}T^{\cdot\cdot}_{\alpha\varrho})\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot}-\tfrac{1}{3}\,{}_{\mathrm{E}}T^{\cdot\cdot}_{\varrho(\alpha}\nabla_{\beta)}\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot}+\tfrac{1}{2}\,\nabla_{(\beta}\nabla_{\alpha}[\overline{K}_{\varrho)}(\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})]$$

$$+\tfrac{1}{2}\,\nabla_{(\beta}[\overline{K}_{\alpha}(\nabla_{\varrho)}\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})]+\tfrac{1}{2}\,\overline{K}_{(\beta}(\nabla_{\alpha}\nabla_{\varrho)}\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})+\nabla_{([\alpha}\overline{K}_{|\delta]|\varrho\cdot}^{\cdot\cdot\kappa}\nabla_{\beta)}\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}$$

$$+\tfrac{1}{2}\,R_{\delta(\alpha\varrho\cdot}^{\cdot\cdot\cdot\kappa}\overline{K}_{\beta)}(\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot})-\tfrac{1}{2}\,R_{\delta(\alpha\varrho|\gamma|}^{\cdot\cdot\cdot\cdot}\overline{K}_{\beta)}(\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot})$$

$$+\nabla_{([\alpha}\overline{K}_{|\delta]\gamma|\varrho}^{\cdot\cdot\cdot}\nabla_{\beta)}\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}+\tfrac{1}{2}\,R_{\delta(\alpha\varrho\cdot}^{\cdot\cdot\cdot\lambda}\overline{K}_{\beta)}(\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot})$$

$$+\nabla_{([\alpha}\overline{K}_{|\delta]|\varrho|\cdot|}^{\cdot\cdot\lambda}\nabla_{\beta)}\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}+\tfrac{1}{2}\,\nabla_{(\beta}(\nabla_{[\alpha}\overline{K}_{|\delta]|\varrho)\cdot}^{\cdot\cdot\kappa})\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}+\tfrac{1}{4}\,\overline{K}_{(\beta}R_{|\delta|\alpha\varrho)\cdot}^{\cdot\cdot\cdot\kappa}\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}$$

$$-\tfrac{1}{2}\,\nabla_{(\beta}\nabla_{[\alpha}\overline{K}_{|\delta]|\varrho)\gamma}^{\cdot\cdot\cdot}\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}-\tfrac{1}{4}\,\overline{K}_{(\beta}R_{|\delta|\alpha\varrho)\gamma}^{\cdot\cdot\cdot\cdot}\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}$$

$$+\tfrac{1}{2}\,\nabla_{(\beta}(\nabla_{[\alpha}\overline{K}_{|\delta]|\varrho)\cdot}^{\cdot\cdot\lambda})\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}+\tfrac{1}{4}\,\overline{K}_{(\beta}R_{|\delta|\alpha\varrho)\cdot}^{\cdot\cdot\cdot\lambda}\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}]$$

$$+\bar{k}^{2}[-\tfrac{1}{4}\,\overline{K}_{\beta}({}_{\mathrm{E}}T^{\cdot\cdot}_{\alpha\varrho})\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot}-\tfrac{1}{3}\,{}_{\mathrm{E}}T^{\cdot\cdot}_{\varrho(\alpha}\overline{K}_{\beta)}(\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})$$

$$+\tfrac{1}{2}\,\nabla_{(\beta}\{\overline{K}_{\alpha}[\overline{K}_{\varrho)}(\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})]\}+\tfrac{1}{2}\,\overline{K}_{(\beta}\{\nabla_{\alpha}[\overline{K}_{\varrho)}(\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})]\}$$

$$+\tfrac{1}{2}\,\overline{K}_{(\beta}[\overline{K}_{\alpha}(\nabla_{\varrho)}\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})]+\overline{K}_{[\delta|\sigma|}^{\cdot\cdot\cdot\kappa}\overline{K}_{(\alpha]\varrho}^{\cdot\cdot\cdot\sigma}\nabla_{\beta)}\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}+\overline{S}_{(\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho\cdot}^{\cdot\cdot\kappa}\nabla_{\beta)}\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}$$

$$+\nabla_{([\alpha}\overline{K}_{|\delta]|\varrho\cdot}^{\cdot\cdot\kappa}\overline{K}_{\beta)}(\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot})-\overline{K}_{[\delta|\sigma\gamma|}^{\cdot\cdot\cdot}\overline{K}_{(\alpha]\varrho}^{\cdot\cdot\sigma}\nabla_{\beta)}\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}+\nabla_{([\alpha}\overline{K}_{|\delta]\gamma|\varrho}^{\cdot\cdot\cdot}\overline{K}_{\beta)}(\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot})$$

$$+\overline{S}_{(\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma\gamma|\varrho}^{\cdot\cdot\cdot}\nabla_{\beta)}\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}+\nabla_{([\alpha}\overline{K}_{|\delta]|\varrho\cdot}^{\cdot\cdot\lambda}\overline{K}_{\beta)}(\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot})+\overline{K}_{[\delta|\sigma\cdot|}^{\cdot\cdot\lambda}\overline{K}_{(\alpha]\varrho}^{\cdot\cdot\sigma}\nabla_{\beta)}\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}$$

$$+\overline{S}_{(\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho\cdot}^{\cdot\cdot\lambda}\nabla_{\beta)}\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}+\tfrac{1}{2}\,\nabla_{(\beta}(\overline{K}_{|[\delta|\sigma\cdot|}^{\cdot\cdot\kappa}\overline{K}_{|\alpha]\varrho)}^{\cdot\cdot\sigma})\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}+\tfrac{1}{2}\,\nabla_{(\beta}(\overline{S}_{\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho)\cdot}^{\cdot\cdot\kappa})\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}$$

$$+\tfrac{1}{2}\,\overline{K}_{(\beta}(\nabla_{[\alpha}\overline{K}_{|\delta]|\varrho)\cdot}^{\cdot\cdot\kappa})\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}-\tfrac{1}{2}\,\nabla_{(\beta}(_{|}\overline{K}_{[\delta|\sigma\gamma|}^{\cdot\cdot\cdot}\overline{K}_{|\alpha]\varrho)}^{\cdot\cdot\sigma})\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}-\tfrac{1}{2}\,\nabla_{(\beta}(\overline{S}_{\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho)\gamma}^{\cdot\cdot\cdot})\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}$$

$$-\tfrac{1}{2}\,\overline{K}_{(\beta}(\nabla_{[\alpha}\overline{K}_{|\delta]|\varrho)\gamma}^{\cdot\cdot\cdot})\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}+\tfrac{1}{2}\,\nabla_{(\beta}(_{|}\overline{K}_{[\delta|\sigma|}^{\cdot\cdot\lambda}\overline{K}_{|\alpha]\varrho)}^{\cdot\cdot\sigma})\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}+\tfrac{1}{2}\,\nabla_{(\beta}(\overline{S}_{\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho)\cdot}^{\cdot\cdot\lambda})\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}$$

$$+\tfrac{1}{2}\,\overline{K}_{(\beta}(\nabla_{[\alpha}\overline{K}_{|\delta]|\varrho)\cdot}^{\cdot\cdot\lambda})\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}]+\bar{k}^{3}[\tfrac{1}{2}\,\overline{K}_{(\beta}\{\overline{K}_{\alpha}[\overline{K}_{\varrho)}(\overline{K}^{\kappa\cdot\lambda}_{\cdot\gamma\cdot})]\}$$

$$+\overline{K}_{[\delta|\sigma|}^{\cdot\cdot\kappa}\overline{K}_{(\alpha]\varrho}^{\cdot\cdot\sigma}\overline{K}_{\beta)}(\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot})+\overline{S}_{(\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho\cdot}^{\cdot\cdot\kappa}\overline{K}_{\beta)}(\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot})$$

$$-\overline{K}_{[\delta|\sigma\gamma|}^{\cdot\cdot\cdot}(\overline{K}_{\alpha]\varrho}^{\cdot\cdot\sigma}\overline{K}_{\beta)}(\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot})+\overline{S}_{(\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma\gamma|\varrho}^{\cdot\cdot\cdot}\overline{K}_{\beta)}(\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot})$$

$$+\overline{K}_{[\delta|\sigma|}^{\cdot\cdot\lambda}\overline{K}_{(\alpha]\varrho}^{\cdot\cdot\sigma}\overline{K}_{\beta)}(\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot})+\overline{S}_{(\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho\cdot}^{\cdot\cdot\lambda}\overline{K}_{\beta)}(\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot})$$

$$+\tfrac{1}{2}\,\overline{K}_{(\beta}(_{|}\overline{K}_{[\delta|\sigma\cdot|}^{\cdot\cdot\kappa}\overline{K}_{|\alpha]\varrho)}^{\cdot\cdot\sigma})\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}+\tfrac{1}{2}\,\overline{K}_{(\beta}(\overline{S}_{\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho)\cdot}^{\cdot\cdot\kappa})\overline{K}^{\delta\cdot\lambda}_{\cdot\gamma\cdot}-\tfrac{1}{2}\,\overline{K}_{(\beta}(_{|}\overline{K}_{[\delta|\sigma\gamma|}^{\cdot\cdot\cdot}\overline{K}_{|\alpha]\varrho)}^{\cdot\cdot\sigma})\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}$$

$$-\tfrac{1}{2}\,\overline{K}_{(\beta}(\overline{S}_{\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho)\gamma}^{\cdot\cdot\cdot})\overline{K}^{\kappa\delta\lambda}_{\cdot\cdot\cdot}+\tfrac{1}{2}\,\overline{K}_{(\beta}(_{|}\overline{K}_{[\delta|\sigma|}^{\cdot\cdot\lambda}\overline{K}_{|\alpha]\varrho)}^{\cdot\cdot\sigma})\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}$$

$$+\tfrac{1}{2}\,\overline{K}_{(\beta}(\overline{S}_{\alpha|\delta\cdot}^{\cdot\cdot\sigma}\overline{K}_{\sigma|\varrho)\cdot}^{\cdot\cdot\lambda})\overline{K}^{\kappa\cdot\delta}_{\cdot\gamma\cdot}]\} \tag{A2}$$

In the above expressions

$$\overline{S}_{\mu\nu\cdot}^{\cdot\cdot\varrho} = \tau_{\mu\nu\cdot}^{\cdot\cdot\varrho} - \tfrac{1}{2}\,\delta^{\varrho}_{\mu}\tau_{\kappa\nu\cdot}^{\cdot\cdot\kappa} - \tfrac{1}{2}\,\delta^{\varrho}_{\nu}\tau_{\mu\kappa\cdot}^{\cdot\cdot\kappa}$$

is the modified spin-tensor of matter and

$$\overline{K}_{\varrho}(v^{\alpha}) := (-)\overline{K}_{\varrho\lambda\cdot}^{\cdot\cdot\alpha}v^{\lambda}, \qquad \overline{K}_{\varrho}(v_{\alpha}) := \overline{K}_{\varrho\alpha\cdot}^{\cdot\cdot\lambda}v_{\lambda}$$

and so on. ${}_{\text{com}}T_{\mu\cdot}^{\cdot\nu}$ means the combined energy-momentum tensor of ECT and

$${}_{\text{E}}T_{\mu\cdot}^{\cdot\nu} := {}_{\text{com}}T_{\mu\cdot}^{\cdot\nu} - \tfrac{1}{2}\,\delta^{\nu}_{\mu}\,{}_{\text{com}}T_{\alpha\cdot}^{\cdot\alpha}.$$

Transforming the formulae (11) and (12) to the form of (A1) and A2) we used the identities which connect $\overset{*}{\nabla}$ with ∇ and $K_{\mu\nu\lambda}^{\cdot\cdot\cdot\kappa}$ with $R_{\mu\nu\lambda\cdot}^{\cdot\cdot\cdot\kappa}$. These identities are given, e. g., in [5]. The sum (A1)+(A2) gives the total superenergy tensor ${}^{\text{t}}_{\text{T}}S_{\mu\cdot}^{\cdot\nu}(P; v^{\varrho})$ of ECT in Trautman's formulation of that theory expressed in terms of U_4 geometry.

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