BEHAVIOUR OF A ROTATING ELECTRICALLY CONDUCTING SPHERE IN A TIME-INDEPENDENT HOMOGENEOUS EXTERNAL MAGNETIC FIELD (COSMIC APPROXIMATION)

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In a previous paper the general theory of the electromagnetic field generated by rotation of an electrically conducting sphere in a time-independent homogeneous external magnetic field was developed. In a second paper the results were applied to a sphere in a laboratory experiment (laboratory approximation). Here the results are specialized for a spherical rotating celestial body which generates a characteristic magnetic dipole and an electric quadrupole (cosmic approximation). The first quantity is used to treat the gyroscopic equation of motion of the body in an external magnetic field, admitting isotropic friction. The physical behaviour of the body shows two interesting features: 1. a precession with a characteristic frequency, 2. parallelization of the rotational axis towards the external field direction due to friction.

1. Some remarks on the general theory and on applications

The problem of an electrically conducting sphere (r_0 radius, σ electric conductivity) with zero electric charge, which rotates with the constant angular velocity $\boldsymbol{\omega} = \boldsymbol{j}\omega_y + \boldsymbol{k}\omega_z$ in an external homogeneous time independent magnetic field $\boldsymbol{B}_0 = \boldsymbol{k}B_0$ was investigated previously [1, 2]. Under the assumptions ($\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{r}$ velocity)

a) $\frac{v^2}{c^2} \ll 1$ (non-relativistic case),

b) $\frac{\omega}{\sigma} \ll 1$ (rather high conductivity σ),

we solved the problem exactly (cgs-system of units and vacuum). This means that our solution is exact up to the order v/c, i. e. apart from the magnetic properties also the electric properties were taken into account.

Historically the problem was initiated by Hertz [3] (old-fashioned notions and therefore a rather intransparent treatment) and continued by Besdin et al. [4] (here is no space for critical remarks and Rädler [5]). If one considers only the magnetic effects of the rotating body, one can use the rough transformation method of Landau and Lifschitz [6]. We

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treated the above problem for two reasons: first, from principal points of view we were interested in its general analytic solution; second, we thought of several cosmic applications which will be investigated in a further paper (behaviour of the solid core of the Earth in the external magnetic field generated by the fluid core, behaviour of a celestial body in the neighbourhood of a neutron star with its extremely strong magnetic field of about 10^{12} Gauss, measured (1976) by Trümper et al.).

Our general results are comprised in the Appendix. In the following we refer to the formulae reproduced there from paper [1]. Since the physical behaviour of the rotating body considered is qualitatively different for the two following cases:

a) $\frac{\sigma\omega r_0^2}{c^2} \ll 1$ (laboratory approximation [2]),

b) $\frac{\sigma\omega r_0^2}{c^2} \gg 1$ (cosmic approximation),

we have to treat both cases separately. This paper is devoted to the cosmic approximation.

2. *Notation*

We use the abbreviations (r radial coordinate)

$$\sigma_0 = \frac{2\pi\sigma}{c^2}, \quad \gamma = \frac{2}{c}\sqrt{\pi\sigma\omega}, \quad \xi = \gamma r. \tag{2.1}$$

For the calculations the functions

$$\begin{aligned} \Omega(\xi) &= A_1 \operatorname{ber}_{\frac{3}{2}}(\xi) + A_2 \operatorname{bei}_{\frac{3}{2}}(\xi), \\ \Gamma(\xi) &= A_1 \operatorname{bei}_{\frac{3}{2}}(\xi) - A_2 \operatorname{ber}_{\frac{3}{2}}(\xi) \end{aligned} \tag{2.2}$$

are very important, where $\operatorname{ber}_{3/2}$ and $\operatorname{bei}_{3/2}$ denote the Kelvin functions defined by

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi}}\left(-\frac{\cos z}{z} + \frac{\sin z}{z^{3/2}}\right) = \operatorname{ber}_{\frac{3}{2}}(\xi) \pm i \operatorname{bei}_{\frac{3}{2}}(\xi) \tag{2.3}$$

with

$$z = \xi e^{\pm\frac{3i\pi}{4}}$$

($I_{3/2}$ Bessel function). Using trigonometric and hyperbolic functions we get the following representations of the Kelvin functions [1]:

$$\begin{aligned} \operatorname{ber}_{\frac{3}{2}}(\xi) = -\sqrt{\frac{2}{\pi}}\,\xi^{-\frac{1}{2}}\Big[&\cos\frac{3\pi}{8}\cos\frac{\xi}{\sqrt{2}}\left(\cosh\frac{\xi}{\sqrt{2}} + \frac{1}{\xi}\sinh\frac{\xi}{\sqrt{2}}\right) \\ &+ \sin\frac{3\pi}{8}\sin\frac{\xi}{\sqrt{2}}\left(\sinh\frac{\xi}{\sqrt{2}} - \frac{1}{\xi}\cosh\frac{\xi}{\sqrt{2}}\right)\Big], \end{aligned} \tag{2.4}$$

$$\begin{aligned} \operatorname{bei}_{\frac{3}{2}}(\xi) = -\sqrt{\frac{2}{\pi}}\,\xi^{-\frac{1}{2}}\Big[&\cos\frac{3\pi}{8}\sin\frac{\xi}{\sqrt{2}}\left(\sinh\frac{\xi}{\sqrt{2}} + \frac{1}{\xi}\cosh\frac{\xi}{\sqrt{2}}\right) \\ &- \sin\frac{3\pi}{8}\cos\frac{\xi}{\sqrt{2}}\left(\cosh\frac{\xi}{\sqrt{2}} - \frac{1}{\xi}\sinh\frac{\xi}{\sqrt{2}}\right)\Big]. \end{aligned} \tag{2.5}$$

3. *Application to a spherical celestial body*

Let us now apply the formulae of the Appendix to a spherical celestial body, i. e. to a sphere of the size of the Earth or even larger. This means that we have to approximate the Kelvin functions in a suitable manner to obtain approximations of both important functions Ω and Γ.

In contrast to a sphere in laboratory experiments, where we found a good approximation by series expansion with respect to ξ (laboratory approximation [2]), the situation here is much more difficult. Dealing with a celestial body we have to consider two regions in which different approximations apply:

a) $\dfrac{\sigma\omega r^2}{c^2} \ll 1$ (laboratory approximation valid in the region about the origin)

b) $\dfrac{\sigma\omega r^2}{c^2} \gg 1$ (cosmic approximation). (3.1)

To illustrate this it is convenient to perform some estimates using figures similar to those of the Earth:

$$\omega = 3 \cdot 10^{-5}\,\mathrm{s}^{-1}, \qquad \sigma = 3 \cdot 10^{15}\,\mathrm{s}^{-1}.$$

The transition region between the two approximations is determined by

$$\frac{\sigma\omega r_t^2}{c^2} = 1, \qquad \text{i.e.} \qquad r_t = \frac{c}{\sqrt{\sigma\omega}} = 1\ \text{km}.$$

Though the region about the origin is extremely small as compared with the celestial body, we must nevertheless distinguish between both regions, because the functions exhibit different behaviour. The cosmic approximation functions in particular do not fulfil the regularity at $r = 0$. For the region about the origin we can adopt the expansion results from the laboratory approximation, but without fixed integration constants, since the boundary value problem for the celestial body must be solved once again from the beginning.

Let us now treat the cosmic approximation. We approximate the Kelvin functions under the condition

$$\alpha = \frac{\xi}{\sqrt{2}} \gg 1. \tag{3.2}$$

Exploiting the limit behaviour

$$\cosh\alpha \to \tfrac{1}{2}e^{\alpha}, \qquad \sinh\alpha \to \tfrac{1}{2}e^{\alpha} \tag{3.3}$$

from (2.4) and (2.5) we are led to

$$\mathrm{ber}_{\frac{3}{2}}(\xi) = -\frac{\xi^{-\frac{1}{2}}e^{\alpha}}{\sqrt{2\pi}}\cos\left(\alpha - \frac{3\pi}{8}\right), \qquad \mathrm{bei}_{\frac{3}{2}}(\xi) = -\frac{\xi^{-\frac{1}{2}}e^{\alpha}}{\sqrt{2\pi}}\sin\left(\alpha - \frac{3\pi}{8}\right). \tag{3.4}$$

This implies instead of (2.2)

$$\Omega = -\frac{\xi^{-\frac{1}{2}}e^{\alpha}}{\sqrt{2\pi}}\left[A_1 \cos\left(\alpha - \frac{3\pi}{8}\right) + A_2 \sin\left(\alpha - \frac{3\pi}{8}\right)\right], \tag{3.5}$$

$$\Gamma = -\frac{\xi^{-\frac{1}{2}}e^{\alpha}}{\sqrt{2\pi}}\left[A_1 \sin\left(\alpha - \frac{3\pi}{8}\right) - A_2 \cos\left(\alpha - \frac{3\pi}{8}\right)\right]. \tag{3.6}$$

Furthermore, from (A.1) and (A.2) we obtain ($\xi_0 = \gamma r_0$)

$$A_1 = -\frac{3\sqrt{2\pi}}{\omega\gamma^{\frac{3}{2}}}\omega_y B_0 \xi_0 e^{-\frac{\xi_0}{\sqrt{2}}} \cos\left(\frac{\xi_0}{\sqrt{2}} - \frac{5\pi}{8}\right), \tag{3.7}$$

$$A_2 = -\frac{3\sqrt{2\pi}}{\omega\gamma^{\frac{3}{2}}}\omega_y B_0 \xi_0 e^{-\frac{\xi_0}{\sqrt{2}}} \sin\left(\frac{\xi_0}{\sqrt{2}} - \frac{5\pi}{8}\right), \tag{3.8}$$

i. e.

$$A_1^2 + A_2^2 = \frac{18\pi\omega_y^2 B_0^2 \xi_0^2 e^{-\sqrt{2}\,\xi_0}}{\omega^2\gamma^3}. \tag{3.9}$$

With the help of these results (3.5) and (3.6) read

$$\Omega = \frac{3\omega_y B_0 r_0 r^{-\frac{1}{2}}}{\omega\gamma} e^{\frac{\gamma}{\sqrt{2}}(r-r_0)} \cos\left[\frac{\gamma}{\sqrt{2}}(r-r_0) + \frac{\pi}{4}\right], \tag{3.10}$$

$$\Gamma = \frac{3\omega_y B_0 r_0 r^{-\frac{1}{2}}}{\omega\gamma} e^{\frac{\gamma}{\sqrt{2}}(r-r_0)} \sin\left[\frac{\gamma}{\sqrt{2}}(r-r_0) + \frac{\pi}{4}\right]. \tag{3.11}$$

Further treatment consists in rather tedious calculations to get the physically interesting quantities, presented in the Appendix without approximation. In the following we shall list our results.

4. Physical quantities inside the sphere

Magnetic field:

$$\tilde{B}_r = \frac{\omega_y B_0}{\omega^2}(\omega_z \sin\Theta \sin\varphi - \omega_y \cos\Theta),$$

$$\tilde{B}_\Theta = \frac{\omega_y^2 B_0}{\omega^2}\left\{1 - \frac{3r_0}{2r} e^{\frac{\gamma}{\sqrt{2}}(r-r_0)} \cos\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\right\}\sin\Theta$$

$$-\frac{3}{2}\frac{\omega_y B_0 r_0}{\omega r} e^{\frac{\gamma}{\sqrt{2}}(r-r_0)} \sin\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\cos\Theta\cos\varphi$$

$$+\frac{\omega_y\omega_z B_0}{\omega^2}\left\{1 - \frac{3r_0}{2r} e^{\frac{\gamma}{\sqrt{2}}(r-r_0)} \cos\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\right\}\cos\Theta\sin\varphi, \tag{4.1}$$

$$\tilde{B}_\varphi = \frac{\omega_y\omega_z B_0}{\omega^2}\left\{1-\frac{3r_0}{2r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\cos\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\right\}\cos\varphi$$

$$+\tfrac{3}{2}\,\frac{\omega_y B_0 r_0}{\omega r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\sin\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\sin\varphi;$$

electric field:

$$\tilde{E}_r = -\frac{r\omega_z B_0}{2c}(1-\cos 2\Theta)-\frac{r\omega_y^2\omega_z B_0}{4c\omega^2}\left\{1-\tfrac{3}{2}\,\frac{r_0}{r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\cos\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\right\}(1+3\cos 2\Theta)$$

$$-\frac{3\omega_y\omega_z B_0 r_0}{4c\omega}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\sin\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\sin 2\Theta\cos\varphi$$

$$+\frac{r\omega_y B_0}{c\omega^2}\left[\omega_z^2+\tfrac{3}{4}(\omega_y^2-\omega_z^2)\frac{r_0}{r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\cos\left(\frac{\gamma}{\sqrt{2}}(r-r_0)\right)\right]\sin 2\Theta\sin\varphi,$$

$$\tilde{E}_\Theta = -\frac{r\omega_z B_0}{2c\omega^2}(\omega^2-\tfrac{3}{2}\,\omega_y^2)\sin 2\Theta+\frac{r\omega_y\omega_z^2 B_0}{c\omega^2}\cos 2\Theta\sin\varphi,$$

$$\tilde{E}_\varphi = \frac{r\omega_y\omega_z^2 B_0}{c\omega^2}\cos\Theta\cos\varphi; \qquad (4.2)$$

electric current density:

$$j_r = 0,$$

$$j_\Theta = \frac{3c\sigma_0\omega_y B_0 r_0}{4\pi\gamma r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}$$

$$\times\left[\frac{\omega_z}{\omega}\cos\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)+\frac{\pi}{4}\right\}\cos\varphi-\sin\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)+\frac{\pi}{4}\right\}\sin\varphi\right],$$

$$j_\varphi = -\frac{3c\sigma_0\omega_y B_0 r_0}{4\pi\gamma r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\left[\frac{\omega_y}{\omega}\cos\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)+\frac{\pi}{4}\right\}\sin\Theta\right.$$

$$\left.+\sin\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)+\frac{\pi}{4}\right\}\cos\Theta\cos\varphi+\frac{\omega_z}{\omega}\cos\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)+\frac{\pi}{4}\right\}\cos\Theta\sin\varphi\right];$$

charge density:

$$\varrho = -\frac{\omega_z B_0}{2\pi c}+\frac{3\omega_y^2\omega_z r_0 B_0}{16\pi c\omega^2 r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\cos\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)\right\}(1+3\cos 2\Theta)$$

$$-\frac{3\omega_y\omega_z r_0 B_0}{8\pi c\omega r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\sin\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)\right\}\sin 2\Theta\cos\varphi$$

$$+\frac{3\omega_y(\omega_y^2-\omega_z^2)r_0 B_0}{8\pi c\omega^2 r}\,e^{\frac{\gamma}{\sqrt{2}}(r-r_0)}\cos\left\{\frac{\gamma}{\sqrt{2}}(r-r_0)\right\}\sin 2\Theta\sin\varphi. \qquad (4.4)$$

5. *Surface charge density and Joule's heat production*

Both these quantities being integral quantities depend on the parameter r_0. Since the relation $r_0 \gg \frac{c}{\sqrt{\sigma\omega}}$ is always valid for the interesting applications, we get rid of the above mentioned separation of the regions. Therefore in a good approximation from (A.9) for the surface charge density

$$\lambda = \frac{r_0 B_0}{16\pi c\omega^2}\left[\omega_z(\omega_z^2+2\omega_y^2)-\omega_z(2\omega_y^2+5\omega_z^2)\cos 2\Theta-\omega_y(3\omega_y^2+7\omega_z^2)\sin 2\Theta\sin\varphi\right] \quad (5.1)$$

results. Furthermore, approximating (A.13) and using (3.9) we find for Joule's heat production of the body (A.11) the expression

$$\frac{d\tilde{Q}}{dt} = \frac{3\omega_y^2 B_0^2 r_0^2 c}{4\sqrt{2}\,\omega\sqrt{\pi\sigma\omega}} \text{ (higher order of magnitude).} \quad (5.2)$$

6. *Physical quantities outside the sphere*

For the magnetic moment (A.16) we obtain

$$m_x = 0, \quad m_y = \frac{\omega_y\omega_z r_0^3 B_0}{2\omega^2}, \quad m_z = -\frac{\omega_y^2 r_0^3 B_0}{2\omega^2}. \quad (6.1)$$

The electric quadrupole moment tensor (A.18) takes the form

$$P_{11} = P_{22} = \frac{\omega_z r_0^5 B_0}{6c}\left(1-\frac{3}{2}\frac{\omega_y^2}{\omega^2}\right), \quad P_{33} = -2P_{11}, \quad (6.2)$$

$$P_{12} = 0, \quad P_{13} = 0, \quad P_{23} = -\frac{\omega_y\omega_z^2 r_0^5 B_0}{2c\omega^2}.$$

Sections 4, 5 and 6 summarize the main results for further application. From the physical point of view, in contrast to the laboratory approximation, where the effects increase proportionally to the conductivity σ, in the celestial body approximation a physically surprising phenomenon occurs, that is the conductivity does not primarily affect the interesting quantities, e. g. the magnetic moment (6.1). This means that these quantities are rather insensitive with respect to a change of the conductivity. But one should keep in mind that there does exist a lower limit for the conductivity, determined by the approximation inequality

$$\frac{\sigma\omega r_0^2}{c^2} \gg 1.$$

7. Angular velocity decrease by Joule heating

Let us now consider a body rotating with a constant angular velocity about a fixed axis and generating the above calculated electromagnetic field. The kinetic energy of the body is

$$T_{\text{kin}} = \tfrac{1}{2}\,\bar{\theta}\omega^2 \ (\bar{\theta} \text{ moment of inertia}). \tag{7.1}$$

Provided that no further forces act, the decrease of kinetic energy is equal to Joule heating of the body. Hence for the relative decrease of the angular velocity the formula

$$\frac{1}{\omega}\frac{d\omega}{dt} = -\frac{1}{2T_{\text{kin}}}\frac{d\tilde{Q}}{dt} = -\frac{1}{\bar{\theta}\omega^2}\frac{d\tilde{Q}}{dt} \tag{7.2}$$

results. Inserting the expression (5.2) we find

$$\frac{1}{\omega}\frac{d\omega}{dt} = -\frac{3B_0^2 r_0^2 c\omega_y^2}{4\sqrt{2}\,\bar{\theta}\omega^3\sqrt{\pi\sigma\omega}}. \tag{7.3}$$

8. The spherical celestial body as a gyroscope

Taking only into account the magnetic effects, i. e. neglecting the electric interaction, the gyroscopic equation of motion of a rigid body in an external magnetic field B_0 reads:

$$\bar{\theta}\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{m}\times\boldsymbol{B}_0 + \boldsymbol{M} \tag{8.1}$$

where a moment $\boldsymbol{M}$ of the friction force is admitted. Here the expressions for the magnetic moment (A.16) written in vectorial form to avoid the distinction of the previously preferred direction, must be inserted (it is assumed that the physical parameters permit the stationary treatment of the problem). The vectorial form of (6.1) reads

$$\boldsymbol{m} = \frac{r_0^3}{2\omega^2}\,\omega X(\omega\times\boldsymbol{B}_0). \tag{8.2}$$

Let us now consider the following model: We describe the motion of the spherical gyroscope with respect to an inertial frame. The gyroscope, surrounded by a fluid, may be embedded concentrically in an external spherical shell, uniformly rotating with the angular velocity $\boldsymbol{\Omega}_0$. Assuming isotropic friction, for the moment of friction force we apply the ansatz:

$$\boldsymbol{M} = \kappa(\boldsymbol{\Omega}_0 - \boldsymbol{\omega}). \tag{8.3}$$

According to calculations of Steenbeck and Helmis [7], the material coefficient κ takes the form

$$\kappa = \frac{8\pi r_0^4\eta}{3d} \tag{8.4}$$

if the thickness d of the fluid layer is rather small compared with the linear dimension of the configuration (η viscosity). One realizes that this moment disappears if both angular velocities coincide. For simplicity we choose $\boldsymbol{\Omega}_0$ parallel to the external magnetic field:

$$\boldsymbol{\Omega}_0 = \boldsymbol{k}\Omega_0. \tag{8.5}$$

Inserting (8.2) and (8.3) into (8.1) we obtain the equation of motion of the gyroscope in component form:

$$\frac{d\omega_x}{dt} = -\frac{\kappa}{\bar{\theta}}\omega_x + \frac{r_0^3 B_0^2}{2\bar{\theta}}\frac{\omega_y\omega_z}{\omega^2},$$

$$\frac{d\omega_y}{dt} = -\frac{\kappa}{\bar{\theta}}\omega_y - \frac{r_0^3 B_0^2}{2\bar{\theta}}\frac{\omega_x\omega_z}{\omega^2}, \qquad \frac{d\omega_z}{dt} = \frac{\kappa}{\bar{\theta}}(\Omega_0-\omega_z). \tag{8.6}$$

The last differential equation can be integrated immediately. The result reads

$$\omega_z = \bar{\omega}_z e^{-\frac{t}{T_\kappa}} + \Omega_0, \tag{8.7}$$

where $\bar{\omega}_z$ is an integration constant and

$$T_\kappa = \frac{\bar{\theta}}{\kappa} \tag{8.8}$$

is a characteristic time constant being a measure for the time needed for the parallelization of the angular velocities. Inserting now the solution (8.7) into (8.6a) and (8.6b) we obtain a linear system of two differential equations, which can be solved in the following way: We multiply the first equation by $2\omega_x$ and the second one by $2\omega_y$ and add them. Introducing

$$\omega_R^2 = \omega_x^2 + \omega_y^2 \tag{8.9}$$

we find the solution

$$\omega_R = \omega_{R_0} e^{-\frac{t}{T_\kappa}} \tag{8.10}$$

being an analogue to (8.7).

If we multiply the second equation with i and add the result to the first one a differential equation occurs which can be integrated. The result is

$$\omega_x + i\omega_y = \bar{\omega}\exp\left[-\frac{t}{T_\kappa} - \frac{ir_0^3 B_0^2}{2\bar{\theta}}\int \frac{\Omega_0 + \bar{\omega}_z e^{-t/T_\kappa}}{\omega_{R_0}^2 e^{-2t/T_\kappa} + (\Omega_0 + \bar{\omega}_z e^{-t/T_\kappa})^2}\,dt\right] \tag{8.11}$$

($\bar{\omega}$ integration constant). As regards the physical interpretation, we realize that a superposition of friction and precession takes place. For small friction a good approximation of (8.11) is given by

$$\omega_x + i\omega_y = \bar{\omega}e^{-\frac{t}{T_\kappa}} e^{i\Omega_P t}, \tag{8.12}$$

where the precession frequency reads

$$\Omega_P = -\frac{r_0^3 B_0^2}{2\bar{\theta}} \frac{\omega_{z0}}{\omega_0^2} \quad (\omega_0^2 = \omega_{R_0}^2 + \omega_{z0}^2). \tag{8.13}$$

If $\omega_{z0} > 0$, the precession angular frequency is negative, i. e. the precession is clock-wise.

Finally, we notice that from (8.3) for the friction heat production

$$\frac{d\tilde{Q}_f}{dt} = \kappa(\Omega_0 - \omega)^2 \tag{8.14}$$

results.

On further application of the above calculations will be reported in a following paper.

APPENDIX

This Appendix contains the results of the general theory [1]. The formulae are written in a different form which is more convenient for practical calculations, namely the Kelvin functions are eliminated in favour of the trigonometric and hyperbolic functions. Under these circumstances the coefficients A_1 and A_2 in (2.2) take the new shape ($\xi_0 = \gamma r_0$):

$$A_1 = \frac{3\omega_y B_0 \sqrt{\pi}\, r_0}{\omega \sqrt{\gamma}} \tag{A.1}$$

$$\times \frac{\cos\frac{3\pi}{8}\left(\cos\frac{\xi_0}{\sqrt{2}}\sinh\frac{\xi_0}{\sqrt{2}} + \sin\frac{\xi_0}{\sqrt{2}}\cosh\frac{\xi_0}{\sqrt{2}}\right) + \sin\frac{3\pi}{8}\left(\sin\frac{\xi_0}{\sqrt{2}}\cosh\frac{\xi_0}{\sqrt{2}} - \cos\frac{\xi_0}{\sqrt{2}}\sinh\frac{\xi_0}{\sqrt{2}}\right)}{\cos(\xi_0\sqrt{2}) - \cosh(\xi_0\sqrt{2})}$$

$$A_2 = \frac{3\omega_y B_0 \sqrt{\pi} r_0}{\omega \sqrt{\gamma}} \tag{A.2}$$

$$\times \frac{\cos\frac{3\pi}{8}\left(\sin\frac{\xi_0}{\sqrt{2}}\cosh\frac{\xi_0}{\sqrt{2}} - \cos\frac{\xi_0}{\sqrt{2}}\sinh\frac{\xi_0}{\sqrt{2}}\right) - \sin\frac{3\pi}{8}\left(\cos\frac{\xi_0}{\sqrt{2}}\sinh\frac{\xi_0}{\sqrt{2}} + \sin\frac{\xi_0}{\sqrt{2}}\cosh\frac{\xi_0}{\sqrt{2}}\right)}{\cos(\xi_0\sqrt{2}) - \cosh(\xi_0\sqrt{2})}.$$

On the surface of the sphere the quantities (2.2) read

$$\Omega(\xi_0) = -\frac{3\omega_y B_0}{\omega\gamma^2\sqrt{r_0}}\left\{1 + \frac{\xi_0}{\sqrt{2}}\,\frac{\sin(\xi_0\sqrt{2}) + \sinh(\xi_0\sqrt{2})}{\cos(\xi_0\sqrt{2}) - \cosh(\xi_0\sqrt{2})}\right\}, \tag{A.3}$$

$$\Gamma(\xi_0) = \frac{3\omega_y B_0\sqrt{r_0}}{\sqrt{2}\,\gamma\omega}\,\frac{\sin(\xi_0\sqrt{2}) - \sinh(\xi_0\sqrt{2})}{\cos(\xi_0\sqrt{2}) - \cosh(\xi_0\sqrt{2})}. \tag{A.4}$$

Results for the interior of the sphere

Magnetic field ($\boldsymbol{B} = \boldsymbol{B}_0 + \tilde{\boldsymbol{B}}$):

$$\tilde{B}_r = \frac{\omega_y}{\omega}\left(r^{-\frac{3}{2}}\Gamma - \frac{\omega_y}{\omega}B_0\right)\cos\Theta + r^{-\frac{3}{2}}\Omega\sin\Theta\cos\varphi - \frac{\omega_z}{\omega}\left(r^{-\frac{3}{2}}\Gamma - \frac{\omega_y}{\omega}B_0\right)\sin\Theta\sin\varphi,$$

$$\tilde{B}_\Theta = -\frac{\omega_y}{\omega}\left(\tfrac{1}{4}r^{-\frac{3}{2}}\Gamma + \tfrac{1}{2}\gamma r^{-\frac{1}{2}}\Gamma' - \frac{\omega_y}{\omega}B_0\right)\sin\Theta + \left(\tfrac{1}{4}r^{-\frac{3}{2}}\Omega + \tfrac{1}{2}r^{-\frac{1}{2}}\Omega'\right)\cos\Theta\cos\varphi$$
$$- \frac{\omega_z}{\omega}\left(\tfrac{1}{4}r^{-\frac{3}{2}}\Gamma + \tfrac{1}{2}r^{-\frac{1}{2}}\gamma\Gamma' - \frac{\omega_y}{\omega}B_0\right)\cos\Theta\sin\varphi,$$

$$\tilde{B}_\varphi = -\frac{\omega_z}{\omega}\left(\tfrac{1}{4}r^{-\frac{3}{2}}\Gamma + \tfrac{1}{2}r^{-\frac{1}{2}}\gamma\Gamma' - \frac{\omega_y}{\omega}B_0\right)\cos\varphi - \left(\tfrac{1}{4}r^{-\frac{3}{2}}\Omega + \tfrac{1}{2}r^{-\frac{1}{2}}\gamma\Omega'\right)\sin\varphi; \quad \text{(A.5)}$$

electric field ($\boldsymbol{E} = \tilde{\boldsymbol{E}}$):

$$\tilde{E}_r = -\frac{r\omega_z}{2c}B_0(1-\cos 2\Theta) - \frac{r\omega_y\omega_z}{4c\omega^2}\left[B_0\omega_y - \frac{\omega}{2}\left(\tfrac{1}{2}r^{-\frac{3}{2}}\Gamma + r^{-\frac{1}{2}}\gamma\Gamma'\right)\right](1+3\cos 2\Theta)$$
$$+ \frac{\omega_z}{4c}\left(\tfrac{1}{2}r^{-\frac{1}{2}}\Omega + \gamma r^{\frac{1}{2}}\Omega'\right)\sin 2\Theta\cos\varphi$$
$$+ \frac{r}{2c\omega^2}\left[2\omega_y\omega_z^2B_0 + \frac{\omega}{2}(\omega_y^2-\omega_z^2)\left(\tfrac{1}{2}r^{-\frac{3}{2}}\Gamma + r^{-\frac{1}{2}}\gamma\Gamma'\right)\right]\sin 2\Theta\sin\varphi,$$

$$\tilde{E}_\Theta = -\frac{r\omega_z}{2c\omega^2}\left[(\omega^2 - \tfrac{3}{2}\omega_y^2)B_0 + \tfrac{3}{2}\omega_y\omega r^{-\frac{3}{2}}\Gamma\right]\sin 2\Theta + \frac{\omega_z}{2c}r^{-\frac{1}{2}}\Omega\cos 2\Theta\cos\varphi$$
$$+ \frac{r}{2c\omega^2}\left[2\omega_y\omega_z^2B_0 + (\omega_y^2-\omega_z^2)\omega r^{-\frac{3}{2}}\Gamma\right]\cos 2\Theta\sin\varphi,$$

$$\tilde{E}_\varphi = \frac{r}{2c\omega^2}\left[2\omega_y\omega_z^2B_0 + (\omega_y^2-\omega_z^2)\omega r^{-\frac{3}{2}}\Gamma\right]\cos\Theta\cos\varphi - \frac{\omega_z}{2c}r^{-\frac{1}{2}}\Omega\cos\Theta\sin\varphi; \quad \text{(A.6)}$$

electric current density:

$$j_r = 0,$$

$$j_\Theta = \frac{c\sigma_0}{4\pi}r^{-\frac{1}{2}}(\omega_z\Omega\cos\varphi - \omega\Gamma\sin\varphi),$$

$$j_\varphi = -\frac{c\sigma_0}{4\pi}r^{-\frac{1}{2}}(\omega_y\Omega\sin\Theta + \omega\Gamma\cos\Theta\cos\varphi + \omega_z\Omega\cos\Theta\sin\varphi); \quad \text{(A.7)}$$

true electric charge density:

$$\varrho = -\frac{\omega_zB_0}{2\pi c} - \frac{\omega_y\omega_z}{16\pi c\omega}\left[\tfrac{3}{2}r^{-\frac{3}{2}}\Gamma - \gamma r^{-\frac{1}{2}}\Gamma'\right](1+3\cos 2\Theta)$$
$$- \frac{\omega_z}{16\pi c}\left[3r^{-\frac{3}{2}}\Omega - 2\gamma r^{-\frac{1}{2}}\Omega'\right]\sin 2\Theta\cos\varphi$$
$$+ \frac{\omega_z^2-\omega_y^2}{8\pi c\omega}\left[\tfrac{3}{2}r^{-\frac{3}{2}}\Gamma - r^{-\frac{1}{2}}\gamma\Gamma'\right]\sin 2\Theta\sin\varphi; \quad \text{(A.8)}$$

electric surface charge density:

$$\lambda = \frac{r_0}{16\pi c\omega^2}\,[\omega_z(\omega_z^2+2\omega_y^2)B_0-\omega_y\omega_z\omega r_0^{-\frac{3}{2}}\Gamma(\gamma r_0)$$
$$-\{\omega_z(2\omega_y^2+5\omega_z^2)B_0+3\omega_y\omega_z\omega r_0^{-\frac{3}{2}}\Gamma(\gamma r_0)\}\cos 2\Theta$$
$$-2\omega_z\omega^2 r_0^{-\frac{3}{2}}\Omega(\gamma r_0)\sin 2\Theta\cos\varphi$$
$$-\{\omega_y(3\omega_y^2+7\omega_z^2)B_0+2(\omega_y^2-\omega_z^2)\omega r_0^{-\frac{3}{2}}\Gamma(\gamma r_0)\}\sin 2\Theta\sin\varphi]; \tag{A.9}$$

Joule's heat production density:

$$\frac{d\tilde{q}}{dt} = \sigma\left(\boldsymbol{E}+\frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}\right)^2; \tag{A.10}$$

Joule's integral heat production of the sphere:

$$\frac{d\tilde{Q}}{dt} = \frac{2\pi\sigma\omega^2}{3c^2}\,(A_1^2+A_2^2)\int\limits_{r=0}^{r_0} r|J_{\frac{3}{2}}(\gamma r e^{\frac{3\pi i}{4}})|^2 dr, \tag{A.11}$$

where

$$A_1^2+A_2^2 = -\frac{9\pi\omega_y^2 B_0^2 r_0^2}{\omega^2\gamma[\cos(\xi_0\sqrt{2})-\cosh(\xi_0\sqrt{2})]} \tag{A.12}$$

and

$$\int\limits_{r=0}^{r_0} r|J_{\frac{3}{2}}(\gamma r e^{\frac{3\pi i}{4}})|^2 dr = \frac{1}{\pi\gamma^2\xi_0}\left[\frac{\xi_0}{\sqrt{2}}\{\sinh(\xi_0\sqrt{2})+\sin(\xi_0\sqrt{2})\}\right.$$
$$\left.+\cos(\xi_0\sqrt{2})-\cosh(\xi_0\sqrt{2})\right]. \tag{A.13}$$

Hence we find

$$\frac{d\tilde{Q}}{dt} = -\frac{3\omega_y^2 B_0^2 r_0 c^2}{8\pi\sigma\omega^2}\left[1+\frac{\xi_0}{\sqrt{2}}\,\frac{\sin(\xi_0\sqrt{2})+\sinh(\xi_0\sqrt{2})}{\cos(\xi_0\sqrt{2})-\cosh(\xi_0\sqrt{2})}\right]. \tag{A.14}$$

Results for the exterior of the sphere

Magnetic dipole field:

$$\hat{\boldsymbol{B}} = \frac{3}{r^3}(\boldsymbol{m}\boldsymbol{e}_r)\boldsymbol{e}_r-\frac{\boldsymbol{m}}{r^3} \tag{A.15}$$

with the magnetic moment

$$m_x = \tfrac{1}{2}\,r_0^{\frac{3}{2}}\Omega(\gamma r_0), \qquad m_y = -\frac{\omega_z}{2\omega}\,r_0^{\frac{3}{2}}\Gamma(\gamma r_0)+\frac{\omega_y\omega_z}{2\omega^2}\,r_0^3 B_0,$$
$$m_z = \frac{\omega_y}{2\omega}\,r_0^{\frac{3}{2}}\Gamma(\gamma r_0)-\frac{\omega_y^2}{2\omega^2}\,r_0^3 B_0; \tag{A.16}$$

electric quadrupole field:

$$\hat{E}_r = \frac{1}{r^4}\left[\tfrac{3}{4} P_{33}(1+3\cos 2\Theta)+3(P_{13}\cos\varphi+P_{23}\sin\varphi)\sin 2\Theta\right.$$

$$\left.+\tfrac{3}{2}\left\{\tfrac{1}{2}(P_{11}-P_{22})\cos 2\varphi+P_{12}\sin 2\varphi\right\}(1-\cos 2\Theta)\right],$$

$$\hat{E}_\Theta = \frac{1}{r^4}\left[\tfrac{3}{2} P_{33}\sin 2\Theta-2(P_{13}\cos\varphi+P_{23}\sin\varphi)\cos 2\Theta\right.$$

$$\left.-\left\{\tfrac{1}{2}(P_{11}-P_{22})\cos 2\varphi+P_{12}\sin 2\varphi\right\}\sin 2\Theta\right], \qquad \text{(A.17)}$$

$$\hat{E}_\varphi = \frac{1}{r^4}\left[2(P_{13}\sin\varphi-P_{23}\cos\varphi)\cos\Theta+\left\{(P_{11}-P_{22})\sin 2\varphi-2P_{12}\cos 2\varphi\right\}\sin\Theta\right]$$

with the quadrupole moment tensor

$$P_{11} = P_{22} = \frac{r_0^5\omega_z}{4c\omega^2}\left[\omega\omega_y r_0^{-\frac{3}{2}}\Gamma(\gamma r_0)+\tfrac{2}{3}B_0\left(\omega^2-\tfrac{3}{2}\omega_y^2\right)\right], \quad P_{33} = -2P_{11},$$

$$P_{12} = 0, \quad P_{13} = -\frac{\omega_z}{4c} r_0^{7/2}\Omega(\gamma r_0),$$

$$P_{23} = -\frac{r_0^5}{4c\omega^2}\left[(\omega_y^2-\omega_z^2)r_0^{-\frac{3}{2}}\omega\Gamma(\gamma r_0)+2\omega_y\omega_z^2 B_0\right]. \qquad \text{(A.18)}$$

From (A.16) we realize the interesting orthogonality relation

$$\boldsymbol{m}\boldsymbol{\omega} = 0. \qquad \text{(A.19)}$$

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